EVOLUTION OF GENERALIZED SET-THEORETIC UNCERTAINTIES IN LINEAR SYSTEMS

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ABSTRACT

This paper considers a new approach in the modeling of set-theoretic uncertainties. P-tensor forms associated with n-dimensional state vectors are used as a modeling mechanism for the representation of polynomic constraints on initial states of the unknown but bounded type. This allows a straightforward computation of the set of possible states for linear systems with uncertain initial state. It is also possible to approximate arbitrarily close the set of reachable states by using the technique developed in this paper. The significance of the results in Set-Theoretic estimation and control problems is discussed.

I INTRODUCTION

The study of set-theoretic uncertainties and their significance in linear systems analysis and design has been restricted, thus far, to two types of sets: ellipsoids and polyhedra (Schweppe [1], Hnylicza [2]). Within this line of thought, authors (Bertsekas and Rohdes [3], Sira-Ramirez [4], [5]) have defined estimation and control problems that provide an alternative to their stochastic -probabilistic formulation. In connection with these problems, from a set-theoretic framework, the computaiton of the set of possible states is a most important and basic problem to be solved. Witsenhausen thus recognized this issue in an early work [6] that has triggered much of the ongoing reserach in this field (See also [7], [8]).

In this paper we treat the problem of computation of the set of feasible states for linear systems, of a time-varying nature, whose initial state is unknown but bounded by a set described by a finite number of polynomic type of constraints on the initial state variables. It is shown that an appropriate modeling technique for such sets is represented by linear restrictions on a family of p-tensor powers of the n-dimensional vector adscribed to the Euclidean space where the uncertainty is being modeled (See Brockett [9]). By considering an associated p-tensor system the evolution problem is transformed into a simple linear problem. We obtain linear evolution equations on a dual associated tensor space which describe the "generalized support vectors" evolution characterizing the state set.

Ellipsoids and polyhedra become only particular cases of "generalized polyhedra". This in turn, constitute a powerful approximation mechanism by which <u>reachable sets</u> of linear systems can be conveniently described.

The paper constitutes a unifying effort within the set theoretic technique for uncertainty modeling, hertofore, unnecesarily restricted to ellipsoids and polyhedra for the sake of mathematical convenience.

Some illustrative examples are presented as well as indications on how the results can be used in set-estimation and set-control problems (Target set and target tube reachability problems [3]).

Section II presents some definitions about p-tensor powers and "Generalized Polyhedra" (GP). In this section we also formulate the main problem. Section III deals with the characterization of the set of possible states for linear systems with initial state bounded by a GP. This section contains some examples and comments. Section IV discusses the results in terms of set-estimation and set control problems. Section V contains some conclusions and suggestions for future research.

II DEFINITIONS

We now present some definitions closely following Brockett [9]. If $\underline{x}$ is an n-vector with components $x_1, x_2, \ldots, x_n$ we denote $\underline{x}^{[p]}$ the $\binom{n+p-1}{p}$ - dimensional vector of p-forms in $x_1, x_2, \ldots, x_n$ (i.e the elements of the vector $\underline{x}^{[p]}$ are of the form $\alpha \Pi_{i=1}^{m} x_i^{p_i}$ with $\Sigma p_i = p$; $p_i \geq 0$ and

$$\alpha^2 = \binom{p}{p_1}\binom{p-p_1}{p_2}\ldots\binom{p-p_1-p_2-\cdots-p_{n-1}}{p_n}$$

If $\underline{y} = A\underline{x}$ then $y^{[p]} = A^{[p]}\underline{x}^{[p]}$ is verified and $A_{[p]}$ is the infinitesimal version of $A^{[p]}$, i.e $d/dt\ \underline{x}(t) = A(t)\underline{x}(t)$ implies $d/dt\ \underline{x}^{[p]}(t) = A_{[p]}(t)\underline{x}^{[p]}(t)$. Some useful properties of the so called tensor powers are: 1) $(AB)^{[p]} = A^{[p]}B^{[p]}$ 2) $(A^q)^{[p]} = (A^{[p]})^q$ whenever A^q is defined and q is an integer. 3) $(A')^{[p]} = (A^{[p]})'$

<u>Lemma 1</u> $d/dt\ \Phi_A^{[p]}(t,t_0) = A_{[p]}\Phi_A^{[p]}(t,t_0)$ whenever $\Phi_A(t,t_0)$ is the state transition matrix associated with $A(t)$.

