ON THE ROBUST CONTROL OF SYSTEMS TO SET-VALUED OBJECTIVES

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Abstract

This paper considers the Target Tube Reachability problem (TTRP) for discrete-time linear systems with set-constrained perturbation inputs and set-bounded parametric uncertainty in the state transition matrix. We present necessary and sufficiency conditions for target reachability and develop a backwards recursive algorithm for the determination of robust attainability regions at each instant of time. Explicit formulas are presented for two important closed convex bounds on the uncertain variables and parameters, namely : ellipsoids and polyhedra. Some important computer-oriented algorithms are also included for the effective characterization of polyhedral bounds of intermediate sets in the proposed backwards recursive scheme leading to target accesibility ascertainment.

1. INTRODUCTION

The majority of engineering problems are influenced by uncertainty. This inescapable fact demands the development of adequate models that take into account the uncertain quantities and give a satisfactory answer to the design problem. There are, generally speaking, two basic alternatives to choose from when such modeling is required:

1) The uncertain quantities may be assumed to have known probabilistic behaviour as random variables or random processes.
2) The unknown quantities may be assumed to lie within specified bounds but otherwise completely unknown.

The first modeling technique has been extensively treated in the literature and more often than not, it reflects the desire of making available to the designer a mathematically tractable formulation for which an enourmous field of knowledge already exists. The solution achieved by using this modeling philosophy guarantees an average behaviour optimal type of design.

The second modeling technique has been explored by fewer authors, but has already provided an extensive set of results and applications in areas where it is less appropriate to use statistical modeling of the uncertainty, i.e processes and plants where it would be extremely expensive to conduct experiments aimed at determining the statistics of certain variables and rather the advice of an experienced operator may lead to determine rough estimates (bounds) of the variables and their perturbations.

The mathematical machinery used in this technique is set-theoretic in nature and represents a conceptually straightforward and practical approach to the design problem under uncertainty.

The first steps of a theory of unknown but bounded signals in linear dynamic systems appeared in Schweppe (1968); in this formulation, ellipsoidal bounds were defined on the uncertain quantities in reference to the recursive state estimation problem for linear dynamic systems. Schweppe and Knudsen (1968) developed a theory of amorphous cloud trajectory prediction based in the bounding ellipsoid technique. Subsequent work by Witsenhausen (1966,1968) led to a more general set-theoretic formulation of the estimation problem using general convex bounded sets.

The reachability problem of target sets in state space using set-theoretic modeling of the uncertainty appeared by the first time in the work of Delfour & Mitter (1969). Their approach is rather general and constitutes the basis for subsequent work in the area by Bertsekas & Rhodes (1971), Glover & Schweppe (1971) and Glover (1971). Glover makes an extensive application of this technique to the Load Frequency Control problem in Electric Power Systems. Hnyilicza (1967) gave the first steps in the use of general convex polyhedra to model uncertainty in the estimation problem. He developed general algorithms for the case of linear dynamic systems. Schlaepfer (1970) used the set-theoretic approach to study estimation problems in Distributed Parameter systems. A set-theoretic methodology for the study of the steady-state security regions was developed as a powerful tool in a variety of Power Systems operation and planning applications by Hnyilicza & Lee & Schweppe (1975). Yared (1976) gave an interesting application of the set-theoretic technique to the Average Frequency Control problem in Electric Power Systems under normal mode of operation.

There has also been some effort in computational issues related to the general reachability problem as well as relations to more sophisticated mathematical approaches. For an account of the first efforts see for instance Morris & Brown (1976), while for the second see Barmish (1977,1979)

This paper deals with the Target Tube Reachability problem (TTRP) for linear dynamic systems evolving on a finite time set and subject to uncertainty in the initial state, perturbation inputs to the plant and the measurement program, as well as in some of the parameters defining the state transition matrix of the systems equation. All the uncertainties considered in the various signals and parameters correspond to set-valued uncertainties i.e only set bounds are known as a model for the uncertainty of the variables under study.

The robust reachability of targets is a basic and realistic problem due to the fact of parameter variations in control of processes of almost any kind. Mathematical simplifications such as slowly varying parameters and other type of models (white noise) for parameter uncertainty are either naive or exaggerated approaches to the problem of design under this type of frequent imprecision. Experience does tell the range of values wich one is to expect in a particular design problem.

Section II of this paper formulates the Robust TTRP for a linear discrete time system in complete generality regarding the involved sets. Section III presents the solution to the problem and the establishing of a backwards recursive algorithm for the determination of target reachability. Section IV particularizes the main algebraic operations involved in the algorithm of the preceeding section to the case of ellipsoids and convex polyhedra. The main contribution of this part lies in the introduction of the notion of robust inverse images of closed, convex bounded sets through linear dynamic maps with uncertain parameters. It is easily seen how convexity is preserved through this type of algebraic operation on the sets. Section V presents some flow diagrams representing computational procedures for performing some of the algebraic operations defined for polyhedral bounds. Some considerations on the computational efforts are indicated at this point. Section VI contains some conclusions and suggestions for further research indicating some recent developments in the general area of set-theoretic analysis of dynamic systems. The list of references used in the preparation of this research is included at the end of the paper.

