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REACHABLE SETS AND SET-THEORETIC EVOLUTION OF THE UNCERTAINTY IN LINEAR SYSTEMS

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Abstract:

In this paper we deal with a new approach to model Set-Theoretic uncertainties with the use of p-tensor forms associated with n-dimensional vectors. This generalization allows computation of reachable sets, for linear systems, arising from highly non-linearly defined initial state constraint sets. The paper represents a unifying effort within the Set-Theoretic technique for uncertainty modeling heretofore unnecessarily restricted to ellipsoids and polyhedra. The paper also introduces a new class of bi-linear systems related to the reachable set computation for forced systems. These new systems have explicit solutions and they are of independent interest. Some examples are included.

INTRODUCCION

The study of set-theoretic uncertainties and their evolution through linear dynamic systems has been restricted, so far, to two types of sets: ellipsoids and polyhedra (Schweppe[14] and Hnyilicza[8]). Within this line of thought, authors[2] have defined estimation and control problems that provide an alternative to the stochastic-probabilistic formulation of such problems. The set-theoretic modeling of uncertainties constitutes a practical alternative in the study of control problems where an intuitive knowledge of the uncertainty is available. This knowledge, usually in the form of bounds on the variables of interest, can be exploited to obtain a "mini-max" type of design.

The set-theoretic technique has also been of valuable assistance in minimum time intercept problems (Athans and Falb[1]), where knowledge of the set of reachable states and the fact that optimal trajectories stay at the boundary of this set (Principle of Optimal Evolution, Halkin[7]) rapidly provides a geometric answer to the problem. This usually results in a need of invoking Pontryagin's Maximum principle [10] .

In this paper we shall treat the computation of the reachable set for a linear, time-varying system by means of a new technique which is based on tensor p-forms associated with n-dimensional vectors[3] and the initial state set (uncertainty) evolution[8]. It will be seen that the generality introduced by this technique allows one to compute te reachable set for even non-convex, unbounded and non-connected initial state sets. At the same time ellipsoids and polyhedra become only particular cases of our method.

We begin our treatment by introducing a set of definitions which are commonly used troughout the paper. Next we develop some theory conducent to the computation of the reachable set for a linear dynamic system subject to initial state uncertainty in the form of a convex polyhedron[15]. This issue allows us to set the stage for the generalizations to be introduced later. At this point a new kind of bi-linear systems and their relation with the reachable set computation is established. We, then, briefly summarize the background related to tensor p-forms closely following Brockett[3,4], Sira[11] and Sandor and Williamson[13]. Next we present a new technique for modeling initial state set uncertainty and compute the reachable set within the permissable generality. The paper ends with some conclusions and discusions of general nature regarding the results and future research topics.

DEFINITIONS

Here we present some of the definitions that will be used in the sequel. A convex set in R^n is a set which contains all the intermediate points connecting, in a straight line, any two points of the set. A closed set in R^n is a set which contains all the limit points of the set. A bounded set in R^n is a set such that the quantity $\sup d(\underline{x},\underline{y})$ (see final section) is finite for all $\underline{x}$ and $\underline{y}$ in the set. We shall usually refer to closed, convex, bounded sets by "ccb" set. A hemispace is defined as the unbounded set of points in R^n which satisfy the linear inequality defined by: $\langle \underline{x}, \underline{h} \rangle \leq 1$. The vector $\underline{h}$ is known as the support vector of the hyperplane $\langle \underline{x}, \underline{h} \rangle = 1$ defining the hemispace. A polyhedron is a ccb set defined as the non-empty intersection of a finite number of hemispaces. i.e; a polyhedron is described as: $\{ \underline{x} \in R^n : \langle \underline{x}, \underline{h}_i \rangle \leq 1 \; ; \; i=1,2,\ldots N \geq n \}$ (Notice that the condition $N \geq n$ on the number of support vectors is only a necessary condition for boundedness). We refer to the set $\{\underline{h}_i ; \; i=1,2,\ldots N \}$ as the support set for the polyhedron. The support set completely characterizes the polyhedron. All the polyhedra we deal with contain the origin as can be easily infered from the definition.