<u>Proof</u> : We have $\underline{x}(t) = \Phi_A(t,t_0)\underline{x}_0$ then $\underline{x}^{[p]}(t) = \Phi_A^{[p]}(t,t_0)\underline{x}_0^{[p]}$. Taking derivatives in this expression we obtain: $d/dt\ \underline{x}^{[p]}(t) = (d/dt\Phi_A^{[p]}(t,t_0))\underline{x}_0^{[p]}$. On the other hand $d/dt\ \underline{x}^{[p]}(t) = A_{[p]}(t)\underline{x}^{[p]}(t) = A_{[p]}(t)\Phi_A^{[p]}(t,t_0)\underline{x}_0^{[p]}$. Then,

$$(d/dt\Phi_A^{[p]}(t,t_0) - A_{[p]}(t)\Phi_A^{[p]}(t,t_0))\ \underline{x}_0^{[p]} = 0$$

for all $\underline{x}_0$. The result follows.

We denote $\underline{x}^{(\underline{p})}$ (notice the vector character of p) the $\binom{n+p}{p}$-dimensional vector (1, $\underline{x}$, ($\underline{x}^{[2]}$)',...,($\underline{x}^{[p]}$)')' (We also call this vector the p-th family power of $\underline{x}$). By extension of the above definitions if $\underline{y} = A\ \underline{x}$ then $\underline{y}^{(\underline{p})} = A^{(\underline{p})}\underline{x}^{(\underline{p})}$ where $A^{(\underline{p})}$ is a block diagonal matrix of the form : diag [1, A, $A^{[2]}$,..., $A^{[p]}$]. It is easy to see that if $d/dt\ \underline{x}(t) = A(t)\ \underline{x}(t)$ then $d/dt\ \underline{x}^{(\underline{p})}(t) = A_{(\underline{p})}(t)\ \underline{x}^{(\underline{p})}(t)$ where $A_{(\underline{p})}(t)$ is the infinitesimal version of the matrix $A^{(\underline{p})}(t)$

The following lemma extends the previous one to the case of p-th family powers of state transition matrices of a linear system.

Lemma 2 $d/dt\, \Phi_A^{(p)}(t,t_0) = A_{(p)}(t)\, \Phi_A^{(p)}(t,t_0)$

Proof: The lemma is a clear consequence of the definitions and the previous lemma.

Definition A Generalized Polyhedron (GP) is a set of the form:

$$\{\ \underline{x} \in R^n : <\underline{h}_i, \underline{x}^{(p)}> \leq 1\ ;\ i=1,2,\ldots,M\ \}$$

$\underline{h}_i$ is called the generalized support vector. The set $\{\underline{h}_i;\ i=1,2,\ldots,M\}$ is the generalized support set. Note that a generalized polyhedron may well represent a non-convex, not necessarily bounded set. Polyhedra, spheres, ellipsoids, zonoids, etc. may be viewed as particular cases of generalized polyhedra.

Comment An ellipsoid $E = \{\underline{x} \in R^n : \underline{x}'Q\,\underline{x} \leq 1\}$ can be expressed as a generalized polyhedron with one generalized support vector defined by $\lambda(Q)$:

$$E = \{\underline{x} \in R^n :< \lambda(Q),\ \underline{x}^{[2]} > \leq 1\ \}$$

where $\lambda(\cdot)$ is a vector valued lexicographic map defined on the set of all symmetric matrices and determined by:

$$\lambda(Q) = [q_{11}, \sqrt{2}q_{12}, \sqrt{2}q_{13}, \ldots, \sqrt{2}q_{1n}, q_{22}, \sqrt{2}q_{23}, \ldots, \sqrt{2}q_{n-1\,n}, q_{nn}]$$

where q_{ij}'s are the elements of Q. The map $(\cdot)$ establishes a one-to one relationship among the set of symmetric matrices and $R^{n(n+1)/2}$.

Definition Given the linear system

$$\frac{d}{dt}\,\underline{x}(t) = A(t)\,\underline{x}(t) \qquad (1)$$

we denote by $\underline{\xi}(t,\underline{x}_0)$ the trajectory starting from the state $\underline{x}_0$. A state $\underline{x}(T)$ is said to be $\underline{x}_0$-feasible if $\underline{\xi}(T;\underline{x}_0) = \underline{x}(T)$ for a certain $T > 0$

Let Σ_0 be a closed set in R^n, we define the Σ_0-feasible set for system (1) at time T, the set:

$$\{\ \underline{x} \in R^n : \underline{x} = \underline{\xi}(T;\ \underline{x}_0) \quad \text{for some } \underline{x}_0 \in \Sigma_0\ \} \qquad (2)$$

We denote this set by $\underline{\xi}(T,\Sigma_0)$. The Σ_0-feasible set at time t is then the set of all $\underline{x}_0$-feasible states at time t for all $\underline{x}_0 \in \Sigma_0$.

