EVOLUTION OF FUZZY SETS IN LINEAR DYNAMIC SYSTEMS

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ABSTRACT

Fuzzy sets are considered as a modeling device for initial state uncertainty in linear dynamic systems. Several linear operations on fuzzy sets are considered and their influence on the membership function. Linear dynamic systems with fuzzy initial states are then analyzed and the membership function evolution equation is derived for the characterization of the fuzzy state at some final time. Fuzzy inputs are also considered and their influence in the set of feasible fuzzy states is established. We obtain a characterization of the set of possible fuzzy states and the implications of this result in fuzzy objectives (targets) reachability problems.

I INTRODUCTION

The purspose of this paper is to introduce and develope the fuzzy set philosophy within the context of linear dynamic systems analysis and design. We use fuzzy sets as a modeling mechanism by which initial states, external signals and objectives of imprecise nature can be conveniently expressed. This article constitutes a sort of generalization of the set-theoretic technique for uncertainty handling (Schweppe [1], Bertsekas [2], Witsenhausen [3], Sira-Ramirez [4]) and set-valued objectives approach to control problems. We utilize fuzzy sets as a way of expressing a subjective knowledge of the initial state variables and external signals affecting the system (i.e statements of the form: " the first state variable, displacement, is rather small while the second, velocity, is too big at the starting time..." or, " we propose to use most of the control power at the final phases of the manuver.." are frequently used in practice to describe the systems state or the control policy). We compute the evolution of the fuzzy initial state set through the state space, i.e we obtain the description at some time t, of the fuzzy state set generated by the dynamic action of the system on the initial states whose only description is represented by a fuzzy set. The membership function of the set of feasible fuzzy states is characterized in terms of the system dynamics and the non-fuzzy (i.e "crisp") influence of external control signals. This simple characterization of the feasible states is the basis for further developements in the area of fuzzy estimation and reachability of fuzzy objectives in linear dynamic systems subject to the action of signals specified only by its membership value function in a fuzzy set. Although fuzzy sets have been introduced with the purpose of describing imprecision rather than uncertainty, (Zadeh [5],[6]) it is however important to realize that a large class of "uncertainties" affecting the performance of linear dynamic systems belong to the one describable by means of fuzzy sets. An experienced designer or plant engineer has always an intuitive feeling for the noisy signals and values of the state at a particular pont in time of the operation of the system. It is therefore possible to prescribe a fuzzy set as the only available knowledge of the external perturbation signals and initial states.

In a large class of systems described by dynamic equations, the objectives at a given point of the planning horizon is better expressed by a general and vague statement. This is particularly frequent in systems where human actions are included, i.e socio-economic systems, ecological systems, etc. Rather often a policy maker establishes a desired goal of, say, the economic system with statements such as : " We are trying to keep inflation *low* for the next ten years" or., " this policy will increase labor supply *rather abruptly* at the end of the economic term..." etc. Even in *mechanistic* systems, it is often desired to keep certain variables within qualitative bounds expressed by objectives which are best modeled by fuzzy sets at certain point in time of the systems evolution. A typical example would be: " For pipe capacitances *small enough* we want exit benzene temperature *low* for *rapid* variations of toluene flow rate..." etc.

It is noteworthy that fuzzy theory has rapidly evolved in directions out of the control systems discipline (Cognitive processes, Grammar, Graph theory, Risk Analysis [7]) while the question of utilizing fuzzy sets as a way of describing ill-defined (poor) knowledge of variables, signals and objectives for dynamic systems has been largely unexplored. This paper represents initial steps in this direction.

In section II we introduce some of the basic definitions, notation and results to be used in the rest of the paper. One of the modest contributions of this part is the introduction of the concept of "vector sum" of fuzzy sets as a generalization of the counterpart operation for "crisp" sets. In this section we present all the appropriate algebraic machinery concerning operations on fuzzy sets. This operations will arise in the processing of the fuzzy information available by the dynamic action of the system., i.e linear transformations, crisp translations, fuzzy translations, addition etc. of fuzzy sets (See also [8] for other types of operations).