Given the linar system:

$$\frac{d}{dt} \underline{x}(t) = A(t)\underline{x}(t) + B(t)\underline{u}(t) \qquad (1)$$

we denote its unique solution, starting at some time t_o from the initial state $\underline{x}_o$ and using a forcing function $\underline{u}(t)$, $0 \leq t \leq T$, by $\underline{\phi}(t; \underline{x}_o, \underline{u}(t))$. By $\underline{\phi}(T; \underline{x}_o, \underline{u}(t))$ we mean the value of this solution at time $t = T$. Let Σ_o be a ccb set in R^n. We define the reachable set at time T from Σ_o for a given $\underline{u}(t)$ the set:

$$\Phi(T; \Sigma_o, \underline{u}(t)) = \{ \underline{x} \in R^n : \underline{x} = \underline{\phi}(T; \underline{x}_o, \underline{u}(t)) \text{ for some } \underline{x}_o \in \Sigma_o \}$$

In other words the reachable set from Σ_o corresponding to $\underline{u}(t)$ is the set of all states generated by the dynamic action of the system[12] on the set Σ_o. We shall simply refer to the "reachable set" when we mean the definition just presented.

PROBLEM FORMULATION

In this section we define the reachable set computation problem for system (1) when the set Σ_o is given by a polyhedron. This serves two purposes; first we motivate the need for a more general formulation to be introduced in the next section. Secondly this allows us to call attention to a new class of bi-linear systems which have explicit solution and surely are systems of independent interest.

Problem 1

Given the linear system (1) and a polyhedron Σ_o given by its support set $\{\underline{h}_{io}\}$ ($i=1,2,\ldots,N \geq n$) find the reachable set $\Phi(T;\ \Sigma_o,\underline{u}(t))$ where $\underline{u}(t)$ is a given piecewise continuous function taking values on R^m and defined on $0 \leq t \leq T$.

Solution to Problem 1

In connection with the solution of problem 1 we first establish the following theorem related to the solution of certain bi-linear system.

Theorem 1. Let $\underline{f}(t)$ be a piecewise continuous bounded function in R^n and $\underline{q}_o$ a known vector. Then the non-linear (bi-linear) system:

$$\frac{d}{dt}\,\underline{q}(t) = F(t)\underline{q}(t) + \langle \underline{q}(t), \underline{f}(t) \rangle\, \underline{q}(t) \quad ; \quad \underline{q}(t_o) = \underline{q}_o \tag{2}$$

has a unique solution at time t given by :

$$\underline{q}(t) = \left(1 - \int_{t_o}^{t} \langle \underline{q}_o\, , \Phi'(\sigma,t_o)\underline{f}(\sigma)\rangle\, d\sigma\right)^{-1} \Phi(t,t_o)\underline{q}_o \tag{3}$$

where $\Phi(t,t_o)$ is the state transition matrix associated with F(t) and ' denotes transpose.

Proof The proof is easily done by direct computation of the time derivative of $\underline{q}(t)$ in (3). We shall prove uniqueness by considering the change of variables $\underline{z}(t) = \Phi(t_o,t)\underline{q}(t)$. This reduces system (2) to the form $d/dt\ \underline{z}(t) = \langle \underline{z}(t), \underline{h}(t)\rangle \underline{z}(t)$ with $\underline{z}(t_o) = \underline{q}(t_o)$ and $\underline{h}(t) = \Phi'(t,t_o)\underline{f}(t)$ Uniqueness of $\underline{z}(t)$ implies uniqueness of $\underline{q}(t)$ due to the non-singularity of the transition matrix. From the boundedness of $\underline{h}(t)$ and some elementary manipulations the function $\underline{w}(\underline{z},\underline{h}) = \langle \underline{z}, \underline{h}\rangle\, \underline{z}$ is shown to be a Lipschitz function. Uniqueness follows from well-known results[5].

Corollary 1 The non-linear system:

$$\frac{d}{dt}\,\underline{y}(t) = -F'(t)\underline{y}(t) - \langle \underline{y}(t), \underline{g}(t)\rangle\, \underline{y}(t) \quad ; \quad \underline{y}(t_o) = \underline{y}_o$$

with $\underline{g}(t)$ a piecewise continuous bounded function has a unique solution given by:

$$\underline{y}(t) = \left(1 + \int_{t_o}^{t} \langle \underline{y}_o, \Phi\,(t_o,\sigma)\underline{g}(\sigma)\rangle\, d\sigma\,\right)^{-1} \Phi'(t_o,t)\ \underline{y}_o$$

Proof. The proof easily follows from well-known properties of transition matrices of adjoint systems and some simple calculations.