The basic problem to be considered in this paper is the following:

Given the linear system (1) and a GP Σ_0 characterized by the generalized support set $\{\underline{h}_{i0};\ i=1,2,\ldots,M\}$ find the Σ_0-feasible set at any time t, $\underline{\xi}(t,\Sigma_0)$.

III MAIN RESULT

The solution to the proposed problem requires use of the definitions and lemmae previously presented. The following theorem is a consequence of this background material.

Theorem The Σ_0-feasible set for system (1) is a GP characterized by the generalized support vectors $\underline{h}_i(t)$; $i=1,2,\ldots,M$ where $\underline{h}_i(t)$ is given by the unique solution at time t of the linear vector differential equation:

$$\frac{d}{dt}\,\underline{h}_i(t) = -A'_{(p)}(t)\,\underline{h}_i(t)$$

with $\underline{h}_i(t_0) = \underline{h}_{i0}$ for all i.

Proof : The form of the Σ_0-feasible set follows from the affine classification theorem [10]. The computation of the generalized support vectors of $\underline{\xi}(t;\Sigma_0)$ is as follows: $\underline{\xi}(t;\Sigma_0) = \{\underline{x} \in R^n : \underline{x} = \Phi_A(t,t_0)\underline{x}_0$ for all $\underline{x}_0 \in \Sigma_0\} = \{\underline{x} \in R^n : \Phi_A(t_0,t)\underline{x} \in \Sigma_0\} = \{\underline{x} \in R^n : <(\Phi_A(t_0,t)\underline{x})^{(p)}, \underline{h}_{i0}> \leq 1$ for all i $\} = \{\underline{x} \in R^n : <\underline{x}^{(p)}, \Phi_A'^{(p)}(t_0,t)\underline{h}_{i0}> \leq 1$ for all i$\}$. Letting $\underline{h}_i(t) \triangleq \Phi_A'^{(p)}(t_0,t)\,\underline{h}_{i0}$, the result follows from lemma 2 and well known properties of time derivative of an inverse matrix.

The preceeding theorem constitutes a useful generalization that we illustrate in the following fixed shwitch-off time problem:

Example Consider the linear system $\dot{x}_1 = x_2$; $\dot{x}_2 = -w^2x_1 + u(t)$ with $|u(t)| \leq 1$ for all t in $[0,T/w]$, $T < \pi$, and $u(t) = 0$ for $t > T/w$. If we denote by R_T the reachable set at time T/w, of the system, for all possible control functions satisfying the above restriction, it is clear that the reachable set after time T/w is identical with the R_T-feasible set $\underline{\xi}(t;R_T)$ for $t > T/w$

Application of Pontryagin's Maximum Principle [11] yields an ellipsoidal zonoid for the reachable set R_T (i.e the intersection of two ellipsoids). The characterization of R_T as a generalized polyhedron is simply:

$$R_T = \{\underline{x} \in R^2 :\ <\underline{x}^{(2)},\ \underline{h}_{iT}> \leq 1\ ;\ i=1,2\ \}$$

where:

$$\underline{h}'_{1T} = [\tfrac{1}{2}(1+\cos T), \tfrac{w^2}{2}(1+\cos T), -\tfrac{w^2}{2}\sin T, \tfrac{w^4}{4}, 0, \tfrac{w^2}{4}]$$

$$\underline{h}'_{2T} = [\tfrac{1}{2}(1+\cos T), -\tfrac{w^2}{2}(1+\cos T), \tfrac{w^2}{2}\sin T, \tfrac{w^4}{4}, 0, \tfrac{w^2}{4}]$$

This set is represented in Fig. 1 for several values of $T < \pi$.