2. PROBLEM FORMULATION

Consider the discrete-time linear dynamic system:

$$\underline{x}(k+1) = (A_0+\sum_{j=1}^{q} p_j(k)A_j)\underline{x}(k)+B\underline{u}(k)+G\underline{w}(k) \quad (1)$$

where $\underline{x}(k) \in R^n$ is a vector called the state of the system at time k, $\underline{u}(k) \in R^m$ is the control vector $\underline{w}(k)$ is a perturbation input signal, $p_j(k)$ is an uncertain parameter of the systems transition matrix A. The matrices A_0 A_j, B and G are real valued matrices of the appropriate dimensions according to:

$\underline{x}(0) \in X_0 \subset R^n$, X_0 is a closed convex bounded set (ccbs)

$\underline{w}(k) \in W_k \;\; \forall k, \;\; W_k \subset R^r$ (ccbs)

$\underline{p}(k) \in P_k \;\; \forall k, \;\; P_k \subset R^q$ (ccbs). The elements of $\underline{p}(k)$ are $p_j(k)$

$\underline{u}(k) \in U_k \;\; \forall k, \;\; U_k \subset R^m$ (ccbs) Set of Admissible Controls.

It is required to find an admissible control sequence $\{ \underline{u}(k),\ k=0,1,\dots,N-1,\ \underline{u}(k) \in U_k \ \forall k \}$ such that the state of the system $\underline{x}(k)$ is found, at each instant of time k of the finite planning horizon 0,1,2,...N-1, within a ccbs X^k (Target Tube) in spite of the values that the variables $\underline{x}_0$, $\underline{w}(k)$ and the parameters $\underline{p}(k)$ may take within their respective restriction (uncertainty) sets.

At each instant of time the controller performs measurements on the state of the system according to the rule (measurement program) :

$$\underline{z}(k) = H\,\underline{x}(k) + \underline{v}(k) \quad (2)$$

where $z(k) \in R^p$ is the measurement vector and $\underline{v}(k) \in R^p$ is a vector quantity of unknown nature, known only to be an element of a prescribed ccb set of R^p called the measurement disturbance set and denoted by V^k. The matrix H has the appropiate dimensions.

The fact that we have chosen a time invariant model does not affect substantially the developments for the more general case.

The form of the state transition matrix and its dependence on the uncertain parameters p_j is quite general and even provides room for the case of non-linear effects of this parameters on the system matrix. It is easy to see that one can always write a matrix with uncertain entries in this form, the upper bound for q would then be n^2. We assume that an ordering of the parameters has been chosen at will to conform the parameter vector $\underline{p}$. Otherwise the form of the set could hardly be defined for P_k.

The presence of uncertain parameters in the matrix B would indeed alter the nature of our results since at some point of the algorithm we shall propose in the next section, this fact would destroy the convexity of the sets making it a very difficult problem from the computational viewpoint. The same applies for the matrix G.

Before stating the main result of this article we shall develop some formulas which are frequent in the operations involved in our algorithm. Also some definitions needed hereafter are now presented.

Definition 1 Let A be a linear map from R^m into R^n and, X a ccb set in R^n. Then the image of X under A is defined as the set:

$$\{ \underline{y} \in R^n : \underline{y} = A \underline{x} \ \forall \ \underline{x} \in X \}$$

and denoted by the symbolism $Y = A X$.

Obviously if X is a ccb set so is Y, symmetry, connectedness and some other important properties of X would be preserved by Y.

Definition 2 Let $\bar{A}$ be a linear map from R^n into R^n and X a ccb set in R^n. $\bar{A}$ not necessarily invertible, then the inverse image of X under $\bar{A}$ is the set :

$$\{ x \in R^n : \bar{A} x \in X \}$$

and denoted by $\bar{A}^{-1} X$.

Definition 3 Let A and B be linear maps in R^n and p certain parameter with values in the range [a,b] then we call the robust inverse image of a set X in R^n, the following set:

$$\{ \underline{x} \in R^n : (A + pB)\underline{x} \in X \ \forall \ p \in [a,b] \}$$

This set is denoted by $(A+pB)^{-1}X$.

The robust inverse image set of X is a ccb set whenever X is ccb.

Definition 4 Let X and Y be two sets in R^n the vector sum of X and Y denoted by $X \oplus Y$ is defined as:

$$\{ \underline{z} \in R^n : \underline{z} = \underline{x} + \underline{y}, \forall \underline{x} \in X \text{ and } \underline{y} \in Y \}$$

The sum of two convex sets is convex, boundedness, closedness etc. are also preserved in this operation.

