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FUZZY STATE ESTIMATION IN LINEAR DYNAMIC SYSTEMS

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ABSTRACT

The problem of state estimation in linear dynamic systems with fuzzy initial states and poorly defined perturbation input signals (i.e fuzzy inputs) is considered. A new notion of vector summ of n-dimensional fuzzy sets provides the basis for state and input "imprecision" propagation trough the systems dynamics. Recursive state estimation formulae are developed to obtain a fuzzy set of possible states at each instant of time.

I. INTRODUCTION

The use of fuzzy sets as models for describing imprecision in the knowledge of initial states was used in [1] for the problem of characterizing dynamic evolution of fuzzy initial states in linear differential systems defined in R^n.

This paper adresses the problem of estimation in the context of linear discrete time vector systems whose initial state, perturbation input signals and observation corrupting noises are poorly known and thus modeled as fuzzy sets. We obtain the basic recursive formulae which produces a fuzzy set estimate of the states which are compatible with the systems dynamics and the measurements program.

The paper parallels previous developements in the area of estimation for set-bounded uncertainties and linear systems (See [2]-[5]). The basic recursive formulae for fuzzy state estimation differ in nothing from the theoretic expressions for "crisp" estimation problems in linear systems. It is the characterization of membership functions, in the case of fuzzy state estimation, which generalizes the set-theoretic approach for systems analysis in the estimation area.

Section II introduces some basic definitions and formulates the fuzzy state estimation problem from a general viewpoint.

Section III determines the membership funciton in a theoretic fashion for the state sets generated by the fuzzy imprecisions. We present here a simple example that emphasizes a new class of fuzzy sets called Gaussian Fuzzy sets. The paper concludes with some suggestions for further research. All the basic formulae and definitions regarding linear transformations, vector addition, fuzzy and "crisp" translations and intersections of fuzzy sets, are presented at the end of the paper in an Appendix. Most of these definitions are straightforward applications of the so-called "Extension Principle" [6].

II. DEFINITIONS AND PROBLEM FORMULATION

Definitions We denote a fuzzy set A with membership function $m_A(\ )$ defined over the universe of discourse U in the n-dimensional euclidean space R^n as:

$$A = \int_U m_A(\underline{u}) \,/\, \underline{u} \tag{2.1}$$

A non-fuzzy set will be termed "crisp" set. Its membership function has the value 1 over the whole domain of the set.

The appendix collects all the basic definitions and algebraic operations regarding fuzzy sets, particularly those more relevant to the estimation problem

We consider a fuzzy set completely specified when we are given the membership function (or compatibility function) over the support of the fuzzy set. This support is defined as the crisp set (i.e non-fuzzy) where the membership function is not zero. We shall denote the support of a fuzzy set A by means of $\dot{A}$.

Problem Formulation Given a linear discrete time dynamic system:

$$\underline{x}(t_k) = \Phi(t_k,t_{k-1})\underline{x}(t_k) + F(t_{k-1})\underline{u}(t_{k-1}) + G(t_{k-1})\underline{w}(t_{k-1}) \tag{3.1}$$

with an initial state characterized by a fuzzy set X_0 in R^n, plant perturbation inputs $\underline{w}(t_k)$ described only by means of a fuzzy set $\Omega_w(t_k)$ in the space R^q. The control signal is a crisp vector defined in R^m of which we have a precise knowledge. All the matrices have the appropriate dimensions.

Given also a measurement program excecuted over the state variables of the system according to:

$$\underline{z}(t_k) = H(t_k)\,\underline{x}(t_k) + \underline{v}(t_k) \tag{3.2}$$

where $\underline{z} \in R^p$, and $\underline{v}(t_k)$ is a source of measurement imprecision which is only known to be characterized by means of a fuzzy set at each instant t_k. This set is denoted by $\Omega_v(t_k)$. Finally, given the information set :

$$I(t_k) = Z(t_k) = \{\underline{z}(t_1), \underline{z}(t_2), \ldots, \underline{z}(t_k)\} \tag{3.3}$$

together with the systems dynamics (3.1) and the measurement program (3.2), the imprecision sets and the control actions: $\underline{u}(t_0), \underline{u}(t_1), \ldots, \underline{u}(t_{k-1})$, it is desired to find the set of feasible fuzzy states at time t_k and characterize the membership function for such set,possibly in a recursive manner,involving all the available information.

III. ESTIMATION PROCEDURE

In this section we outline in a general form the estimation process that has to be accomplished on-line on the basis of the previous information and the aquired a posteriori information generated by the systems dynamical evolution and the outputs of the measurement device.