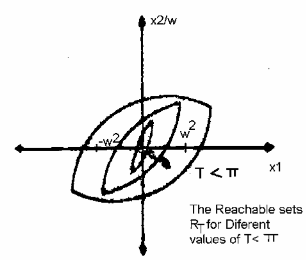


figure 1

The R_T-reachable set for all time $t > T/w$ is then a GP of the form $\{\underline{x} \in R^2 : <\underline{x}^{(2)}, \underline{h}_i(t)> \leq 1$ $i = 1,2\}$ where $\underline{h}_i(t)$ satisfies the differential equation:

$$\frac{d}{dt}\,\underline{h}_i(t) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -\sqrt{2} & 0 \end{bmatrix} \underline{h}_i(t)$$

whose unique solutions, taking as initial conditions respectively $\underline{h}_{1T}$ and $\underline{h}_{2T}$ are :

$$\underline{h}_1'(t) = [\tfrac{1}{2}(1+\cos T), w^2\cos\tfrac{T}{2}\cos(\tfrac{T}{2}-wt), w\cos\tfrac{T}{2}\sin(\tfrac{T}{2}-wt), \tfrac{w^4}{4}, 0, \tfrac{w^2}{4}]$$

$$\underline{h}_2'(t) = [\tfrac{1}{2}(1+\cos T), -w^2\cos\tfrac{T}{2}\cos(\tfrac{T}{2}-wt), -w\cos\tfrac{T}{2}\sin(\tfrac{T}{2}-wt), \tfrac{w^4}{4}, 0, \tfrac{w^2}{4}]$$

The R_T-feasible set is shown in Figure 2 for some value of $t > T/w$.

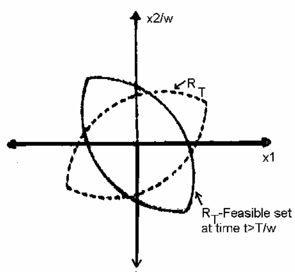


Figure 2

Corolary 1 Consider the linear system $\dot{\underline{x}}(t) = A(t)\underline{x}(t) + B(t)\underline{u}(t)$ with $\underline{u}(t)$ a known piecewise continuous function of time taking values in R^m. If the set Σ_0 is characterized by a GP centered around some vector $\underline{q}_0$ and described by :

$$\Sigma_0 = \{\underline{x} \in R^n : \langle (\underline{x} - \underline{q}_0)^{(p)}, \underline{h}_{i0} \rangle \leq 1 \ \forall i\}$$

then the Σ_0-feasible set is characterized by a generalized support set $\underline{h}_i \ \forall i$, and a "generalized centroid" $\underline{q}(t)$ which are, respectively, the unique solutions of the differential equations :

$$\frac{d}{dt}\underline{h}_i(t) = -A'_{(p)}(t)\,\underline{h}_i(t) \quad \text{with } \underline{h}_i(t_0) = \underline{h}_{i0}$$

and

$$\frac{d}{dt}\underline{q}(t) = A(t)\underline{q}(t) + B(t)\underline{u}(t) \text{ with } \underline{q}(t_0) = \underline{q}_0$$

Proof The proof of this corolary is straightforward and constitutes an extension of theorem 1 to the case of forced systems.

Corolary 2 If the set Σ_0 is characterized by a homogeneous p-degree form in the initial state variables; $\Sigma_0 = \{\underline{x} \in R^n : \langle \underline{x}^{[p]}, \underline{h}_{i0} \rangle \leq 1 ; i=1,\dots M\}$ then the feasible set of the forced system is also characterized by a set of homogeneous p-degree forms with support vectors $\underline{h}_i(t)$ and centroid $\underline{q}(t)$ given, respectively, by the unique solution of the differential equations:

$$\frac{d}{dt}\underline{h}_i(t) = -A'_{[p]}(t)\,\underline{h}_i(t) \quad \text{with } \underline{h}_i(t_0) = \underline{h}_{i0} \quad \forall i$$

and :

$$\frac{d}{dt}\underline{q}(t) = A(t)\underline{q}(t) + B(t)\underline{u}(t) \quad \text{with } \underline{q}(t_0) = \underline{0}$$

Corolary 3 Given the linear system $\dot{\underline{x}}(t) = A(t)\underline{x}(t) + B(t)\underline{u}(t)$ and an ellipsoid $\Sigma_0 = \{\underline{x} \in R^n : \langle \underline{x}, Q_0\,\underline{x} \rangle \leq 1\}$. The set of feasible states is the ellipsoid :

$$\{\underline{x} \in R^n : \langle (\underline{x} - \underline{q}(t))^{[2]}, \underline{h}(t) \rangle \leq 1\}$$

where $\underline{q}(t)$ satisfies the same equation as in Corolary 2 and $\underline{h}(t)$ on the other hand satisfies:

$$\frac{d}{dt}\underline{h}(t) = -A'_{[2]}(t)\,\underline{h}(t) \qquad (4)$$

with $\underline{h}(t_0) = \underline{h}_0 = \lambda(Q_0)$

with $\lambda(\cdot)$ being the lexicographic map defined in Section I.