Theorem 2 The reachable set $\Phi(T;\ \Sigma_o, \underline{u}(t))$ in Problem 1 is a polyhedron characterized by the support set $\{\ \underline{h}_i(T)\ \}$ ($i = 1,2,\dots,N \geq n$) where $\underline{h}_i(T)$ is given by the unique solution at time T of the bi-linear system:

$$\frac{d}{dt}\,\underline{h}_i(t) = -\,A'(t)\underline{h}_i(t) - <\underline{h}_i(t),\ B(t)\underline{u}(t)>\ \underline{h}_i(t)\quad ;$$

$$\underline{h}_i(t_o) = \underline{h}_{io} \qquad (4)$$

for all i.

Proof. The solution of (1) has the well known form $\underline{x}(t) = \Phi(t,t_o)\underline{x}_o + \int_{t_o}^{t}\Phi(t,\sigma)B(\sigma)\underline{u}(\sigma)d\sigma$ where $\Phi(t,t_o)$ is the state transition matrix associated with A(t). At every fixed moment of time this equation establishes an automorphism between the set of initial conditions Σ_o and the set of states $\underline{x}(t)$ [9]. This indicates that the initial state set is affinely equivalent to the region of reachable states of system (1). This confirms that these regions belong to the same class i.e; the form of the reachable set is uniquely determined by the form of the initial state set Σ_o. Then according to the affine classification theorem [9] the reachable set is a polyhedron. Moreover, the interior points of the initial state set become interior points of the reachable set. The computation of the support set is as follows: from the variation of constants formula we have $\underline{x}(t_o) = \underline{x}_o = \Phi(t_o,T)\underline{x}(T) - \int_{t_o}^{t}\Phi(t_o,\sigma)B(\sigma)\underline{u}(\sigma)d\sigma$. Substituting this value in the equation for the i-th hemispace defining the polyhedron Σ_o we obtain the equation which determines the i-th boundary for the reachable set at time T :

$$<\underline{x}(T),\Phi'(t_o,T)\underline{h}_{io}> \ \leq\ 1 + <\underline{h}_{io}\ ,\ \int_{t_o}^{T}\Phi(t_o,\sigma)B(\sigma)\underline{u}(\sigma)d\sigma> \ \triangleq\ m(T)$$

using the bi-linear property of inner products we obtain after some other manipulations:

$$<\underline{x}(T),\Phi'(t_o,T)\underline{h}_{io}\ /(\ 1+\int_{t_o}^{T}<\underline{h}_{io}\ ,\Phi(t_o,\sigma)B(\sigma)\underline{u}(\sigma)\ >d\sigma)> \ \leq\ 1$$

The result follows from the corollary to theorem 1.

Note when $1 + \int_{t_o}^{t} <\underline{h}_{io}\ ,\Phi(t_o,\sigma)B(\sigma)\underline{u}(\sigma)\ > d\sigma = 0$ for some i, this implies that the i-th hyperplane defining the polyhedron contains the origin. This hyperplane, however, is still orthogonal to the support vector $\Phi'(t_o,T)\ \underline{h}_{io}$. Notice that under the assumption that our polyhedra contain the origin, the possibility $m(T) < 0$ is precluded. In such a case, however, we can use $m(T) = \text{sgn}\, \dot{m}(T)\ |m(T)|$. In general, it is easy to show that the differential equation satisfied by the support set is: (sgn m(t) stands for signum function)

$$\frac{d}{dt}\underline{h}_i(t) = -A'(t)\underline{h}_i(t) - \text{sgn}\, m(t) < \underline{h}_i(t), B(t)\underline{u}(t) > \underline{h}_i(t)$$

$$\underline{h}_i(t_o) = \underline{h}_{io}$$

The reachable set is then described by: $\{ \underline{x} \in R^n : < \underline{x}, \underline{h}_i(t) > \ \leq \text{sgn}\, m(t)$ for all i } . In this manner we have, thus, relaxed the condition of having a polyhedron that contains the origin for all time $t > t_o$.

Corollary 2. The reachable set $\Phi(T; \Sigma_o, \underline{0})$ of system (1) is a polyhedron whose support set $\{\underline{h}_i(t)\}$ ($i=1,2,\ldots N \geq n$) is the unique solution of the linear system

$$\frac{d}{dt}\underline{h}_i(t) = -A'(t)\underline{h}_i(t) \qquad \text{with} \quad \underline{h}_i(t_o) = \underline{h}_{io} \qquad (5)$$

for each i.

Proof Immediate from the preceeding theorem upon making $\underline{u}(t) \equiv 0$, $0 \leq t \leq T$.