The estimation procedure is accomplished in two steps as follows:

1) Prediction step : The available information at time t_{k-1} is propagated through the systems dynamics to obtain the set of possible states at time t_k. We assume we have available an estimate fuzzy set at time t_{k-1} denoted by $X(t_{k-1}|t_{k-1})$ This set is mapped through the state transition matrix and added to the fuzzy imprecision set generated by the signal $\underline{w}(t_{k-1})$. The resulting

fuzzy set suffers a crisp translation (See Appendix) caused by the control action. This prediction step really amounts to compute the set of fuzzy reachable states at time t_k from all we imprecisely know about the state at time t_{k-1} in a compatible fashion with the system dynamics and its postulated imprecisely known signals.

2) Estimation step (information update) : Measurements are preformed at time t_k on the systems state $\underline{x}(t_k)$ via measurement program (3.2). We compute, on the basis of this measurement value $\underline{z}(t_k)$ the fuzzy set of states compatible with this imprecise measurement.

Below we give some of the formulae (in a fuzzy set-theoretic fashion) that need be used in the above steps of the estimation process.

$$X(t_k|t_{k-1}) = \Phi(t_k,t_{k-1})\, X(t_{k-1}|t_{k-1}) \oplus \{F(t_{k-1})\underline{u}(t_{k-1})\} \oplus G(t_{k-1})\, \Omega_w(t_{k-1}) \qquad (3.1)$$

$$X(t_k|t_k) = X(t_k|t_{k-1}) \cap - H^{-1}(t_k)\, [\Omega_v(t_k) \oplus \{-\underline{z}(t_k)\}] \qquad (3.2)$$

The formulae (3.1) and (3.2) correspond respectively to the prediction and the estimation step outlined above.

The recursive process represented by these formulae is started with the condition :

$$X(t_0|t_0) = X_0 \qquad (3.3)$$

Let $V(t_k)$ and $m_{V(k)}(\cdot)$ denote, respectively, the support set and the membership function of $\Omega_v(t_k)$. Similarly let $W(t_k)$ and $m_{W(k)}(\cdot)$ define $\Omega_w(t_k)$, while $X(t_{k-1}|t_{k-1})$ and $m_{X(k-1|k-1)}(\cdot)$ define $X(t_{k-1}|t_{k-1})$. Then, using (3.1) , (3.2) and the results of the Appendix we can obtain the membership function and the support sets of $X(t_k|t_k)$ as:

$$m_{X(k|k)}(\underline{x}) = \{ m_{X(k|k-1)}(\underline{x}) \wedge m_{V(k)}[-H(t_k)\underline{x} + \underline{z}(t_k)]\}$$

where:

$$m_{X(k|k-1)}(\underline{x}) = \sup_{\underline{q}} \{ m_{X(k-1|k-1)}[\Phi(t_{k-1},t_k)\,\underline{x} - F(t_{k-1})\underline{u}(t_{k-1}) - \underline{q}\,] \wedge m_{W(k-1)}[G^{+}(t_{k-1})\,\underline{q}]\}$$

where " + " denotes pseudoinverse and the reversal of arguments in Φ implies inversion of the matrix.

On the other hand we have :

$$X(t_k|t_k) = X(t_k|t_{k-1}) \cap -H^{-1}(t_k)[\, V(t_k) \oplus \{-\underline{z}(t_k)\}]$$

with :

$$X(t_k|t_{k-1}) = \Phi(t_k,t_{k-1})\, X(t_{k-1}|t_{k-1}) \oplus F(t_{k-1})\underline{u}(t_{k-1}) \oplus G(t_{k-1})\, W(t_{k-1})$$

Example

We introduce a class of fuzzy sets which we term Gaussian Fuzzy sets defined by three quantities; a real positive number called the amplitude, a vector called the center and a positive definite matrix called the dispersion matrix. The triple (a, $\hat{\underline{u}}$, Q) totally defines such sets which have membership functions :

$$A \leftrightarrow m_A(\underline{u}) = a \exp -(\underline{u} - \hat{\underline{u}})' Q^{-1} (\underline{u} - \hat{\underline{u}})$$

Consider the linear dynamic system :

$$\underline{x}(k) = \Phi\underline{x}(k-1) + B\,\underline{u}(k-1) + G\,\underline{w}(k-1)$$

and measurement program :

$$\underline{z}(k) = H\,\underline{x}(k) + \underline{v}(k)$$

with : (" ~ " means fuzzy membership)

$\underline{x}(0) \in X(0) \sim (1,\underline{0},\Psi)$; $\underline{w}(k) \in \Omega_w(k) \sim (1,\underline{0},Q(k))$; $\underline{v}(k) \in \Omega_v(k) \sim (1,\underline{0},R(k))$.