The evolution equations for an ellipsoidal set-theoretic uncertainty in the initial state, not making use of tensor forms of second degree, have the following form:

$$\frac{d}{dt}Q(t) = -A'(t)\,Q(t) - Q(t)A(t) \qquad (5)$$

$$\text{with } Q(t) = Q_0$$

The centroid equations remain the same.

Comment Let $-A = P$, comparing (4) and (5) and by virtue of the Comment in the introduction, we have:

1) $\underline{h}(t) = \lambda(Q(t))$ and
2) $\lambda(P'Q + QP) = P'_{[2]}\,\lambda(Q)$

It can be shown that implication 2) is completely general with λ defined as in Section I. This fact constitutes a useful formula in direct solution methods for algebraic and differential Lyapunov equations

Comment One of the difficulties of the set-theoretic approach in modeling uncertainties in the form of ellipsoids is constituted by the fact that the intersection of two ellipsoids is not an ellipsoid. Approximations are then necessary to mantain mathematical tractability offered by the ellipsoids equations. Our approach allows us to deal with this fact without approximations at the expense of some data storing. The intersection of two ellipsoids $E_1 = \{\underline{x} \in R^n : \langle \underline{x}, Q_1\,\underline{x} \rangle \leq 1\}$ and $E_2 = \{\underline{x} \in R^n : \langle \underline{x}, Q_2\,\underline{x} \rangle \leq 1\}$ is simply expressed as $\{\underline{x} \in R^n : \langle \underline{h}_i, \underline{x}^{[2]} \rangle \leq 1 ;\ i=1,2\}$ where $\underline{h}_1 = \lambda(Q_1)$ and $\underline{h}_2 = \lambda(Q_2)$.

It is a well-known fact that iso-crone surfaces can be computed in terms of the reachable set of an associated negative time linear system easily obtainable from the original system on which the time-optimal problem has been defined. The following example shows that p-tensor forms can be conveniently used to arbitrarily approximate iso-crone sets (i.e reachable sets)

Example Consider the linear system : $\dot{x}_1 = -\mu_1 x_1 + u(t)$; $\dot{x}_2 = -\mu_2 x_2 + u(t)$ where $|u(t)| \leq 1$

The isocrone set for some time T is characterized by :

$$\{\underline{x} : \left[\frac{1+\alpha_2 \pm \mu_2 x_2}{2}\right]^{\mu_1} = \left[\frac{1+\alpha_1 \pm \mu_1 x_1}{2}\right]^{\mu_2}\}$$

If μ_1/μ_2 is rational then the isocrone set is easily characterized as the boundary of a GP. The description is then exact. On the other hand if the quantity μ_1/μ_2 is irrational then the isocrone set can be approximated arbitrarily close by the boundary of a GP. The higher the dimension of the support vectors the better the approximation.

P-tensor forms are easily seen to constitute an efficient mechanism for initial state uncertainty modeling regardless of the non-convex or even non-connected, unbounded nature of the set. The following example shows how to characterize sets of this nature in terms of GP's.

Example

Consider the time-varying linear system:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 & f(t) \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

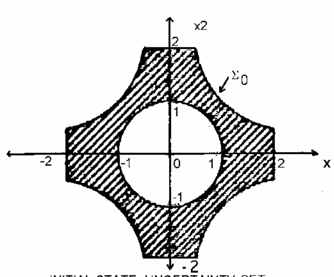
with the initial state unknown but bounded by the set:

$$\Sigma_0 = \{\underline{x} \in R^2 : |x_1 x_2| \leq 1 \ ; \ |x_1| \leq 2 ; \ |x_2| \leq 2 \ ; \ x_1^2 + x_2^2 \geq 1 \ \}$$

and let $f(t) = u(t)$ specified by:

$$u(t) = \begin{cases} +1 & \text{for } t \in [0,\beta) \\ -1 & \text{for } t \in (\beta,T] \end{cases}$$

It is required to compute the set of possible states at time T. The set Σ_0 , shown in Figure 3 can be expressed as a GP.