Definition 5 Let X and Z be two ccb sets in R^n, X necessarily containing Z, we define the Pontryaguin difference of X and Z and denote it by $X - Z$ as a set Y such that $Z \oplus Y = X$.

3. NECESSARY AND SUFFICIENT CONDITIONS FOR TARGET REACHABILITY IN A ROBUST SENSE

In this section we establish the main result concerning reachability of the given target tube X^k, under the stated hypothesis of set-bounded uncertainty of a collection of variables including initial states, perturbation signals to the plant and parameters of the state transition map.

The result is established as a necessary and sufficient condition for robust reachability of the tube. This condition is obtained as a set inclusion condition that has to be fulfilled in order for the target to be reachable in a guaranteed fashion. We make here use of the previous definitions regarding images, inverse images and robust images of ccb sets under linear maps.

Proposition 1 *The necessary and sufficient condition for an admissible control sequence* u(k), k=0,1,2,...,N-1. *such that target tube reachability is achieved in a robust sense (i.e for all possible values of the parametric uncertainty) is that the set inclusion relation :*

$$X_R^0 \supset X_0$$

is valid. The set X_R^0 *is computed (off-line) by means of the following backwards recursive algorithm:*

$$X_M^{k+1} = X_R^{k+1} - G W^k$$

$$X_A^k = X_M^{k+1} \oplus (-B U_k)$$

$$X_{RA}^k = \bigcap_{p(k) \in P_k} (A_0 + \textstyle\sum_{j=1}^{q} p_j(k) A_j)^{-1} X_A^k$$

$$X_R^k = X_{RA}^k \cap X^k$$

with the "initial condition" :

$$X_R^N = X^N$$

The set X_M^k is called the Modified Target set at time k, and conforms a Modified Target Tube for each and every k in the finite planning horizon. This set represents the region of the state space from which no value of the plant perturbation input can force the state at time k-1 out of the Reduced Target set X_R^k. The Modified Target Set is thus a robust set with respect to the additive action of the plant uncertain input signal on the present value of the propagated state through the state transition matrix.

The set X_A^k is called the Attainability Target set and represents the set in state space for which an admissible control vector u(k) can be found such that the next state is found within the Modified Target Set at the next instant of time. This Attainability Target set thus contains everything that is possible to transfer to a secure region in the state space from which the additive disturbances do not take the state out of the Reduced Target set at the next instant of time.

The set X_{RA}^k is called the Robust Attainability set at time k. This set represents that portion of the Attainable set which is immune to the parametric uncertainty multiplicative action on the states at time k, so that reachability of the Modified Target set at the next instant of time can be achieved. This is the smallest set from which the state, no matter what parameters or perturbation signals nature chooses to apply, its elements will always be capable of transition to the Reduced Target set.

The set X_R^k is the Reduced Target set at time k and as might have been already infered, this set contains that part of the state space which has to be achieved by the state (i.e reached) and at the same time contains those states which can be guaranteed to posses an admissible control sequence for reachability of the rest of the tube prescribed as a target. This is a compromise set where you have what you want to reach and what you must reach to insure a long term satisfactory behaviour of the systems state trajectory.

The Dynamic Programming philosophy underlying this backwards recursive process is evident. Its off-line character allows room for computer studies and feasibility experiments. In this article we shall consider some of these topics in connexion with polyhedral constraints in the fifth section.

All the basic algebraic operations involved in the above *a priori* recursive algorithm do not destroy the convexity of the original data sets.

Suppose that target reachability has been concluded with the aid of the previous algorithm. We must then proceed to find a control sequence that actually produces target reachability. We shall now indicate how to compute a set in the control space, at each instant of time k, (i,e a tube) which has the property that any of its elements produces, at the proper time, state transitions that ensure robust reachability of the target tube. We term such tube the Strategy Control tube or Robust Strategy Control Tube. This set is given by:

$$\bar{U}^k = B^{-1} [X_M^{k+1} - \bigcup_{\underline{p}(k)\in P_k} (A_0 + \textstyle\sum_{j=1}^{q} p_j(k) A_j) X_{k|k}]$$

where $X_{k|k}$ is either a singleton, in the case of perfect measurements, or an estimate set of the state at time k produced by a processing of the observations, on the state values in an on-line fashion, when these observations are noise-corrupted.

In general the Strategy Control tube is constituted by a sequence of sets which are non-convex. This is easily inferred from the above formula due to the presence of set unions (this operation is known to destroy convexity in general). However it is possible to prescribe the convex hull for the non-convex set in Pontryaguin difference with the Modified Target set at time k+1 and thus obtain a "strong" Strategy Control set.