Comment: The corollary establishes the dual character of the support set description for polyhedra and their dynamic evolution through linear (unforced) systems. This character, however, does not hold in the general (forced) case where the governing equation for the support set evolution is clearly bi-linear.

Example

Consider the linear system $\dot{x}_1 = x_2$; $\dot{x}_2 = -x_1 + u$ with $u(t) = 1$ for all $t \in [0 , \beta)$ and $u(t) = -1$ for all $t \in (\beta , T]$. The initial state set Σ_o is characterized by the support set:

$$\underline{h}'_{10} = (1, 1) , \ \underline{h}'_{20} = (-1, 1), \ \underline{h}'_{30} = (-1,-1), \ \underline{h}'_{40} = (1,-1)$$

(see figure 1(a)).

The reachable set at time T is a polyhedron characterized by the support set:

$$\underline{h}'_1(T) = \left(\frac{\cos T + \sin T}{2(\cos\beta + \sin\beta) - \cos T - \sin T} , \frac{\cos T - \sin T}{2(\cos\beta + \sin\beta) - \cos T - \sin T}\right)$$

$$\underline{h}'_2(T) = \left(\frac{-\cos T + \sin T}{2-2(\cos\beta + \sin\beta) + \cos T - \sin T} , \frac{\sin T + \cos T}{2-2(\cos\beta + \sin\beta) + \cos T - \sin T}\right)$$

$$\underline{h}'_3(T) = \left(\frac{-\cos T - \sin T}{2-2(\cos\beta + \sin\beta) + \cos T + \sin T} , \frac{\sin T - \cos T}{2-2(\cos\beta + \sin\beta) + \cos T + \sin T}\right)$$

$$\underline{h}'_4(T) = \left(\frac{\cos T - \sin T}{2(\cos\beta - \sin\beta) - \cos T + \sin T} , \frac{-\sin T - \cos T}{2(\cos\beta - \sin\beta) - \cos T + \sin T}\right)$$

The representation of this set is given in figure 1(a) for several values of T. We also represent in fig. 1(b) the reachable set for a fixed value of T and several values of the switching instant β. Some of the features indicated in the preceeding note are present in this example.

GENERALIZATIONS

We now present some definitions closely following references [3,4,11,13]. If $\underline{x}$ is an n-vector with components $x_1, x_2, \ldots, x_n$ we denote $\underline{x}^{[p]}$ the $\binom{n+p-1}{p}$ dimensional vector of p-forms in $x_1, x_2, \ldots, x_n$ (i,e; the elements 6f the vector $\underline{x}^{[p]}$ are of the form $\alpha \prod_{i=1}^{n} x^{p_i}$ with $\Sigma p_i = p$, $p_i \geq 0$ and $\alpha = \sqrt{\binom{p}{p_1}\binom{p-p_1}{p_2} \ldots \binom{p-p_1-p_2-\ldots p_{n-1}}{p_n}}$). If $\underline{y} = A\,\underline{x}$ then $\underline{y}^{[p]} = A^{[p]}\underline{x}^{[p]}$ is verified and $A_{[p]}$ is the infinitesimal version of $A^{[p]}$, i.e; $A_{[p]} = \lim_{h\to 0} ((I + hA)^{[p]} - I^{[p]})$. Thus $d/dt\, \underline{x}(t) = A(t)\underline{x}(t)$ implies that $d/dt\, \underline{x}^{[p]}(t) = A_{[p]}(t)\, \underline{x}^{[p]}(t)$. Some useful properties of the so-called p-tensor powers are: 1) $(AB)^{[p]} = A^{[p]}B^{[p]}$; 2) $(A^q)^{[p]} = (A^{[p]})^q$ for q integer and A^q defined; 3) $(A')^{[p]} = (A^{[p]})'$.

<u>Lemma</u> 1 $d/dt\, \Phi_A^{[p]}(t,t_o) = A_{[p]}(t)\, \Phi_A^{[p]}(t,t_o)$ whenever $\Phi_A(t,t_o)$ is the transition matrix associated with the matrix A(t).

<u>Proof</u> $\underline{x}(t) = \Phi(t,t_o)\underline{x}_o$ then $\underline{x}^{[p]}(t) = \Phi^{[p]}(t,t_o)\, \underline{x}_o^{[p]}$. Now, $\frac{d}{dt}\underline{x}^{[p]}(t) = \frac{d}{dt}\Phi^{[p]}(t,t_o)\, \underline{x}_o^{[p]} = A_{[p]}(t)\, \underline{x}^{[p]}(t) = A_{[p]}(t)\, \Phi^{[p]}(t,t_o)\, \underline{x}_o^{[p]}$. The result follows.