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Section III presents some results related to the evolution of fuzzy sets through linear dynamic systems. It is also analyzed here, how the formulae can be exploited to treat fuzzy estimation problems fuzzy objectives reachability and perturbed reachability of fuzzy objectives. It is concluded that a fuzzy set reachability problem generates a crisp set reachability problem for which a good amount of literature is already available (Bertsekas and Rhodes [9], Delfour and Mitter [10], Sira-Ramirez [4] [11],[12])

Section IV presents some simple examples of fuzzy sets evolution through linear dynamic systems.

Section V provides some conclusions and suggestions for future research. Some works related to this article and references here cited are listed at the end of the paper.

II DEFINITIONS AND BASIC RESULTS

In this section we introduce the basic notation and definitions which will be used throughout the paper. Some of the basic results are also stated in the form of propositions.

Definition 1 We denote a fuzzy set A with membership function $\mu_A(\ )$ defined over the universe of discourse U in the n-dimensional euclidean space R^n as:

$$A = \int_U \mu_A(\underline{u}) / \underline{u} \qquad (2.1)$$

A non-fuzzy set will be termed "crisp". Its membership function has value 1 over the whole domain of the set.

A straightforward application of the "extension principle" [6], allows us to define linear transformations , crisp translations, etc. of fuzzy sets in R^n.

Definition 2 We define a crisp translation of a fuzzy set A in the direction of the vector $\underline{v}$, as the fuzzy set given by:

$$A \oplus 1/v = \int_{\mathcal{A}\oplus \underline{v}} \mu_A(\underline{u}-\underline{v}) / \underline{u} \qquad (2.2)$$

where $\mathcal{A}$ stands for the support of the fuzzy set A (i.e the set of points where the membership function of A is not zero). "$\oplus$" denotes vector sum [1].

As a generalization of the preceeding definition, consider the fuzzy singleton $\underline{v}$ with membership function $\mu_B(\underline{v})$ i.e $B = \{\mu_B(\underline{v}) / \underline{v}\}$.

Definition 3 A fuzzy translation of the fuzzy set A in the direction and extent of the fuzzy singleton B defined above, is defined as a fuzzy set described by:

$$A \oplus \{\mu_B(\underline{v})/\underline{v}\} = \int_{\mathcal{A}\oplus \underline{v}} [\mu_A(\underline{u}-\underline{v}) \wedge \mu_B(\underline{v})] / \underline{u} \qquad (2.3)$$

where the symbol "$\wedge$" stands for the minimum of the two function values specified to its sides.

As a further generalization of the previous definitions we introduce now the direct sum or "vector sum" definition of two fuzzy sets. This definition generalizes that of the vector sum of two crisp sets. (We define the vector sum of two crisp sets M and N as the set of elements which can be expressed ,non-uniquely, as the sum of elements in the sets M and N respectively)

Definition 4 Let A be a fuzzy set with compatibility function μ_A (i.e membership function) and similarly, let B be a fuzzy set characterized by μ_B. We define the vector sum (or direct sum) of A and B as the fuzzy set specified by:

$$A \oplus B = \int_{\mathcal{A}\oplus \mathcal{B}} \sup_{\underline{v}} [\mu_A(\underline{u}-\underline{v}) \wedge \mu_B(\underline{v})] / \underline{u} \qquad (2.4)$$

where $\mathcal{A}\oplus\mathcal{B}$ denotes that the universe of discourse is the vector sum of the crisp sets that serve as supports of the fuzzy sets A and B respectively. The "sup" operation is necessary to eliminate the possibility of having ill-defined membership values for those elements that can be expressed in a non-unique fashion as sum of elements in $\mathcal{A}$ and $\mathcal{B}$. This supremum operation usually results in a maximization operation.

The above formula (2.4) constitutes a natural " convolution operation" on the membership functions of two fuzzy sets. Naturally, this definition includes the case where A and B are crisp sets. The formula (2.4) is, not surprisingly, reminiscent of that which establishes the probability density function of the sum of two independent random variables. Note also that the roles of $\underline{u}$ and $\underline{v}$ can be interchanged.