As pointed out above, the sets $X_{k|k}$ are the outcome of an estimation process that is performed according to the systems dynamics and the compatibility of the set of possible states with those rendered by the measurement program. The estimation process is accomplished in the following manner and evidently is an *on-line* process:

$$X_{k|k} = X_{k|k-1} \cap -H^{-1}(V^k \oplus \{-\underline{z}(k)\})$$

$$X_{k|k-1} = \bigcup_{p\in P_k} (A_0 + \textstyle\sum p_j A_j) X_{k-1|k-1} \oplus \{B\bar{u}(k-1)\} \oplus GW_{k-1}$$

with the "initial condition" :

$$X_{0|0} = X_0$$

where the control $\bar{u}(k-1)$ is any element in the Strategy Control set $\bar{U}^{k-1}$.

As in the case of the Strategy Control sets, the estimation process involves dealing with non-convex sets. In this case it would be necessary to compute an inner bound to the first summand in the previous formula so that certain strength is added to the on-line process of simultaneous estimation and control.

4. ELLIPSOIDAL AND POLYHEDRAL CASES

In this section we particularize the main formulas and algebraic operations included in the backwards recursive process presented in the previous section. We shall present formulas in a general and summary fashion and provide comments where necessary.

Ellipsoidal case

Definition 6 An ellipsoid E in an Euclidean n-space is defined as the set:

$$\{ \underline{x} \in R^n : (\underline{x}-\bar{\underline{x}})' Q (\underline{x}-\bar{\underline{x}}) \leq 1 \}$$

where the vector $\underline{x}$ is called the centroid and Q is the dispersion matrix (for some other definitions of ellipsoids see Sira(1977))

Proposition 2 Let X and Y be two ellipsoids with centroid in the origin and dispersion matrices Q and R respectively. The vector sum of X and Y is not an ellipsoid but it can be out-bounded by the ellipsoid :

$$\bar{S} = \{ \underline{x} \in R^n : \underline{x}' S \underline{x} \leq 1 \}$$

where:

$$S^{-1} = b^{-1} Q^{-1} + (1-b)^{-1} R^{-1}$$

with b a design parameter chosen from the interval (0,1). (For proof See Schweppe (1968))

Proposition 3 The Pontryaguin difference of the ellipsoids X and Y (X contains Y) is not an ellipsoid but an inner bound ellipsoid is given by:

$$\bar{D} = \{ \underline{x} \in R^n : \underline{x}' D \underline{x} \leq 1 \}$$

where:

$$D^{-1} = Q^{-1} - b(1-b)^{-1} R^{-1}$$

with b an arbitrary parameter lying in the open interval (0,1). (Proof in Schweppe (1968))

Proposition 4 The intersection of the ellipsoids X and Y is not an ellipsoid but an outer ellipsoidal bound is represented by:

$$\{ \underline{x} \in R^n : \underline{x}' J \underline{x} \leq 1 \}$$

where:

$$J = (1+g)^{-1} Q + g(1+g)^{-1} R$$

and g is a design parameter in the open infinite interval $(0,\infty)$. (See Glover & Schweppe)

Proposition 5 An inner bound ellipsoid of the non-empty intersection of X and Y is given by:

$$\{ \underline{x} \in R^n : \underline{x}' K \underline{x} \leq 1 \}$$

where:

$$K = a_1 Q + a_2 R$$

with a_1 and a_2 chosen via the following procedure:

$$a_1 = (1-m_1)/(1-m_1 m_2)$$
$$a_2 = (1-m_2)/(1-m_1 m_2)$$

whenever $m_1 < 1$, $m_2 \leq 1$ or $m_1 \leq 1$ and $m_2 < 1$ and:

$$m_1 = \min_{\underline{x}' Q \underline{x} \leq 1} \underline{x}' R \underline{x}$$

$$m_2 = \min_{\underline{x}' R \underline{x} \leq 1} \underline{x}' Q \underline{x}$$

all other possibilities of values for m_1 and m_2 correspond to either inclusion of X in Y, Y in X or empty intersection (See Glover & Schweppe (1971)).

Proposition 6 Let M be an nxn invertible matrix. Then the image of X under M is an ellipsoid whose dispersion matrix Q_M satisfies $M'Q_M M = Q$, i.e

$$MX = \{\underline{x} \in R^n : \ \underline{x}' \ (M')^{-1} Q \ M^{-1} \ x \leq 1 \}$$

If M is non-invertible, then the pseudo-inverse should be used. In this case the ellipsoid is said to be "degenerate" since there is a proper sub-space of R^n which wholly contains the image of X under such map. (Sira,77)

Proposition 7 Let M be a not necessarily invertible matrix, then the inverse image of X under M is an ellipsoid given by:

$$M^{-1}X = \{\underline{x} \in R^n: \ \underline{x}'M'QM \ \underline{x} \leq 1 \}$$

