DETERMINING THE FEASIBLE REGION FOR NON-STATIONARY NON-LINEAR FEEDBACK SYSTEMS

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ABSTRACT

This paper presents a Carleman's linearization based method for obtaining arbitrarily close approximations to the feasible region of non-linear time-varying feedback control systems. The set of feasible states is characterized as a Generalized Polyhedron [15] whose support vectors evolution equations are explicitely established. Examples and implications of the results are included.

I INTRODUCTION

The set of feasible states for dynamic unforced systems is defined as the set of all possible solutions of the systems differential equation, at certain prescribed instant of time, arising from every initial state confined within the bounds of a compact set in the state space of the system.

Thus far, the studies dealing with the feasible set have been restricted to the linear case (Witsenhausen [1], Podchukaev [2], Sira-Ramirez [3]) and as initial state sets only ellipsoids and polyhedra have been considered (Schweppe [4], Hnyilicza [5], Bertsekas [6]). As applications of the feasible set concept, authors ([4],[5],[6], Schlaepfer [7], Sira-Ramirez [8]) have defined estimation and control problems from a non-probabilistic framework whose importance in applications is the basis of currently on-going research (Barmish [9],[10],[11],[12], Schmittendorf [13], Chukwu [14]).

This work tries to lift the linear case, as well as the ellipsoids and polyhedra, restrictions on set-theoretic uncertainties evolution problems by proposing the use of Carleman's linearization technique for feasible set computation on a class of non-linear analytic feedback systems. The technique allows arbitrarily close approximations to the feasible region of the non-linear system by considering the feasible set of a high-dimensional linear system with sparse system matrix.

Section II presents some basic definitions related to tensor powers of n-dimensional vectors, families of tensor powers, their use in defining Generalized Polyhedra (GP) as well as some other basic results regarding associated tensor systems to basic linear systems. (See also Brockett [19],[20]) This section contains a basic theorem due to Loparo and Blankenship [18] which constitutes the essence of later results about feasible regions of non-linear analytic feedback systems.

Section III deals with the basic problem treated in this paper, namely to determine the feasible region of a non-linear analytic system whose initial state is characterized by a GP (See also [15], [16]). This problem is readily solved using Loparo's and Blankenship result. The evolution equations for the Generalized Support Vecotrs (GSV) defining the feasible set are obtained via standard linear techniques.

Section IV discusses the implications of the results in set-theoretic estimation and control problems for non-linear systems. Some suggestions for further research are given at the end of this section.

II BASIC DEFINITIONS AND RESULTS

In this section we present some definitions which can also be found in [15],[16], and [19].

If $\underline{x}$ is an n-vector with components $x_1,x_2,\dots x_n$ we denote $\underline{x}^{[p]}$ the $\binom{n+p-1}{p}$-dimensional vector of p-forms in $x_1,x_2,\dots x_n$ (i.e the elements of of $\underline{x}^{[p]}$ are of the form : $\alpha \, \Pi_{i=1}^{m} x_i^{p_i}$ with $\Sigma p_i = p$; $p_i \geq 0$ and

$$\alpha = \sqrt{\binom{p}{p_1}\binom{p-p_1}{p_2}\cdots\binom{p-p_1-p_2-\cdots-p_{n-1}}{p_n}}$$

If $\underline{y} = A\,\underline{x}$ then $\underline{y}^{[p]} = A^{[p]}\underline{x}^{[p]}$ is verified and $A_{[p]}$ denotes the infinitesimal version of $A^{[p]}$, i.e $d/dt\ \underline{x}(t) = A(t)\ \underline{x}(t)$ implies $d/dt\ \underline{x}^{[p]}(t) = A_{[p]}(t)\ \underline{x}^{[p]}(t)$. Some useful properties of these "tensor powers" are : 1) $(AB)^{[p]} = A^{[p]}B^{[p]}$: 2) $(A^q)^{[p]} = (A^{[p]})^q$ whenever A^q is defined and q is an integer. 3) $(A')^{[p]} = (A^{[p]})'$

Lemma 1 $d/dt\ \Phi_A{}^{[p]}(t,t_0) = A_{[p]}\Phi_A{}^{[p]}(t,t_0)$ whenever $\Phi_A(t,t_0)$ is the state transition matrix associated with $A(t)$.

A proof of this simple fact can be found in [15].