We denote $\underline{x}^{(\underline{p})}$ (notice the vector character of p) the $\sum_{j=0}^{p}\binom{n+j-1}{j}$ dimensional vector $(1\ ,\ \underline{x}',\ (\underline{x}^{[2]})', \ldots, (\underline{x}^{[p]})'\ )'$. By extension of the above definitions if $\underline{y} = A\,\underline{x}$ then $\underline{y}^{(\underline{p})} = A^{(\underline{p})}\underline{x}^{(\underline{p})}$ where $A^{(\underline{p})}$*is a block-diagonal matrix of the form $\mathrm{diag}(1, A, A^{[2]}, A^{[3]}, \ldots, A^{[p]})$. It is easy to see that if $d/dt\, \underline{x}(t) = A(t)\underline{x}(t)$ then $d/dt\, \underline{x}^{(\underline{p})} = A_{(\underline{p})}(t)\underline{x}^{(\underline{p})}$ where $A_{(\underline{p})}$ is the infinitesimal version of $A^{(\underline{p})}$ and is clearly seen to be equal to $\mathrm{diag}(0, A, A_{[2]}, \ldots, A_{[p]})$. We can thus obtain the following lemma:

<u>Lemma 2</u> $d/dt\, \Phi^{(\underline{p})}(t,t_o) = A_{(\underline{p})}(t)\, \Phi^{(\underline{p})}(t,t_o)$ whenever $\Phi(t,t_o)$ is the transition matrix associated with A(t).

<u>Proof:</u> The lemma is a clear consequence of the definitions and Lemma 1.

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* This notation is not to be confused with the notation for <u>compound matrices</u>[3,6] which are defined as the matrices built up of all their pxp minors ordered lexicographically. This is the reason why we give p a vector character.

Definition. A generalized polyhedron is a set of the form:

$$\{ \underline{x} \in R^n: \langle \underline{h}_i, \underline{x}^{(p)} \rangle \leq 1 , i=1,2,\ldots,M \}$$

$\underline{h}_i$ is called a generalized support vector. The set $\{\underline{h}_i\}$ is the generalized support set . Note that a generalized polyhedron may well represent a non-convex, not necessarily bounded set (even non-connected sets may arise) in R^n. Polyhedra, spheres, ellipsoids, zonoids etc. may be viewed as particular cases of a generalized polyhedron as we have defined it.

We now formulate an extension of problem 1 with $\underline{u}(t) = 0$. The forced case is not treated here for reasons of space.

Problem 2

Given the linear system $\dot{\underline{x}}(t) = A(t)\underline{x}(t)$ and a generalized polyhedron Σ_o characterized by the generalized support set $\{\underline{h}_{i0}\}$ ($i= 1,2,\ldots, M$) find the reachable set $\Phi(T; \Sigma_o, \underline{0})$.

The solution to problem 2 requires use of the definitions and lemmae previously given. The following theorem is a consequence of this background material.

Theorem 3

The reachable set $\Phi(T; \Sigma_o,\underline{0})$ for problem 2 is a generalized polyhedron characterized by the generalized support set $\{ \underline{h}_i(T) \}$ $(i=1,2,\ldots M)$ where $\underline{h}_i(T)$ is given by the unique solution at time T of the linear system:

$$\frac{d}{dt} \underline{h}_i(t) = -A'_{(p)}(t)\, \underline{h}_i(t) \quad ; \quad \underline{h}_i(t_o) = \underline{h}_{io} \qquad (6)$$

Proof: The form of the reachable set is justified by the arguments given in the proof of Theorem 2 in the previous section. The computation of the generalized support set is as follows: $\Phi(T; \Sigma_o,\underline{0}) = \{\underline{x} \in R^n: \underline{x} = \Phi(t,t_o)\underline{x}_o$ for $\underline{x}_o \in \Sigma_o\} = \{\underline{x} \in R^n: \Phi(t_o,t)x \in \Sigma_o\} = \{\underline{x} \in R^n: \langle(\Phi(t_o,t)\underline{x})^{(p)}, \underline{h}_{io}\rangle \leq 1$ for $i =1,2,\ldots M\} = \{\underline{x} \in R^n : \langle \underline{x}^{(p)}, \Phi'^{(p)}(t_o,t)\underline{h}_{i0}\rangle \leq 1 ; i = 1,2,\ldots,M\}$. Letting $\underline{h}_i(t) = \Phi'^{(p)}(t_o,t)\underline{h}_{i0}$, the result follows from Lemma 2 and well-known properties of time derivative of an inverse matrix.