Remark 1 It is a basic principle that the result can never be more precise than the data. Equally true, the sum of two fuzzy sets can not be a crisp set. This obvious fact is easily established by inspection of formula (2.4)

Definition 5 Let A and B be two fuzzy sets with membership functions μ_A and μ_B. The Pontryaguin difference of A and B is defined as a fuzzy set C such that $B \oplus C = A$. We often refer to this set C as the difference of A and B and denote it by $A \ominus B$.

Remark 2 Note that according to remark 1 it is not always possible to define the Pontryaguin difference of two sets. For instance suppose A is crisp and B is fuzzy. There is no fuzzy, or crisp, set C which added to B reproduces A. As a simple example consider the question of finding the set of numbers such that when added to those *close to zero* result in the set of numbers exactly between zero and one. It is therefore a necessary condition for the existence of $A \ominus B$ the existence of the support set $\mathcal{C} = \mathcal{A} \ominus \mathcal{B}$, but it is not sufficient.

As a simple application of the extension principle introduced by Zadeh [6], we define now the direct and inverse images of a fuzzy set A in R^n under a linear transformation.

Definition 6 Given a non-singular linear transformation P , and a fuzzy set A with membership function μ_A, we define the direct image under P of the fuzzy set A, the fuzzy set given by:

$$P(A) = \int_{P(\mathcal{A})} \mu_A(P^{-1}\underline{u}) / \underline{u} \qquad (2.5)$$

This simple result allows for the characterization of fuzzy initial state set evolution in linear dynamic systems.

Definition 7 The inverse image of a fuzzy set A, with membership function μ_A, under a linear map Q, not necessarily invertible, is the fuzzy set described by means of:

$$Q^{-1}(A) = \int_{Q^{-1}(A)} \mu_A(Q\underline{u}) / \underline{u} \qquad (2.6)$$

The inverse operation above on the linear map Q is only notational. Sira[4],[11],[12] shows that the actual inverse never needs be computed for the case of crisp sets such as ellipsoids, polyhedra and Generalized Polyhedra.

III FUZZY SETS AND LINEAR DYNAMIC SYSTEMS

In this section we shall analyze the evolution of initial state fuzzy sets due to the dynamic action of linear time-varying continuous systems whose state is described by a vector differential equation of the following type:

$$\frac{d}{dt}\underline{x}(t) = A(t)\underline{x}(t) + B(t)\underline{u}(t) \qquad (3.1)$$

$$\text{with } \underline{x}_0 \tilde{\varepsilon} X_0$$

where "$\tilde{\varepsilon}$" denotes fuzzy membership to the fuzzy set X_0 characterized by μ_0.

We shall study the structure of the set of possible fuzzy states at a later time $t \geq t_0$ and infer from this structure a possible way of controlling a system to attain a fuzzy objective prescribed at some time T in the planning horizon.

Proposition 1 Given a linear dynamic system described by (3.1) with X_0 a fuzzy set as above, representing all available knowledge of the initial state $\underline{x}(t_0) = \underline{x}_0$. Then the set of feasible states at time t, is a fuzzy set characterized by a membership function μ_t given by:

$$\mu_t = \mu_0(\Phi^{-1}(t,t_0)(\underline{x}-\underline{v})) \qquad (3.2)$$

where $\underline{v} = \int_{t_0}^{t} \Phi(t,\sigma)B(\sigma)\underline{u}(\sigma)d\sigma$

i.e the set of fuzzy states at time t is a linear transformation of a translation of the initial state fuzzy set. The transformation is determined by the state transition matrix $\Phi(t,t_0)$ and the translation is defined by the history of the control signal $\underline{u}(t)$ in the interval $[t_0,t]$.

The proof of the above proposition is an immediate consequence of the definitions in the previous section and the fact that the solution of (3.1) is given by the variation of constants formula (Brockett [13].

The open loop control signals $\underline{u}(t)$ are seen to affect only the position of the set of fuzzy final states of the system. The signals do not influence the shape of the membership function μ_0. This fact is important in establishing a relationship among fuzzy objectives reachability problems and target set reachability problems. We shall deal with these problems later in this section.