INITIAL STATE UNCERTAINTY SET
FIGURE 3

$$\Sigma_0 = \{\underline{x} \in R^2 : \langle \underline{x}^{(2)} , \underline{h}_{i0} \rangle \leq 1 \ ; \ i=1,2,\ldots,7 \ \}$$

with:

$\underline{h}'_{10} = (0,0,0,0,\ 2/2,0\ )$; $\underline{h}'_{20} = (0,0,0,0,-\ 2/2,0\ )$

$\underline{h}'_{30} = (0,1/2,0,0,0,0\ )$; $\underline{h}'_{40} = (0,-1/2,0,0,0,0\ )$

$\underline{h}'_{50} = (0,0,1/2,0,0,0\ )$; $\underline{h}'_{60} = (0,0,-1/2,0,0,0\ )$

$\underline{h}'_{70} = (2,0,0,-1,0,-1)$

the centroid is located at the origin.

At time T , the set of possible states is a GP with the support set given by the vectors :

$$\underline{h}'_1(T) = (mn, me^T, e^T[n+(T-2\beta)m], 0, \tfrac{\sqrt{2}}{2}\, e^{2T}, e^{2T}(T-2\beta))$$

$$\underline{h}_2(T) = -\underline{h}_1(T)$$

$$\underline{h}'_3(T) = (n/2, (e^T/2)(T-2\beta), 0,0,0\ ) \ ; \ \underline{h}_4(T) = -\underline{h}_3(T)$$

$$\underline{h}'_5(T) = (m/2, 0, e^T/2, 0,0,0\ ) \ ; \ \underline{h}_6(T) = -\underline{h}_5(T)$$

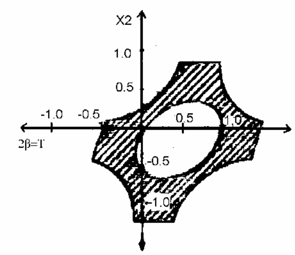
$$\underline{h}'_7(T) = (2-(n^2+m^2), -2e^T n, -2e^T[(T-2\beta)n+m], -e^{2T}, \sqrt{2}\, e^{2T}(T-2\beta), e^{-2T}[1+(T-2\beta)^2]\ )$$

where $m = 1 + e^T - 2e^\beta$ and $n = 1 + e^T(T-1)$

The centroid at time T is given by:

$$q_1(T) = mT - n - 2\beta(1-e^\beta) \ ; \quad q_2(T) = -me^{-T}$$

Figure 4 shows this set for $T = 0.69$



THE Σ_0– FEASIBLE SET
FIGURE 4

We shall now examine a simple extension of the main result in this paper to the case of forced systems in which the input function takes unknown values, bounded at each instant of time by a GP. This GP, however, is assumed to arise as the output of a linear dynamic system whose initial state is also a GP. We shall characterize the set of feasible states as the sum of two GP's (vector sum).

Consider the linear system:

$$\frac{d}{dt}\underline{x}(t) = A(t)\underline{x}(t) + B(t)\underline{w}(t) \quad \text{with} \quad \underline{x}_0 \in \Sigma_0 \qquad (6)$$

and $\underline{w}(t)$ is an unknown but bounded signal contained at each instant of time t by a GP which we denote by $\Omega(t)$. We assume that $\underline{w}(t)$ is given by:

$$\underline{w}(t) = H(t)\ \underline{z}(t) \qquad (7)$$

where

$$\frac{d}{dt}\underline{z}(t) = F(t)\ \underline{z}(t) \quad \text{with} \quad \underline{z}_0 \in \Xi_0 \qquad (8)$$

The set $\Omega(t)$ is the direct image under $H(t)$ of the set $\Xi(t)$ obtained as the Ξ_0-feasible set of the dynamic system defined by $F(t)$. $\underline{w}(t)$ is an m-vector and $\underline{z}(t)$ is an r-vector. H and F have the appropriate dimensions. The following proposition characterizes the feasible set for all possible forcing functions $\underline{w}(t)$ restricted instantaneously by $\Omega(t)$.

Proposition Let Σ_0 and Ξ_0 be GP's characterized respectively by the support sets $\underline{h}_{i0}$ and

$\underline{s}_{i0}$ and centered around the origin:

$$\Sigma_0 = \{\underline{x} \in R^n : <\underline{x}^{(p)}, \underline{h}_{i0}> \leq 1 \quad \forall\, i \in \underline{M}\}$$
$$\Xi_0 = \{\underline{z} \in R^r : <\underline{z}^{(q)}, \underline{s}_{i0}> \leq 1 \quad \forall\, i \in \underline{N}\}$$