We denote $\underline{x}^{(\underline{p})}$ (notice the vector character of p) the $\binom{n+p}{p}$-dimensional vector ($\underline{x}', (\underline{x}^{[2]})', \ldots, (\underline{x}^{[p]})'$)' (We also call this vector the p-th family power of $\underline{x}$). By extension of the above definitions if $\underline{y} = A\,\underline{x}$ then $\underline{y}^{(\underline{p})} = A^{(\underline{p})}\ \underline{x}^{(\underline{p})}$ where $A^{(\underline{p})}$ is a block-diagonal matrix of the form: diag [A , $A^{[2]}$, ..., $A^{[p]}$]. It is easy to see that if $d/dt\ \underline{x}(t) = A(t)\underline{x}$ then $d/dt\ \underline{x}^{(\underline{p})}(t) = A_{(\underline{p})}(t)\ \underline{x}^{(\underline{p})}(t)$ where $A_{(\underline{p})}(t)$ is the infinitesimal version of $A^{(\underline{p})}(t)$.

The following lemma extends the previous one to the case of p-th family powers of state transition matrices for linear systems.

Lemma 2 $d/dt\ \Phi_A{}^{(\underline{p})}(t,t_0) = A_{(\underline{p})}(t)\ \Phi_A{}^{(\underline{p})}(t,t_0)$

Definition 1 A Generalized Polyhedron (GP) is a set of the form:

$$\{\ \underline{x} \in R^n : \langle \underline{h}_i, \underline{x}^{(\underline{p})} \rangle \leq 1\ ;\ i = 1,2,\ldots,M\ \}$$

$\underline{h}_i$ is called a generazlized support vector (GSV). The set $\{\underline{h}_i : i=1,2,\ldots,M\}$ is the generalized support set. Note that a GP may well represent a non-convex, not necessarily bounded set. Polyhedra, spheres, ellipsoids, zonoids, etc. may be viewed as particular cases of GP

Definition 2 Given the linear system: $d/dt\ \underline{x}(t) = A(t)\ \underline{x}(t)$ we denote by $\underline{\xi}(t,\underline{x}_0)$ the trajectory starting from $\underline{x}_0$. A state $\underline{x}(T)$ is said to be $\underline{x}_0$-feasible if $\underline{\xi}(T,\underline{x}_0) = \underline{x}(T)$ for a certain $T > 0$. Let Σ_0 be a closed set in R^n, we define the Σ_0-feasible set (region), for the given linear system, at time T the set : $\{\underline{x} \in R^n :\ \underline{x} = \underline{\xi}(T,\underline{x}_0)$ for some $\underline{x}_0 \in \Sigma_0\ \}$. We denote this set by $\underline{\xi}(T, \Sigma_0)$. The Σ_0-feasible set at time t is then the set of all $\underline{x}_0$-feasible states at time t for all $\underline{x}_0 \in \Sigma_0$

Proposition 1 The Σ_0-feasible set for the linear system $d/dt\ \underline{x} = A(t)\ \underline{x}$ is a GP characterized by the support vectors $\underline{h}_i(t)$; $i=1,2,\ldots,M$, whenever Σ_0 is a GP characterized by the GSV $\underline{h}_{i0}$. Moreover $\underline{h}_i(t)$ is given, for each i , by the unique solution at time t of the linear vector differential equation: $d/dt\ \underline{h}_i(t) = -A'_{(\underline{p})}(t)\ \underline{h}_i(t)$ with $\underline{h}_i(t_0) = \underline{h}_{i0}$

Proof The form of the Σ_0-feasible set follows from the affine classification theorem [10]. The computation of the GSV of $\underline{\xi}(t,\Sigma_0)$ is as follows :
$\underline{\xi}(t,\Sigma_0) \triangleq \{\underline{x} \in R^n : \underline{x} = \Phi_A(t,t_0)\underline{x}_0$ for all $\underline{x}_0 \in \Sigma_0\} \triangleq \{\underline{x} \in R^n : \Phi_A(t_0,t)\underline{x} \in \Sigma_0\}$
$\triangleq \{x \in R^n : <(\Phi_A(t_0,t)\underline{x})^{\underline{(p)}}, \underline{h}_{i0}> \leqq 1$; for all i $\}$. Letting $\underline{h}_i(t) = \Phi_A'^{\underline{(p)}}(t_0,t)\ \underline{h}_{i0}$ we have $\underline{\xi}(t, \Sigma_0) \triangleq \{\underline{x} \in R^n : <\underline{x}^{\underline{(p)}}, \underline{h}_i(t)> \leqq 1$ for all i $\}$
The result follows from the definition of $\underline{h}_i(t)$, lemma 2 and well known properties of time derivatives of an inverse matrix.