As a corolary to the above proposition, if the linear system is:

$$\frac{d}{dt}\underline{x}(t) = A(t)\,\underline{x}(t) \quad \text{with } \underline{x}_0 \tilde{\varepsilon} X_0$$

then the set of possible states X(t) is given by:

$$X(t) = \int_{R^n} \mu_0(\Phi^{-1}(t,t_0)\underline{x}) / \underline{x} \qquad (3.3)$$

We shall now treat the problem of obtaining the description of the fuzzy set of final states when the signals affecting the behaviour of the system are of imprecise (fuzzy) nature.

Consider the system described by (3.1) with $\underline{u}(t)$ being a vector at time t, whose only description is given by the fact that it belongs to certain prescribed fuzzy set Ω. If additionaly our dynamic system has an initial state described by a fuzzy set then the set of fuzzy reachable states at time t is also characterized by a fuzzy set. Let $L(\underline{u})$ denote the linear operation defined by:

$$L(\underline{u}) = \int_{t_0}^{t} \Phi(t,\sigma)B(\sigma)\underline{u}(\sigma)d\sigma \qquad (3.4)$$

where $\Phi(t,\sigma)$ is the state transition matrix associated with A(t). Brockett [13] shows that the range space of L( ) coincides with the range space of a so-called Gramian matrix $W(t_0,t)$. The null spaces of both operators also coincide. We can construct then a matrix V whose columns constitute an orthonormal basis for the range space of $W(t_0,t)$. Therefore the solution of (3.1) can also be expressed in terms of the matrix V(t) as:

$$\underline{x}(t) = \Phi(t,t_0)\underline{x}_0 + V(t)\,\underline{y}$$

where $\underline{y}$ is uniquely determined by $\underline{u}(t)$ whenever the Gramian matrix $W(t_0,t)$ is invertible (i.e the system is controllable). Corresponding then to the set of fuzzy controls Ω we can find a set Ψ of fuzzy character where the variables $\underline{y}$ exist. The problem of computing the fuzzy reachable set is then reduced to compute the support and membership function of :

$$X(t) = \Phi(t,t_0)X_0 \oplus V(t)\,\Psi \qquad (3.5)$$

This can be done by using the formulae developed in the previous section for linear transformations and vector addition of fuzzy sets. Formula (2.5) should use the pseudo-inverse of the matrix V(t) for the case we are treating.

The following proposition summarizes and concludes the discussion above for the case of controllable linear systems.

Proposition 2 Let X_0 be a fuzzy initial state set and Ω a fuzzy set of external influences to the linear dynamic system (3.1). Then the set of reachable fuzzy states X(t) is expressed, whenever the system is controllable, as:

$$X(t) = \Phi(t,t_0)X_0 \oplus \Phi(t,t_0)W(t_0,t)\Phi'(t,t_0)\Theta \qquad (3.6)$$

Where Θ is the inverse image fuzzy set of Ω under the linear map B'(t). $W(t_0,t)$ is given by:

$$W(t_0,t) = \int_{t_0}^{t} \Phi(t_0,\sigma)B(\sigma)B'(\sigma)\Phi'(t_0,\sigma)d\sigma \qquad (3.7)$$

Proof The proof of the proposition follows from the preceeding discussion and the fact that the reachable set for (3.1) can be expressed by elements of the form $\Phi(t,t_0)\underline{x}_0 + \Phi(t,t_0)W(t_0,t)\underline{\eta}$ where $\underline{\eta}$ is related to $\underline{u}$ by means of $\underline{u} = B'(t)\Phi'(t_0,t)\underline{\eta}$ (See [13]).

We now turn to the formulation of fuzzy target (or fuzzy objective) reachability problem for a linear system such as (3.1).

Formulation of Fuzzy Objectives Reachability Problem

Suppose we are given a linear dynamic system described by a vector differential equation (3.1), with a fuzzy description of the initial state represented by a fuzzy set X_0. We are also given a fuzzy set X_T specified at a particular time $T > t_0$. We are asked to find conditions which will ensure the existence of a control signal, or at least a fuzzy set to which it may belong, such that the fuzzy objective X_T is reached by the system state at time T. The control signals are bounded by a crisp set Ω.