If M is not full rank, the ellipsoid $M^{-1}X$ is also degenerate since this ellipsoid would not have any excursions into the null space of M i.e it would be circumscribed to the orthogonal complement of this proper sub-space of R^n. (Proof in Sira (1977))

Proposition 8 Let X be an ellipsoid, then the robust inverse image of X :

$$Z = \bigcap_{\underline{p} \in P} (A_0 + \textstyle\sum_{j=1}^{q} p_j A_j)^{-1} \ X$$

is not an ellipsoid. P is an ellipsoid characterized by $P = \{\underline{p} \in R^q: \ p'P \ p \leq 1 \}$. However an inner bound ellipsoid to $\overline{Z}$ is given by:

$$\overline{Z} = \{ \underline{x} \in R^n: \ \underline{x}'(A_0 + \textstyle\sum p_j^* A_j)' Q (A_0 + \textstyle\sum p_j^* A_j) \underline{x} \leq 1 \}$$

where the p_j^*'s are the components of a vector $\underline{p}^*$ given by:

$$\underline{p}^* = (k \ P - K_B)^{-1} \ \underline{b}$$

with k being a solution of the algebraic equation:

$$\underline{b}' (k^2 I - 2kP^{-1} K_B + K_B P^{-1} K_B)^{-1} \underline{b} = 1$$

which makes p^* lie within P (k is a scalar and the algebraic equation is of 2n-th order) The vector $\underline{b}$ has components $b_i = Tr(A_i' \ Q \ A_0)$ while K_B is a q x q symmetric matrix whose i,j-th entry is $k_B(i,j) = Tr(A_i' \ Q \ A_j)$. Where Tr stands for the trace operator.

Proof The proof of this proposition constitutes a simple excersise in an constrained algebraic optimization problem. We simply sketch the proof at this time.

Consider the trace of the dispersion matrix of $\overline{Z}$, it is easy to show that this trace is a measure of the extension of the ellipsoid. Maximizing this functional subject to the ellispoidal restriction of the parameter vector p leads to a Lagrangian optimization problem whose solution is given by p^*. The Lagrange multiplier k , could then be obtained by substitution of the solution vector $\underline{p}^*$ in the restriction equation.

The basic operation needed for the computation of the Robust Strategy Control Set and the updated estimate set is the robust union of ellipsoids. We define now this operation.

Definition 7 The Robust Union under the set P of parameters of the ellipsoid X is given by:

$$U = \bigsqcup_{\underline{p} \in P} (A_0 + \textstyle\sum_{j=1}^{q} p_j A_j) \ X$$

This operation evidently does not yield, in general, an ellipsoid. However several approximations can be worked out (i.e bounds). The simplest one being a sphere whose radius should equal the value of the spectral radius (See Brockett (1970)) of the matrix

$$(A_0 + \textstyle\sum p_j A_j)'^{-1} Q \ (A_0 + \textstyle\sum p_j A_j)^{-1} \qquad (3)$$

The computation of the maximization operation would have to be restricted over the condition $\underline{p} \in P$.

A second alternative is the maximization of the support functional of the generic ellipsoid whose dispersion matrix is given by (3) with respect to all possible values of p. This is far easier to achieve and the result is identical to the one given in the previous proposition except for the fact that Q has to be replaced by Q^{-1}. We leave it to the reader the verification of this simple result

We now proceed to summarize the basic formulae involved in the main algorithm for the polyhedral case.

Polyhedral Case

Definition 8 A polyhedron (also called polytope) in R^n is defined as:

$$\{ \underline{x} \in R^n : \ \langle \underline{x}, \underline{x}_i^* \rangle \leq 1 \ ; \ \forall \ i=1,2,\ldots,N_x \}$$

$N_x \geq n + 1$. $\underline{x}_i^*$ are called support vectors.

Notice that our definition of polyhedra makes them always contain the origin of coordinates. This does not represent a loss of generality and gives us the additional bonus of greater mathematical tractability without lossing the essential features of the problems.

Proposition 9 Let X and Y be polyhedra characterized by support vectors $\underline{x}_i^*$ and $\underline{y}_i^*$ respectively. Then the sum $X \oplus Y$ is a polyhedron with support vectors $\underline{z}_i^*$ given by: (Proof in:Sira (1977))

$$\underline{z}_i^* = \underline{x}_i^* / (1+m_i^*) \quad \text{where } m_i^* = \max_{\underline{y} \in Y} \langle \underline{y}, \underline{x}_i^* \rangle$$

$$i = 1,2,\ldots, N_x$$

$$\underline{z}_{N_x+j}^* = \underline{y}_j^* / (1+ n_j^*) ; \ n_j^* = \max_{\underline{x} \in X} \langle \underline{y}, \underline{y}_j^* \rangle$$