Using the definitions and approximations developed in the appendix, we can recursively characterize, in an approximate way, the fuzzy estimate of the state at time k, $X(k|k)$ as a gaussian fuzzy set characterized by the triple $(a(k|k),\ \hat{\underline{x}}(k|k),\ \Sigma(k|k))$ where :

$$a(k|k-1) = a(k-1|k-1) \;,$$

$$a(k|k) = a(k|k-1)\exp -\{[\hat{\underline{x}}(k|k) - \hat{\underline{x}}(k|k-1)]'\Sigma^{-1}(k|k-1)[\hat{\underline{x}}(k|k) - \hat{\underline{x}}(k|k-1)]\} \;;$$

$$\hat{\underline{x}}(k|k-1) = \Phi\hat{\underline{x}}(k-1|k-1) + F\,\underline{u}(k-1) \;;$$

$$\hat{\underline{x}}(k|k) = [(1-\rho(k))\Sigma^{-1}(k|k-1) + \rho(k)H'R^{-1}(k)H]^{-1} [\rho(k)H'R^{-1}(k)\underline{z}(k) + (1-\rho(k))\Sigma^{-1}(k|k-1)\hat{\underline{x}}(k|k-1)]$$

$$\Sigma(k|k) = (1-d^2(k))[(1-\rho(k))\Sigma^{-1}(k|k-1) + \rho(k)H'R^{-1}(k)H]^{-1} \;;$$

$$\Sigma(k|k-1) = [1/(1-\beta(k-1))]\,\Phi\Sigma(k-1|k-1)\Phi' + G\,[1/\beta(k-1)]Q(k-1)\,G' \;;$$

$$d^2(k) = [\underline{z}(k) - H\,\underline{x}(k|k-1)]'\{(1/(1-\rho(k)))H\Sigma(k|k-1)H' + (1/\rho(k))\}^{-1}[\underline{z}(k) - H\,\underline{x}(k|k-1)] \;;$$

with :

$$1 \geq \rho(k) \geq 0 \;;\quad 1 \geq \beta(k) \geq 0 \quad \forall\ k$$

and

$$\hat{\underline{x}}(0|0) = \underline{0} \;;\quad \Sigma(0|0) = \Psi \;;\quad a(0|0) = 1 \qquad (3.4)$$

The recursive estimation formulae (3.4) generalize those of set-estimation algorithms using ellipsoids. By means of approximation formulae these have been found to be identical to those developed by Schweppe [7]. The appendix presents the general formulae which do not preserve the gaussian character through operations such as vector addition and intersections of gaussian fuzzy sets.

IV. CONCLUSIONS AND SUGGESTIONS FOR FURTHER RESEARCH

In this paper we have given preliminary steps toward an estimation technique for systems with ill-defined or imprecise knowledge of initial states and external input signals for the plant or the measurement sub-system. Fuzzy sets are used as the basic device for imprecision modeling. The basic structure of the recursive algorithms generalizes that of the corresponding set-bounded uncertainties, yet they differ in nothing from the conceptual viewpoint. We have presented general estimation formulae which characterize the membership funcition of the fuzzy estimate set of the state in a recursive fashion. These formulae are applicable to any class of fuzzy sets charcterizing the imprecisions affecting the system knowledge. We made a particular application of these algorithm for the case of "gaussian fuzzy sets". By appropiate approximation techniques these are shown to be identical , except for a special feature called the membership function amplitude, to the recursive algorithm developed in [7] for estimation with ellipsoidal bounded uncertainties.

An interesting area for research lies in the fuzzy-theoretic control of linear dynamic systems to fuzzy targets and state space tubes. Computational implementation capable of handling more general classes of fuzzy sets should be developed for estimation and control problems.

The mathematical tractability of gaussian fuzzy sets may be used to advantage in a class of problems where sources of system imprecision make it undesairable even from a philosophical viewpoint to use random processes.