We understand that the reachability of a fuzzy objective by an essentially fuzzy state means that the fuzzy set containing the fuzzy state is included in the fuzzy objective. i.e from a fuzzy set-theoretic viewpoint, the membership function of the set of fuzzy states has to be less or equal than the membership function of the fuzzy target set.(See [6])

Let μ_0 be the membership function characterizing X_0 and let μ_T be the corresponding function for the fuzzy set X_T. Then the application of a control policy $\underline{u}(t)$ on the system renders a set of feasible states of fuzzy nature at time T which is expressed as :

$$X(T) = \int_{R^n} \mu_0(\Phi^{-1}(T,t_0)(\underline{x} - \underline{v}(T))/\underline{x} \qquad (3.8)$$

where $\underline{v}(T)$ is the unique solution at time T of the differential equation:

$$\frac{d}{dt}\underline{v}(t) = A(t)\,\underline{v}(t) + B(t)\underline{u}(t), \quad \underline{v}(t_0) = \underline{0} \qquad (3.9)$$

Consider the crisp set:

$$R(T) = \{\underline{v} \in R^n: \ \mu_0(\Phi^{-1}(T,t_0)(\underline{x}-\underline{v})) \leq \mu_T(\underline{x}) \ \}$$

The following proposition establishes that a fuzzy objective reachability problem for a linear system generates an equivalent target set reachability problem.

Proposition 3 The fuzzy objective reachability problem defined above has a solution if and only if there exists a control signal $\underline{u}(t)$ in Ω , such that the target set R(T) is reached by the state of system (3.9) at time T.

Proof The proof of the proposition is immediate from the preceeding discussion and definitions.

The preceeding Target Set Reachability problem has been treated by several authors. (See for example Delfour and Mitter [10], Barmish and Schmitendorf [14], Barmish, Flemming and Thorp [15], Chukwu and Silliman [16] and others [4],[9],[11] and [12])

Notice that the existence of a crisp set R(T) as defined above is a necessary condition for the existence of solutions to the fuzzy objectives reachability problem (FORP).It is clear that such condition is non sufficient due to the fact that (3.9) might not be controllable to the target R(T) in case the set exists.

Consider now a perturbed system described by the vector first order differential equation:

$$\frac{d}{dt}\underline{x}(t) = A(t)\underline{x}(t) + B(t)\underline{u}(t) + G(t)\underline{w}(t) \qquad (3.10)$$

with $\underline{x}_0 \in X_0$ and $\underline{w}(t) \in \Omega_w$ for all t. A problem similar to the FORP can be defined for this perturbed case. The signal $\underline{w}(t)$ acts as a noisy input to the linear dynamic system whose only description is represented by the fuzzy set Ω_w with membership function μ_w. As before we set as an objective for the perturbed system (3.10) the reachability of a fuzzy set X_T at certain time T. The control signal $\underline{u}(t)$ is bounded by a crisp set Ω_u .

It is not difficult to see that the above problem, which we shall refer to as the perturbed fuzzy objectives reachability problem (PFORP),can be reduced to a simple FORP where the fuzzy objective has to be modified by the influence of the fuzzy input signals in the final reachable set.

Suppose we denote by $R_w(T)$ the fuzzy feasible set of the system:

$$\frac{d}{dt}\underline{\tilde{x}}(t) = A(t)\underline{\tilde{x}}(t) + G(t)\underline{w}(t), \quad \underline{\tilde{x}}(t_0) = \underline{0}\ ;\ w(t) \underset{\sim}{\in} \Omega_w \qquad (3.11)$$

then the PFORP is equivalent to the following FORP (unperturbed):

Find a control signal within Ω_u such that the fuzzy state of the system:

$$\frac{d}{dt}\underline{x}(t) = A(t)\underline{x}(t) + B(t)\underline{u}(t), \quad \underline{x}(t_0) \underset{\sim}{\in} X_0 \qquad (3.12)$$

belongs to the fuzzy set $X_T \ominus R_w(T)$.