Definition 3 Let $\underline{f}(\underline{x},t)$ be an R^n-valued uniformly bounded analytic function of $\underline{x}$ and piecewise continuous in t. Then we can express $f(\underline{x},t)$ as an infinite series in terms of the tensor powers of $\underline{x}$. Furthermore if $\underline{f}(\underline{0},t) = 0$ for all t, then the series takes the form $\underline{f}(\underline{x},t) = \sum_{k=1}^{\infty} F_k(t)\ \underline{x}^{[k]}$

Definition 4 Let $d/dt\ \underline{r}(t) = \sum_{k=2}^{\infty} F_k(t)\ \underline{r}^{[k]}(t)$ then for any $k > 0$ we have:

$$d/dt\ \underline{r}^{[k]}(t) = \sum_{j=k+1}^{\infty} R_{kj}\ \underline{r}^{[j]}(t)$$

and thus R_{kj} are implicitely defined.

Theorem 1 (Loparo and Blankenship [18])

Consider the non-linear system $d/dt\ \underline{x}(t) = \underline{f}(\underline{x}(t), t)$ with $\underline{x}_0 \in R^n$ with $\underline{f}$ as in definition 3, and let $\varepsilon > 0$, $T < \infty$ be given. Then the non-linear system has a unique continuous solution and there exist an integer $p = p(\varepsilon,T,\underline{x}_0)$, a linear map A_p such that the unique solution of the linear differential equation:

$$\frac{d}{dt}\underline{y}_p(t) = A_p(t)\ \underline{y}_p \quad \text{with} \quad \underline{y}_p(0) = \underline{x}_0^{[p]} \quad \text{and :}$$

$$A_p = \begin{bmatrix} F_1(t) & F_2(t) & F_3(t) \ldots\ldots & F_p(t) \\ \underline{0} & F_{1[2]}(t) & R_{23}(t) \ldots\ldots & R_{2p}(t) \\ \underline{0} & \underline{0} & F_{1[3]}(t) \ldots\ldots & R_{3p}(t) \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \\ \underline{0} & \underline{0} & \ldots\ldots & F_{1[p]}(t) \end{bmatrix}$$

satisfies the property:

$$\sup_{0 \leqq t \leqq T} \| \underline{x}(t) - \underline{\tilde{x}}_p(t) \| < \varepsilon$$

where $\underline{\tilde{x}}_p(t) = [\, I_{nxn} : \underline{0}\,]\ \underline{y}_p$

Proof The proof of this theorem can be found in [18] in complete detail.

In essence the theorem guarantees that for a large class of non-linear analytic systems, we can always find a sufficiently close approximation by finding the solution of a high dimensional linear system whose parameters are computable in terms of the coefficients of the Volterra series expansion of the non-linear map. The forcing initial state is related through a tensor power to the original initial state of the non-linear system. The projection of the trajectory solution, for this high dimensional system, into the original state space produces the desired approximation. Loparo and Blankenship [18] utilize this result for estimating the domain of attraction of non-linear feedback systems.

III FEASIBLE REGIONS FOR NON-LINEAR SYSTEMS

In this section we make use of Loparo's result to approximate arbitrarily close the feasible region of a non-linear system whose initial state is bounded by a compact GP. The theorem given in this section is a simple generalization of theorem 1 and follows from the results in section II.

Consider the non-linear system:

$$\frac{d}{dt}\underline{x}(t) = \underline{f}(\underline{x}(t), t) \quad \text{with} \quad \underline{x}_0 \varepsilon \Sigma_0 \tag{3.1}$$

where Σ_0 is a compact GP defined by:

$$\Sigma_0 = \{\underline{x} \varepsilon R^n : \langle \underline{x}^{(q)}, \underline{h}_{i0} \rangle \leq 1;\ i=1,2,\ldots,M\} \tag{3.2}$$

Then the Σ_0-feasible set at time T can be approximated to within a desired error by utilizing the result of theorem 1 since each and everyone of the feasible trajectories arising from an initial state in Σ_0 can be approximated. The set of all the approximating solutions at time T, constitutes an approximate representation of the feasible set originated by Σ_0