V. REFERENCES

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APPENDIX

General Definitions and Operations with Fuzzy Sets:

Consider fuzzy sets A and B with membership functions m_A , m_B and support sets $\mathcal{A}$, $\mathcal{B}$ respectively. We define a crisp translation of a fuzzy set A in the direction and extent of a non-fuzzy vector $\underline{v}$ as the fuzzy set whose membership function is $m(\underline{u}) = m_A(\underline{u}-\underline{v})$ and support set $\mathcal{A} \oplus \underline{v}$. The vector sum of two fuzzy sets A and B is defined as the fuzzy set C with memebership function $m_C(\underline{u}) = \sup_{\underline{v}}[\, m_A(\underline{u} - \underline{v}) \wedge m_B(\underline{v})\,]$ with support set $\mathcal{A} \oplus \mathcal{B}$. This definition is equivalent to add in vector form the α-level sets([6])of each fuzzy set and adscribe the membership function value α to the boundaries of such crisp sum. The formula for vector addition of fuzzy sets constitutes a natural "convolution operation" on the membership functions of the involved fuzzy sets and it includes the case where A and B are crisp. Not suprisingly, the formula is reminiscent of that which establishes the probability density function of two independent random variables. Given a non-singular linear transformation P, and a fuzzy set A we define the direct image under P of the fuzzy set A , the fuzzy set with compatibility function $m_{P(A)}(\underline{u}) = m_A(P^{-1}\underline{u})$ and support set $P\mathcal{A}$. The inverse image of a fuzzy set A under a linear map Q , not necesarily invertible, is fully described by $m_A(Q\,\underline{u})$ and support $Q^{-1}\mathcal{A}$. (the inverse operation on the support is merely notational since Q^{-1} needs not be ever computed). The intersection of two fuzzy sets A and B is a fuzzy set C with memebership function $m_C(\underline{u}) = m_A(\underline{u}) \wedge m_B(\underline{u}) \triangleq \mathrm{Min}\{m_A(u), m_B(\underline{u})\}$. This fuzzy set is the largest fuzzy set contained in both A and B. The support set of $A \cap B$ is $\mathcal{A} \cap \mathcal{B}$

Gaussian Fuzzy Sets in R^n:

A gaussian fuzzy set A , with support R^n, amplitude a, dispersion Q and center $\underline{\hat{u}}$ is defined as:

$$A = \int_{R^n} a \exp\{-(\underline{u} - \underline{\hat{u}})'\, Q^{-1}\, (\underline{u} - \underline{\hat{u}})\}/\, \underline{u}$$

The entire set is briefly refered to as a triple $(a, \underline{\hat{u}}, Q)$.

Crisp translation :

$$A \oplus \underline{v} = (a, \underline{\hat{u}}-\underline{v}, Q)$$

Direct image under P :

$$PA = P(\underline{a},\underline{\hat{u}},Q) = (a,\; P\,\underline{\hat{u}}\,,\; PQP')$$

Inverse image under M :

$$M^{-1}A = M^{-1}(a,\underline{\hat{u}},Q) = (a,\, M^{-1}\underline{\hat{u}},\, M'^{-1}Q\, M^{-1})$$

Vector Addition

$$A = (a,\, \underline{\hat{u}}_1,\, Q_1) \quad ; \quad B = (b,\, \underline{\hat{u}}_2,\, Q_2)$$

then A + B is not, in general,a Gaussian Fuzzy set, for instance, when A = m B for any real m then indeed the sum is a gaussian fuzzy set. Otherwise, we have to work out approximation formulas where the gaussian character is preserved .

For the case of vector addition an approximate gaussian fuzzy set can be obtained using the following formulas:

$$A \oplus B \cong (\min(a,b), \underline{\hat{u}}_1 + \underline{\hat{u}}_2,\; \frac{1}{\alpha} Q_1 + \frac{1}{1-\alpha} Q_2)$$

where α is any scalar parameter that satisfies $0 < \alpha < 1$

Intersection

The intersection of two gaussian fuzzy sets A and B is a fuzzy set which, in general, is not gaussian. As in the preceeding case, approximation formulas for the membership function can be given. One such gaussian approximation to this intersection is given by:

$$A \cap B \cong (c,\, \underline{\hat{u}}^*,\, Q)$$

where :

$$c = a \exp -\{(\underline{\hat{u}}^*-\hat{u}_1)Q_1^{-1}(\underline{\hat{u}}^* - \underline{\hat{u}}_1)\}$$

$$\underline{\hat{u}}^* = [\rho Q_1^{-1} + (1-\rho)Q_2^{-1}]^{-1}[\rho Q_1^{-1}\underline{\hat{u}}_1 + (1-\rho)Q_2^{-1}\underline{\hat{u}}_2]$$

$$Q = (1-d^2)\,[\rho\, Q_1^{-1} + (1-\rho)Q_2^{-1}]^{-1}$$

and

$$d^2 = (\underline{\hat{u}}_1-\underline{\hat{u}}_2)'[\rho^{-1}Q_1 + (1-\rho)^{-1}Q_2]^{-1}(\underline{\hat{u}}_1-\underline{\hat{u}}_2)$$

The support set of the approximations for the intersection and the addition of gaussian fuzzy sets, is the entire space R^n.