($\underline{M}$ and $\underline{N}$ denote the sets $1,2,\ldots, M$ and $1,2,\ldots, N$ respectively). Then the set of all possible states at time t of system (6) subject to (7) and (8) is given by:

$$\Sigma(t) = \Sigma_1(t) \oplus \Sigma_2(t) \qquad (9)$$

where $\oplus$ denotes set-theoretic vector sum of the involved sets, and :

$$\Sigma_1(t) \triangleq \{\underline{x} \in R^n : <\underline{x}^{(p)}, \underline{h}_i(t)> \leq 1 \quad \forall\, i \in \underline{M}\}$$

with $\underline{h}_i(t)$ being the unique solution at time t of

$$\frac{d}{dt}\underline{h}_i(t) = -A'_{(p)}(t)\,\underline{h}_i(t) \; ; \; \underline{h}_i(t_0) = \underline{h}_{i0} \;\forall\, i$$

and :

$$\Sigma_2(t) \triangleq \{\underline{x} \in R^n : <\underline{x}^{(q)}, \underline{\sigma}_i(t)> \leq 1 \;\forall\, i \in \underline{N}\}$$

where $\underline{\sigma}_i(t)$ satisfies :

$$M'^{(q)}(t,t_0)\,\underline{\sigma}_i(t) = \underline{s}_i(t) \quad \forall\, i$$

and $\underline{s}_i(t)$ is the unique solution of :

$$\frac{d}{dt}\underline{s}_i(t) = -F'_{(q)}(t)\,\underline{s}_i(t) \; ; \; \underline{s}_i(t_0) = \underline{s}_{i0} \;\forall\, i$$

$M(t,t_0)$ is given by :

$$M(t,t_0) = \int_{t_0}^{t} \Phi_A(t,\xi)B(\xi)H(\xi)\Phi_F(\xi,t)d\xi$$

Φ_A and Φ_F are the state transition matrices associated with $A(t)$ and $F(t)$ respectively.

The sum of two sets as in (9) is defined by :

$$\Sigma_1 \oplus \Sigma_2 \triangleq \{\underline{y} \in R^n : \underline{y} = \underline{x} + \underline{z} \quad \forall\, \underline{x} \in \Sigma_1 \text{ and } \underline{z} \in \Sigma_2 \}$$

Proof: The proof of the proposition is readily made by standard state augmentation techniques and some straightforward manipulations. We remark here that the vector sum of two GP is also a GP whose generalized support set is related to the support sets of the involved GP's through somewhat complex optimization problems. It is an important fact that ,generally, in (9) Σ_2 is a degenerate GP (i.e one lying in a proper subspace of R^n).

IV SET-ESTIMATION AND SET-CONTROL PROBLEMS

In this section we examine the signifiance of the results in the previous section in problems of set-estimation and set-control nature.

We have seen that GP's can be effectively used in modeling reachable sets for linear dynamic systems. A set-estimation problem generally consists of a reachable set computation problem and the computation of the set of compatible states in light of the measurements. The estimation of the state is then carried by intersecting the reachable set with the set of states compatible with the measurements.

Assume we perform continuous measurements on a linear system state which is set-theoretic uncertain. The measurements are assumed to be corrupted by noise of unknown but bounded nature:

$$\underline{y}(t) = C(t)\underline{x}(t) + \underline{v}(t)$$

where $\underline{y} \in R^k$ and $\underline{v}(t) \in \Psi(t)$, a GP for each t, and centered around the origin with generalized support set $\underline{\zeta}_i(t)$

$$\Psi(t) \triangleq \{\underline{v} \in R^k : <\underline{v}^{(s)}, \underline{\zeta}_i(t)> \leq 1 \;\forall\, i \in \underline{P}\}$$

Then, the set of states compatible with the measurement $\underline{y}(t)$ obtained at time t from the system is also a GP given by :

$$\{\underline{x} \in R^n : <(\underline{x} - \underline{x}^*)^{(s)}, \underline{m}_i(t)> \leq 1 \quad \forall\, i \in \underline{P}\}$$

where $\underline{x}^*(t)$ satisfies $C(t)\,\underline{x}^*(t) = \underline{y}(t)$ and $\underline{m}_i(t) = (-1)^s C^{(s)}(t)\underline{\zeta}_i(t)$.