Theorem 2 Consider the non-linear system (3.1) with $\underline{f}$ as in definition 3 and the initial state x_0 confined within the bounds of a compact set described by (3.2). Then the set:

$$\{\underline{x} \varepsilon R^n : \langle \underline{x}^{(p)}, \tilde{\underline{h}}_i(T) \rangle \leq 1;\ i= 1,2,\ldots, M\} \tag{3.3}$$

where $\tilde{\underline{h}}_i(T)$ is given by the unique solution at time T of the linear differential equation :

$$\frac{d}{dt}\tilde{\underline{h}}_i(t) = - A_p'(t)\, \tilde{\underline{h}}_i(t) \quad \text{with} \quad \tilde{\underline{h}}_i(0) = [\underline{h}_{i0};\ \underline{0}] \tag{3.4}$$

$p = p(\varepsilon,T,\Sigma_0)$ (with ε a prespecified approximation error) > q

$p = q$ if $p(\varepsilon,T,\Sigma_0) \leq q$

approximates to within a desired ε the Σ_0-feasible set of system (3.1) subject to initial state constraints of type (3.2).

Proof Theorem 2 is a direct consequence of Theorem 1 established by Loparo and Blankenship. It should be noted however that because of the possible difference that may arise among the parameters $p(\varepsilon,T,\Sigma_0)$ and q, one should always choose the greater,so that (3.4) always makes sense and the specified error can even be made smaller. Notice that p can also be considered as a function of each index i. It is not difficult to see that among all possible support vectors defining the initial state set, one should choose as a value of p that which is rendered maximum as a function of the support vectors considered one by one.

Even if the initial state set is not a GP, as often occurs when trascendental functions are used to define such sets, it can be shown under certain mild conditions that these can also be approximated by a GP to within an ε. Obviously in such cases our approach would require a second approximation to the problem of finding a description for the Σ_0-feasible set.

We now present some simple examples which underline some of the salient features of non-linear maps, induced by non-linear systems, on compact, convex sets. It will be seen how convexity is destroied by the action of a non-linear map.

Example 1 Consider the following simple non-dynamic example given by the action of a non-linear map on a convex polyhedron.

Let: $y_1 = x_1^2 + x_2$ and the set:

$y_2 = \sqrt{2}\, x_1$ $\{ \underline{x} \in R^2 : < \underline{x} , \underline{h}_i > \leq 1 ; i=1,2,3,4 \}$

with: $\underline{h}'_1 = (1,1)$; $\underline{h}'_2 = (1,-1)$

$\underline{h}'_3 = (-1, 1)$ $\underline{h}'_4 = (-1,-1)$

Fig. 1 shows the first and second order approximations to the set into which the polyhedron is mapped by action of the non-linear relation defined above.

The above example clearly shows how convexity of a compact set is eliminated by a non-linear map even in the simplest case of non-linearity . The Carleman linearization approach for this example rapidly renders an image which is practically impossible to differentiate from the actual exact image set.

Example 2 Consider the Van der Pol differential equation:

$$x'' + x - (1- x^2) x' = 0$$

which can also be written in state space form in the usual way :

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = - x_1 + (1- x_1^2)x_2$$

Suppose we are given a set of initial states represented by the unit circle:

$$x_1^2 + x_2^2 \leq 1$$

which is a Generalized Polyhedron in R^2 with support vector given by:

$$\underline{h}'_1 = (0,0,1,0,1)$$

The feasible set can then be easily computed using our preceeding approach by succesive approximations, i.e with p= 2,3,.. , for a certain prespecified terminal time T. The first approximation to the reachable set (p=2) turns out to be an ellipse while from the third approximation on, we obtain a figure closely resembling the limit cicle contour. The procedure is easily implemented on a digital computer given the linear nature of the equations. Figure 2 gives an idea of the shape of the feasible set for the Van der Pol system. The non-connectedness of this set is clearly shown in this figure.

The above examples show how convexity and simple connectedness of a compact initial state set are lost by the action of non-linear maps either generated or not by a non-linear differential equation. However, it should be emphasized that this need not be the case in every situation. The follo-win example indicates that in certain case these two important features of some sets, could be preserved by the action of the non-linear system.