The preceeding theorem is a useful generalization that we illustrate by means of the following example:

Example

Consider the linear system: $\dot{x}_1 = x_2$; $\dot{x}_2 = u$, with u restricted to be zero after some given time T and also $|u(t)| \leq 1$ for $0 \leq t \leq T$. Assuming $x_1(0) = x_2(0) = 0$, we are asked to find the reachable set for all $t > T$.

Applying Pontryagin's Maximum principle[10] it is easily found that the reachable set at time T is a parabolic zonoid (intersection of two parabolic regions) which we can characterize as a generalized polyhedron of the form : $\{\underline{x} \in R^2 : < \underline{h}_{iT} , \underline{x}^{(2)} > \leq 1 \; ; \; i = 1,2 \}$ where:

$$\underline{h}'_{1T} = (0, 4/T^2, -2/T, 0, 0, 1/T^2) \; ; \; \underline{h}'_{2T} = (0, -4/T^2, 2/T, 0, 0, 1/T^2)$$

The set of reachable states for $t > T$ is also a parabolic zonoid characterized by the support set:

$$\underline{h}'_{1t} = (0, 4/T^2, (4/T^2)(T-t) - 2/T, 0, 0, 1/T^2) \; ;$$

$$\underline{h}'_{2t} = (0, -4/T^2, (-4/T^2)(T-t) + 2/T, 0, 0, 1/T^2)$$

Figure 2(a) and (b) show the reachable sets before and after $t = T$

The following is a particularization of Theorem 3 :

Corollary 3 If the set Σ_o is characterized by a homogeneous p-degree form in the state variables: $\{\underline{x} \in R^n : < \underline{h}_{io}, \underline{x}^{[p]} > \leq 1 \; ; \; i = 1,2, \ldots M \} = \Sigma_o$, then the reachable set is also characterized by an homogeneous p-degree form with support set $\underline{h}_i(t)$ given by the unique solution of the system: $d/dt \, \underline{h}_i(t) = -A'_{[p]}(t) \, \underline{h}_i(t)$ with $\underline{h}_i(t_o) = \underline{h}_{i0}$ for all i.

Comment: Notice that for the ellipsoidal case in which these sets are usually represented by $\{\underline{x} \in R^n : < \underline{x}, F \underline{x} > \leq 1 \}$*there is only one generalized support vector of dimension $n(n+1)/2$ obtained from the matrix F. For this we define a lexicographic map $\lambda : R^{n \times n} \rightarrow R^{n(n+1)/2}$ determined by $\lambda(F) = (f_{11}, f_{12}, \ldots, f_{1n}, f_{22}, f_{23}, \ldots, f_{2n}, \ldots, f_{n-1\,n-1}, f_{n-1\,n}, f_{nn})$ then the ellipsoid is also characterized by $\{\underline{x} \in R^2 : < \lambda(F), \underline{x}^{[2]} > \leq 1\}$.

Comment: One of the difficulties of the set-theoretic approach in characterizing uncertainties in the form of ellipsoids is constituted by the fact that the intersection of two ellipsoids is not an ellipsoid. Approximations are then necessary[14] in the form of ellipsoids. Our approach allows us to deal with this reallistic fact without approximations. The intersection of two ellipsoids $E_1 = \{\underline{x} : < \underline{x}, F_1 \underline{x} > \leq 1\}$ and $E_2 = \{ \underline{x} : < \underline{x}, F_2 \underline{x} > \leq 1 \}$ is simply expressed by $\{ \underline{x} : < \underline{h}_i , \underline{x}^{[2]} > \leq 1 \; ; \; i = 1,2\}$ where $\underline{h}_1 = \lambda(F_1)$ and $\underline{h}_2 = \lambda(F_2)$.