$$j=1,2,\ldots,N_y$$

Proposition 10 Let X and Y be as before with X containing Y. Then the Pontryaguin difference set X - Y is defined as a polyhedron characterized by support vectors $\underline{z}_i^*$ where: (Proof in Sira (1977))

$$\underline{z}_i^* = \underline{x}_i^* \ / \ (1-m_i^*) \ ; \ m_i^* = \max_{\underline{y} \in Y} \langle \underline{y}, \underline{x}_i^* \rangle$$

$$i=1,2,\ldots, N_x$$

Proposition 11 Let X and Y be two convex polyhedra given as before, then the intersection $X \cap Y$ is a convex polyhedron whose support vectors are $\underline{z}_i^*$ given as: (Sira 1977)

$$\underline{z}_i^* = \underline{x}_i^* / m_i^* \ ; \ m_i^* = \min \{1, \max_{\underline{y} \in Y} \langle \underline{y}, \underline{x}_i^* \rangle \}$$

$$i = 1,2,\ldots, N_x$$

$$z^*_{N_x+j} = y^*_j / n^*_j \quad ; \; n^*_j = \min\{1, \max_{x \in X} \langle x, y^*_j \rangle\}$$

$$j = 1,2,\ldots,N_y$$

Proposition 12 Let M be an nxn invertible matrix and X a polyhedron with support vectors x^*_i. Then, the image of X under the linear map M is given by: (Sira (1977))

$$MX = \{z \in R^n : \langle z, z^*_i \rangle \leq 1;\; i=1,\ldots,N_x\}$$

with:

$$z^*_i = M'^{-1} x^*_i \quad \forall \; i.$$

Proposition 13 Let M be an nxn matrix not necessarily invertible, and X a polyhedron as before. The inverse image of X under M is also a polyhedron whose support vectors are z^*_i. These are given by: (Sira (1977))

$$z^*_i = M' x^*_i \quad \forall \quad i.$$

Proposition 14 Let X be a polyhedron with support vectors x^*_i, then the robust inverse image of X :

$$Z = \bigcap_{p \in P} (A_0 + \sum_{j=1}^q p_j A_j)^{-1} X$$

where P is a hyperbox in R^q with coordinate constraints $|p_j| \leq \beta_j$, is a polyhedron with a finite number of hyperfacets given by the expression:

$$Z = \{z \in R^n : \langle z, (A_0 + \sum_{j=1}^q \pm\beta_j A_j)' x^*_i \rangle \leq 1 \quad \text{for all } i = 1,2,\ldots, N_x\}$$

where all possible combinations of signs should be taken in the q summands of the above formula. An alternative expression for the above polyhedron is the following:

$$Z = \{z \in R^n : \langle z, A_0' x^*_j \rangle + \sum_{j=1}^q \beta_j |\langle z, A_j' x^*_j \rangle| \leq 1 \quad j = 1,2,\ldots, N_x\}$$

The number of facets (hyperplanes) bounding this polyhedron is lesser or equal than $2^q N_x$. This clearly shows that the amount of computation is substantially increased when the number of parameters is quite large in a particular problem. However, not all of the hyperfacets are "active" in the sense that they actually tightly bound the polyhedron.

The problem of computing the Robust Union of an infinite number of sets parametrized by a vector p :

$$U = \bigcup_{p \in P} (A_0 + \sum_{j=1}^q p_j A_j) X$$

is quite a complicated problem when X is a polyhedron and no general formulas are known, at this point, except for some particularly simple case treated by Barmish (1979). However we shall outline a procedure for finding an outer bound to the infinite union, solving a finite number of non-linear optimization problems with constraints.

a) Produce an expression for the vectors:

$$(A_0 + \sum_{j=1}^q p_j A_j)^{-1} x^*_i \quad \text{for each } i.$$

(The existence of computer programs such as MACSYMA, See Boger *et al* (1975) easily allows one to handle this task quite effectively.

b) Minimize in norm each of the vectors over the existence range of the vector p. This step essentialy entails solving N_x - constrained optimization problems where the objective functions are rational functions of multinomials in the q parameters.

c) Choose separately the solution found in the previous step for each i, and substitute the original support vector x^*_i by x^*_i/m^*_i where m^*_i is the solution found in the previous step. This polyhedron bounds the infinite union.

Obviously this proceedure is quite limited and can only be followed in low dimensional cases without much complications. We remark however that many efficient computer packages are available for implementation of these ideas. For a complete treatment of the TTRP from the computational viewpoint see Sira (1977).