In turn then, a PFORP is reduced to a Target Set Reachability problem. The Pontryaguin difference set of the sets X_T and $R_w(T)$ has to exist for the problem to make sense. We suppose that the difference set exists i.e the fuzziness present in the noisy signal $\underline{w}$ is not as large as to totaly preclude the state objective to be recognizable. As before the fact that the modified objective is recognizable does not guarantee that it is reachable.

Since the PFORP is equivalent to a FORP we shall indicate how to find a fuzzy set of control strategies for a FORP. First let us introduce a definition.

Definition 8 Let V(t) be a matrix whose columns form an orthonormal basis for the range space of the linear operator defined in (3.4). Then the set of fuzzy feasible translations for the FORP defined above is given by:

$$\zeta = V^{-1}[\ X_T \ominus \Phi(T,t_0)X_0\ ] \cap \Omega \qquad (3.13)$$

i.e the set of fuzzy feasible translations is the inverse image under V of the difference of the fuzzy objective and the initial state evolution. The translations must also be permissible as (3.13) establishes by the intersection operation. The computation of the fuzzy set ζ allows the prescription of fuzzy policies for the operation of the system. This type of answer is of importance in the control of *humanistic* systems where general qualitative controls are useful to the designer since they serve as a guide to asses behaviour of the system in the future.

Fuzzy estimation problems arise when either the state of the system is fuzzy or the measurements performed on the state are fuzzy. Both situations frequently arise also at the same time. In such cases a fuzzy state set is obtained as an estimate of the state of the system. Fuzzy estimation can be considered as a generalization of the so-called Set-theoretic estimation discipline within the control systems literature. A good amount of work has been carried in the past on problems of this nature (See Schweppe [1],[17],[18], Schlaepfer [19], Schlaepfer and Schweppe [20]). Little has been explored,however, in the fuzzy set context. We shall only deal, briefly,with estimation problems defined on fuzzy systems of discrete nature rather than continuous.

Suppose we have a linear system described by (3.10), with fuzzy initial state set X_0 and noisy signal $\underline{w}$ described by a fuzzy set Ω_w. Suppose we measure the state of the system in an imprecise fashion. i.e we observe the system state through a device which introduces an error whose only description is given by a fuzzy set. We assume that the measurements are performed at discrete instants of time and it is our interest to extract all possible information regarding the state at those particular instances.

$$\underline{y}(t_k) = H(t_k)\,\underline{x}(t_k) + \underline{v}(t_k) \qquad (3.14)$$

where $\underline{y}$, the measurement has p components and H has the appropriate dimensions. The perturbation vector $\underline{v}(t_k)$ can be supposed to belong to a fuzzy set which specifies the imprecision of the measurement device. We denote this fuzzy set as Ω_v .

Due to the nature of the measurements , the dynamic system can be expressed as:

$$\underline{x}(t_{k+1}) = \Phi(t_{k+1},t_k)\underline{x}(t_k) + \tilde{F}(t_k)\underline{u}(t_k) + \tilde{G}(t_k)\underline{w}(t_k) \qquad (3.15)$$

We are also assuming that the control signals remain constant during the sampling period. $\tilde{F}$ and $\tilde{G}$ are easily computed in terms of B, G and the state transition matrix.

The estimation procedure is accomplished in two basic steps:

1) Prediction step. The available information at time t_k is propagated through the system dynamics to obtain the set of possible states at time t_{k+1} . The state at time t_k is introduced in the system equations as an entire fuzzy set and transformed by means of the state transition matrix. The reachable set at time t_{k+1} computed in this fashion will be denoted by $X(k+1|k)$.
2) Update step. Measurements are performed at time t_{k+1} on the system state via measurement program (3.14). The fuzzy set of states compatible with this measurement is then computed.