CONCLUSIONS

In this paper we have introduced a new technique for uncertainty modeling in a set-theoretic fashion. The generality allowed by the method makes it no longer necessary to restrict the uncertainty sets to ellip-

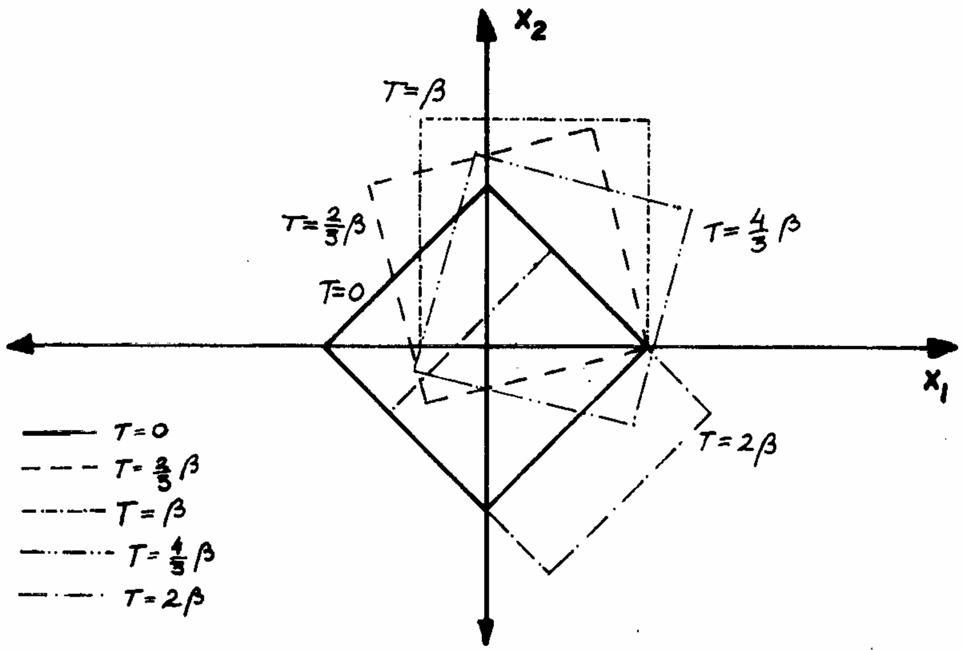
* F is a symmetric,positive definite matrix.

soids and polyhedra. Possibilities for modeling a larger class of sets are thus open while mantaining the inherent linearity characterizing the system under study. It has been shown that the evolution of the initial state set uncertainty represented by a rather general set can be computed in terms of the evolution of its "generalized support set" through a linear "p-tensor" system associated with the dynamical systems where the uncertainty has to be considered. The paper also introduced a new kind of bi-linear systems of independent interest but here adscribed to reachable set computation for a forced linear system with set-constrained initial state.

A generalization of theorem 3 for forced systems has been worked out by the author in an unpublished manuscript. The forced case entails a good deal of definitions which could not be presented here. Extensions of these results to discrete time-linear systems appear to bring a simplyfying development into the picture. Another useful generalization to the results here presented is represented by the inclusion of generalized bounds on the forcing term $\underline{u}(t)$ (i.e; generalized polyhedron).