5 SOME COMPUTATIONAL CONSIDERATIONS

One of the most serious problems that often arise with the computer implementation of the necessary and sufficient conditions for existence of a solution to the TTRP, is the fact that the backwards recursive algorithm generates a sequence of polyhedra with a growing number of hyperfacets (bounding hyperplanes). This fact promts one to develop efficient computational procedures to reduce the number of constraints defining a particular polyhedron, since some of these constraints might be redundant (i.e. non-active) and thus unnecessarily occupying memory locations in the computer. This is particularly frequent in the case of computing intersections of polyhedra. A good number of the constraints appearing in one of them does not intervene in the final description of the true intersection.

In this section we shall indicate some computational procedures usually followed to eliminate redundant constaints in polyhedra descriptions. Some of the ideas are taken from Sira (1977) and Hnyilicza (1967).

Constraints reduction algorithms

We now describe two general constraint reduction algorithms which yield more economical description of polyhedra given by a number of constraints, some of which may be redundant. These algorithms are modified versions of the "bounding hyperbox algorithm" found in Hnyilicza & Lee & Schweppe (1975).

Definition 9 A Hiperbox H in R^n is a rectangular parallelopiped defined by : $H = \{x \in R^n : x^j_{min} \leq x_j \leq x^j_{max};\; j = 1,2,\ldots,N\}$. Let P be a polyhedron, then a bounding hyperbox is a hyperbox that contains P.

Bounding Hyperbox algorithm.

Assume we are given a polyhedron P defined by: $P = \{x \in R^n : \langle x, x^*_i \rangle \leq 1 \;\forall\; i=1,\ldots, K\}$. We now construct the tightest of all bounding hyperboxes containing P in the following way:

Let e_j be the vector $(0,0,..,1,0,..,0)'$ with 1 in the j^{th} component. Then the minimal bounding hyperbox is given by:

$$H^0 = \{ \underline{x} \in R^n : \; c^j_{min} \leq x^j \leq c^j_{max} \; ; \; j=1,..N\}$$

where c^j_{min} and c^j_{max} are generated by process:

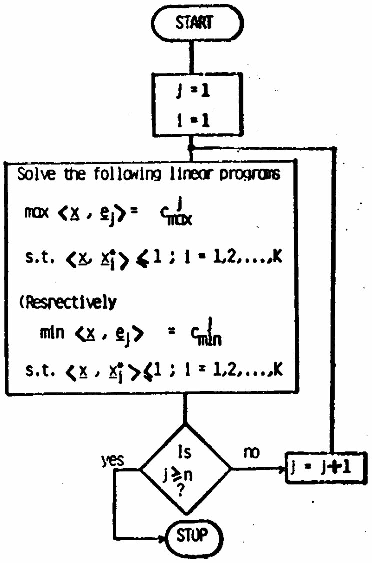

Fig. 1

An initial guess for the bounding hyperbox may also be considered before calculating the above minimal bounding hyperbox. In any case, the constraint reduction algorithm needs an initial bounding hyperbox to get started.

The following process calculates the vertices of the bounding hyperbox which lie in the same orthant as the direction in which each support vector $\underline{x}_i^*$ points and eliminates those constraints which do not intersect the bounding hyperbox. This yields a more economical description of the polyhedron P. (See fig.2)

True minimal description algorithm

The algorithm of Fig. 3 eliminates all redundant constraints for a polyhedron P. This algorithm should be used after an initial constraint reduction algorithm based in the bounding hyperbox has been applied. (See Fig.3)

Fig.4 shows an algorithm to find the minimal description of the intersection of two polyhedra with support sets $\underline{x}_i^*$ (i=1,..,K) and $\underline{y}_j^*$ (j=1,..,N) for the polyhedra P_1 and P_2 respectively.

6. CONCLUSIONS AND SUGGESITONS FOR FURTHER RESEARCH

In this paper we have presented extensions of previous results to the parametric uncertainty case in a TTRP. A backwards recursive algorithm was proposed for Target Tube reachability ascertainment.