The estimate set at time t is obtained by intersecting this set with the reachable set at time t. The intersection of two GP's is obviously a GP whose support set is computed as follows:

Suppose we have two GP with descriptions

$$\Sigma_1 \triangleq \{\underline{x} \in R^n : <\underline{x}^{(p)}, \underline{h}_i> \leq 1 \quad \forall \quad i \in \underline{M}\}$$
$$\Sigma_2 = \{\underline{x} \in R^n : <\underline{x}^{(q)}, \underline{s}_i> \leq 1 \quad \forall \quad i \in \underline{N}\}$$

with $p \geq q$ then :

$$\Sigma_1 \cap \Sigma_2 \triangleq \{\underline{x} \in R^n : <\underline{x}^{(p)}, \underline{r}_i> \leq 1 \quad \forall\, i \in (\underline{M+N})\}$$

where

$$\underline{r}_i = \underline{h}_i \quad ; \quad i=1,2,\ldots, M$$
$$\underline{r}_i = [\,\underline{0}\,, \underline{s}_i]' \quad \underline{i} = M+1, M+2,\ldots, M+N.$$

The dimension of the vector $\underline{0}$ above depends on the value of p-q in the obvious manner. Some of the constraints defining the intersection set may turn out to be redundant. A simplification of the number of constraints has to be carried to obtain the most "economical" description of the intersection set.

Notice that the generalized support set of the set of compatible states is precomputable. Only the centroid of this set has to be computed on-line according to the measurements obtained from the system.

The basis for the analysis of Target set and Target Tube reachability problems defined on continuous time systems subject to set-theoretic uncertain initial states is constituted by Corolary 1 in the previous section. If we assume that $A(t)$ is of the form:

$$A(t) = A_0 + \sum_{i=1}^{m} v_i(t)\, A_i$$

where A_0 and A_i are constant, then the evolution equations of the initial state uncertainty for any control function $\underline{u}(t)$ take the form:

$$\frac{d}{dt}\underline{h}_i(t) = -[\, A'_{0(\underline{p})} + \sum_{i=1}^{m} v_i(t) A'_{i(\underline{p})} \,] \underline{h}_i(t)$$
$$\underline{h}_i(t_0) = \underline{h}_{i0} \quad \forall\, i$$

and

$$\frac{d}{dt}\underline{q}(t) = A(t)\,\underline{q}(t) + B(t)\underline{u}(t) \quad ; \underline{q}(t_0) = \underline{q}_0$$

i.e the external additive signals $\underline{u}(t)$ only influence the centroid evolution equations, having nothing to do with the "spread" of the uncertainty set at any time. On the contrary the structural signals $v_i(t)$ affect both the centroid and the support set evolution equations. The structural variables could (whenever possible) be conveniently manipulated to obtain desired final values for the support set at some time T. Whether or not such functions exist is a controlability problem defined on a bi-linear system for which a good amount of results is available in the existing literature. Even if the desired values of the support vectors are not attainable, it still makes sense to define an optimal control problem on the bi-linear structure which asks for the optimal signals $v_i^*(t)$ for which a wheighted sum of differences among desired final values of the support set and the final state of the bi-linear system is to be minimized. Once this structural signals are determined, one can formulate a standard optimal regulator problem on the centroid equations to obtain a close value of the centroid to some pre-specified desired centroid of the Target Set.

Target Tube Reachability problems can also be treated by using this sort of "independence result" among the evolution equations for the set of possible states. In this class of problems a Target Tube (modeled as a time varying GP) is given and control signals are required which keep the state of the system within this tube for a certain period of time. A reasonable approach to handle this problem is to define optimal control problems on the above equations. For the first a quadratic regulator philosophy with either bounded structural controls or penalties on $v_i(t)$ can furnish a meaningful "engineering solution" to the problem of minimizing the lack of intersection among the target tube and the tube of possible states. Once the parametric controls have been found by either controllability or optimal control results, a second optimization problem can be formulated on the centroid evolution equation. A tracking regulator problem represents a reasonable approach which completes a solution method for Target Tube Reachability problems defined on linear systems with variable parameters.

V CONCLUSIONS

In this paper we have considered a new technique for set-constrained uncertainty modeling. The generality allowed by the method makes it nolonger necessary to restrict the uncertainty sets to ellipsoids and polyhedra for the sake of mathematical convenience. Possiblities of modeling a larger class of sets are thus open while mantaining the benefits of linearity characterizing the system under study. Initial state uncertainty evolution has been computed in terms of the evolution equations of "generalized support vectors" through a linear p-tensor system associated with the linear dynamic system where the uncertainty is being considered. The implications of the results has been examined in set-estimation and set-control problems. It has been established that target set reachability problems are meaningfuly treated by considering the set of possible states evolution. The variable structure systems offer special interest due to the possibilities of uncertainty dispersion control. Several extensions of the results to energy-type of uncertainty modeling could be attempted. The more general problem of characterization of the reachable set for linear systems with controls bounded by a GP is still unsolved.

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