Below we give some of the formulae (in a fuzzy set-theoretic fashion) that need be used in the above steps.

$$X(t_{k+1}|t_k) = \Phi(t_{k+1},t_k)X(t_k) \oplus \{F(t_k)\underline{u}(t_k)\} \oplus G(t_k)\,\Omega_w \qquad (3.16\text{ -}17)$$

$$X(t_{k+1}|t_{k+1}) = X(t_{k+1}|t_k) \cap -H(t_k)[\Omega_v \oplus -\underline{y}(t_k)]$$

The estimation process is started with the condition :

$$X(t_0|t_0) = X_0 \qquad (3.18)$$

We shall not present the specific formulae in terms of membership functions since these can be easily infered from standard concepts and operations on fuzzy sets available in the existing literature, Zadeh [6].

IV EXAMPLES

In this section we shall give some simple examples of the academic type.

Example 1 Consider the linear time varying system:

$$\frac{d}{dt}\underline{x}(t) = A(t)\,\underline{x}(t) \qquad (4.1)$$

with the initial state fuzzy set expressed as:

$$X_0 = \int_{R^n} e^{-\underline{x}'Q_0^{-1}\underline{x}} \;/\; \underline{x} \qquad (4.2)$$

where Q_0 is a positive definite symmetric matrix.

The set of fuzzy feasible states at time T is easily seen to be characterized by :

$$X(T) = \int_{R^n} e^{-\underline{x}'\,Q^{-1}(T)\,\underline{x}} \;/\; \underline{x} \qquad (4.3)$$

where Q(T) is the unique solution at time T of the matrix differential equation:

$$\frac{d}{dt}Q(t) = A(t)Q(t) + Q(t)A'(t) \qquad (4.4)$$

with $\qquad Q(t_0) = Q_0$

The matrix Q(t) is simmetric and positive definite for all $t \geq t_0$. It is interesting to note that with this class of examples a parallel to the set-theoretic technique can be established by considering the α-level sets (Zadeh [6]) of the membership functions.

Example 2 Consider the linear system:

$$\frac{d}{dt}\underline{x}(t) = A(t)\underline{x}(t) + B(t)\underline{u}(t) \qquad (4.5)$$

with the initial state characterized by the fuzzy set:

$$\int_{R^n} e^{-\underline{x}'Q_o^{-1}\underline{x}} \;/\; \underline{x} \qquad (4.6)$$

Consider the fuzzy objective set at time T.

$$X_T = \int_{R^n} e^{-(\underline{x}-\underline{m})'S^{-1}(\underline{x}-\underline{m})} \;/\; \underline{x} \qquad (4.6)$$

A control policy $\underline{u}(t)$ in the interval $[t_0,T]$ renders a fuzzy set of feasible states:

$$X(T) = \int_{R^n} e^{-(\underline{x}-\underline{v})Q^{-1}(T)(\underline{x}-\underline{v})} \;/\; \underline{x} \qquad (4.7)$$

where Q(t) satisfies (4.4) and $\underline{v}$ is the solution at time T of the vector differential equation:

$$\frac{d}{dt}\underline{v}(t) = A(t)\underline{v}(t) + B(t)\underline{u}(t)$$

$$\underline{v}(t_0) = \underline{0}$$

It is easy to show that the fuzzy objective is whenever the solution of (4.4) at time T satisfies

$$Q(T) \geq S \qquad (4.9)$$

and the system (4.8) is controllable to the state $\underline{m}$ in a finite time $T-t_0$. In this case the set R(T) is reduced to a singleton of non-fuzzy character represented by the vector $\underline{m}$. If $\underline{u}(t)$ is somehow restricted the problem is reduced to a constrained controllability problem which is readily solved using the results of [14].

V CONCLUSIONS

Initial steps have been given in the utilization of fuzzy set theory in the analysis and design of linear dynamic systems. It has been shown that set objectives reachability problem for linear systems,as well as estimation problems can be set in a fuzzy environment and qualitative analysis be carried out by using standard concepts already developed within the fuzzy theory. We have shown that a fuzzy objective (i.e qualitative) reachability problem reduces to a crisp set reachability problem. This kind of problems have been extensively treated in the past by many authors and a good number of results and techniques are available in the existing literature. The fuzzy philosophy of systems design could therefore greately benefit from those results. Some of the developements parallel those of the set-theoretic technique (ellipsoidal uncertainties) for certain class of membership functions (exponential quadratic). Some simple examples have been presented for illustration purposes.

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