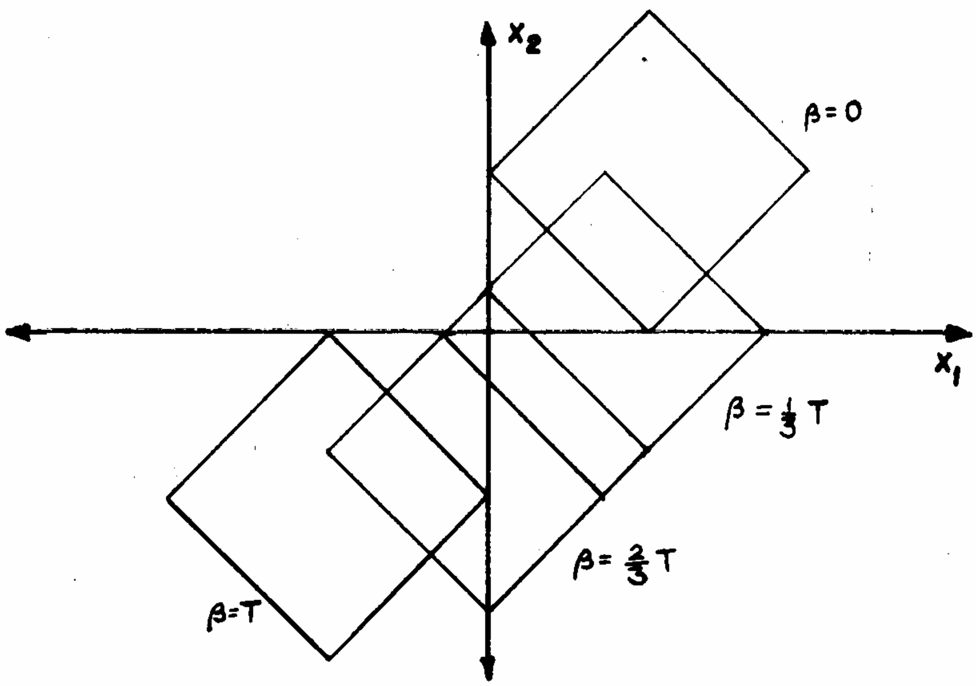
LIST OF SYMBOLS

$\{\}$	denotes :	A set
ϵ		belonging to...
$<\,,\,>$		Inner product
$\Phi(t,t_o)$		Transition matrix
$\Phi(T;\ \Sigma_o,\underline{u}(t))$		Reachable set at time T from the initial state set Σ_o with forcing function $\underline{u}(t)$
M'		Transpose of the matrix M
$\triangleq$		definition
$\lvert\ \rvert$		Absolute value
$\dot{\underline{x}}$		Time derivative of $\underline{x}$
$\binom{m}{n}$		$m!\ /\ n!(m-n)!$
$\underline{x}^{[p]}$		p-tensor form associated with $\underline{x}$
$\underline{x}^{(\underline{p})}$		family of j-tensor forms associated with $\underline{x}$ for $j = 0,1,\ldots,p$
$\sup d(\underline{x},\underline{y})$		suppremum of the distance between $\underline{x}$ and $\underline{y}$



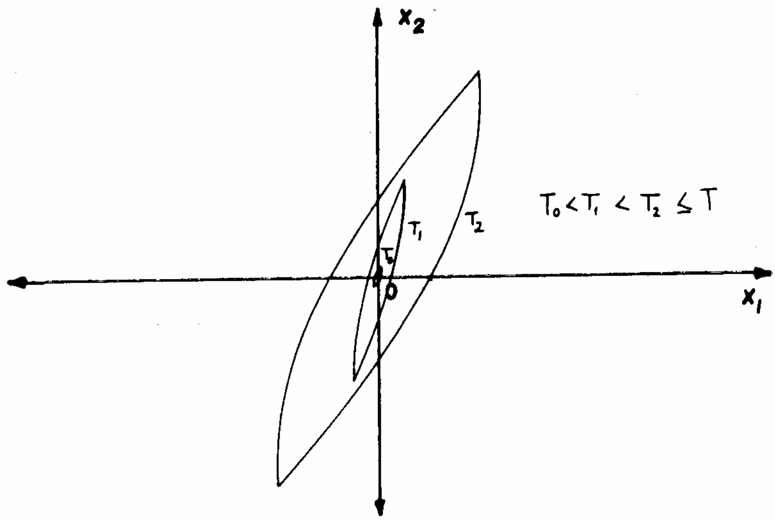
EVOLUTION OF THE REACHABLE SET FOR FIXED β

FIG. 1 (a)



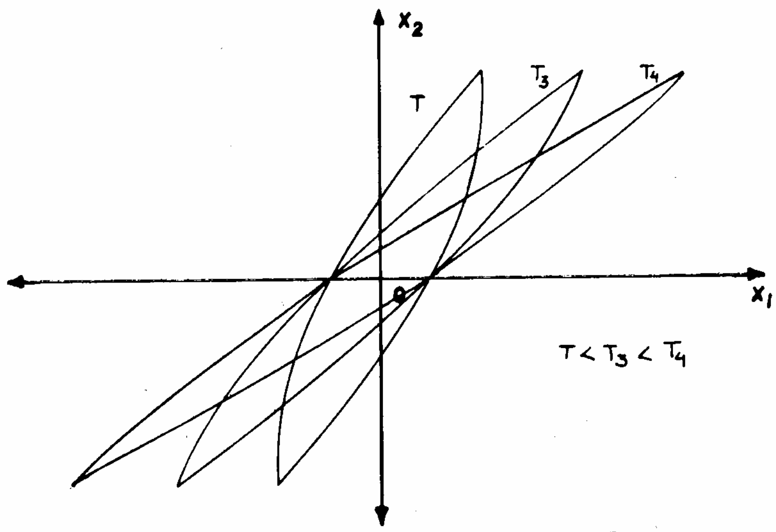
THE REACHABLE SET AT TIME T FOR DIFFERENT β'S

FIG. 1 (b)



THE REACHABLE SETS FOR $T_i \leq T$

FIG. 2 (a)



THE REACHABLE SET EVOLUTION AFTER T

FIG. 2 (b)

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