Example 3 The second order equation:

$$y'' + ay + by^3 = 0 \quad ; \quad \text{with } a > 0 , \ b > 0$$

can be rewritten as a system :

$$y' = x$$
$$x' = -ay - by^3$$

It is easy to see that the initial state set $x^2 + y^2 \leq 1$ generates , afeter a sufficiently large time, a feasible set represented by the region:

$$x^2 + ay^2 + \frac{1}{2} by^4 \leq K$$

which is a convex, connected region closely resembling an ellipse.

IV CONCLUSIONS AND SUGGESTIONS FOR FUTURE RESEARCH

This paper has examined an approach for computing the feasible regions of a non-linear analytic system. The approach is based in a Carleman's linearization technique and it only involves computations associated with linear dynamic systems for which a vast amount of computer based resources are readily available. The drawbacks of the method rely on the fact that a high dimensional linear system has to be solved for obtaining the required approximation. On the other hand the computations associated with this large sistem are substantially facilitated due to the sparse nature of the system matrix (tipically a 20x20 matrix generated by the procedure, contains only about 70 non-zero elements).

The results outlined in this paper have implications in several areas related to the study and control of non-linear systems. First of all, the computation of the domain of attraction for non-linear systems([18]) can benefit from the introduction of the idea of starting directly with a generalized polyhedron as an intial guess for this region. This paper has shown that the hansling of such sets is mathematically convenient and requires no substantial extra effort.

The feasible set concept has, by itself, a number of implications in set-estimation and set-control problems. This paper simply widens the range of applicability of the theory by demonstrating that closely enough approximations can be worked for "modified target sets", "attainability sets" etc. (See [6]) related to set-reachability problems associated with non-linear systems.

The method allows for research into the field of target set reachability problems by non-linear feedback systems. Approximate control strategies can be obtained by considering parametric control of the linear approximation system when the feedback map is included explicitely in the original non-linear equations. (See [18] for this simple extension).

The idea of utilizing Generalized polyhdera in some of the existing problems related to set-theoretic estimation and control should be persued further due to the flexibility that these sets offer for the handling of the mathematical equations.

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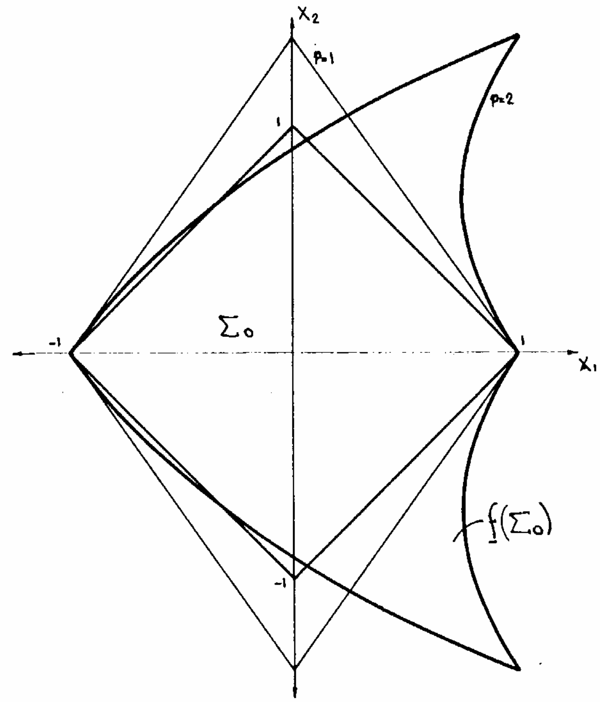

FIGURE 1

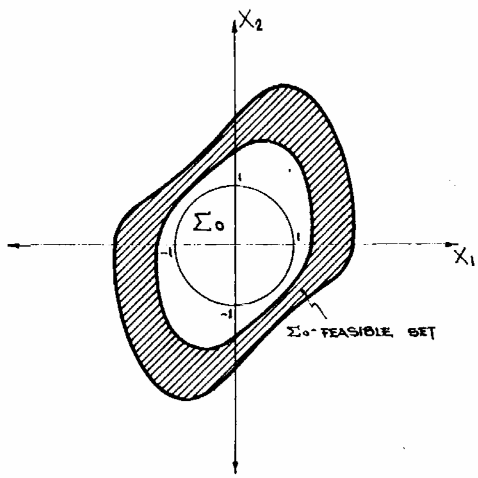

FIGURE 2
Σ_o-Feasible Set far van der pol system