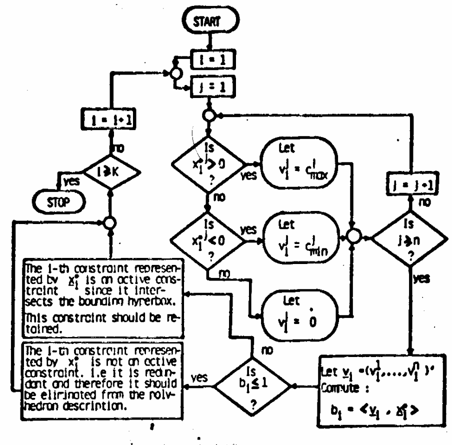

Fig.2

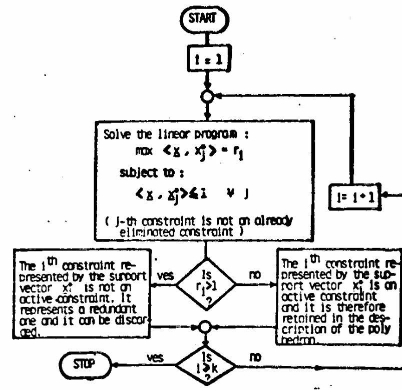

Fig.3

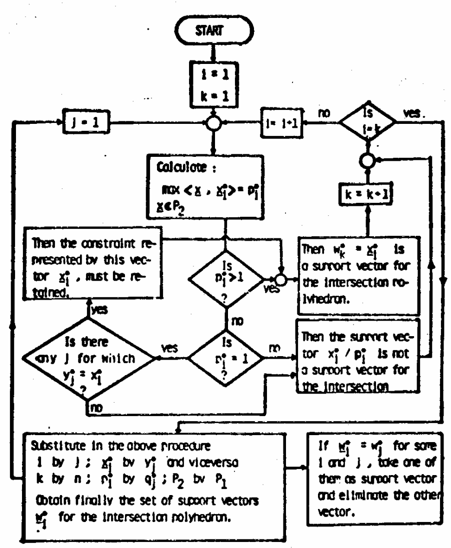

Fig.4

The main idea that allows for treatment of the parametric uncertain case lies in the concept of robust inverse images and robust direct images which basically entail respectively an infinite intersection and an infinite union of closed convex bounded sets over a parameter set.

We have particularized our main results for the most important cases of set-valued uncertainties, namely : Ellipsoids and Polyhedra. Explicit formulas were presented which allow computer implementation of the basic recursive scheme of section III.

Some new and interesting problems have arisen of these preliminary studies. Potential areas for further developments are among others the following . a) Computational experiences with practical problems. b) Development of new methods for efficiently approximating from outside and inside infinite unions and intersections of the two main classes of sets. c) Connection of some of our developments with bi-linear systems reachability of target sets and tubes. d) Exploration with other types of convex sets (cones, geometric theory) and even non-convex sets (Generalized Polyhedra, Sira (1978, 1979 a, 1979 b, 1979 c) e) Relation of this work with Fuzzy Sets (See also Sira(1979 d)).

EFERENCES

Barmish, B ., (1977) . " Envelope Correspondence & Its Applications to the Design of Guaranteed Error Controllers", SIAM J. Control & Opt., Vol. 15: 1001-1015.

Barmish, B ., (1979) ."The Propagation of Parametric Uncertainty Via Politopes", IEEE Trans. Auto. Control, Vol.AC-24: 346-349.

Bertsekas, D., & Rhodes, I., (1971) ."On the Minimax Reachability of Target Sets and Target Tubes", Automatica, Vol.7: 233-247.

Bogen, R., et al., (1975) ."MACSYMA Reference Manual", Project MAC, M.I.T. Cambridge. Mass.

Brockett, R., (1970) . Finite Dimensional Linear Systems, John Wiley & Sons, Inc. chapter 4.

Delfour, M., & Mitter, S., (1969) . "Reachability of Perturbed Systems and Min Sup Problems", SIAM J. on Control, Vol. 7, No.4: 521-533.

Glover, D., (1971) . "Control Systems with Unknown but Bounded Disturbances-Applications to Electric Power Systems", M.I.T. PhD Thesis.

Glover, D., & Schweppe, F., (1971) . "Control of Linear Dynamic Systems with Set-Constrained Disturbances", IEEE Trans. Auto. Control, Vol.AC-16, No.2 : 411-423.

Hnyilicza, E., (1967) . "A Set-Theoretic Approach to State Estimation", M.S. Thesis. M.I.T.

Hnyilicza, E., & Lee, S., & Schweppe, F., (1975) "Steady State Security Regions; Set-Theoretic Approach", 9th Power Ind. Comptr. Appl. Conf. New Orleans. LA.

Morris, R., & Brown, R., (1976) ."Memory & Time Conserved Calculation of State and Control Constrained Optimal Control Trajectories", IFAC Symp. on Large-Scale Syst. Th. & Appl. :35-42.

Schlaepfer, F., (1970) . "Set-Theoretic Estimation of Distributed Parameter Systems", MIT . Electr. Syst. Lab. Rept., ESL-R-413.

Schweppe, F., (1968) ." Recursive State Estimation: Unknown but Bounded Errors and System Inputs", IEEE Trans. Auto. Control. Vol. AC-13, No.1: 22-28.

Schweppe, F., & Knudsen, H., (1968) . "The Theory of Amorphous Cloud Trajectory Prediction", IEEE Trans. Inform. Theory, Vol. IT-14 : 415-427.

Sira, H., (1978) . "Reachable Sets and Set-Theoretic Evolution of the Uncertainty in Linear Systems", Annals 2°Brazilian Congress on Auto. Control. SBA, CNPq, UFSC.:677-688.

Sira, H., (1979a) . "Evolution of Generalized Uncertainties in Bi-linear Systems", Proc. 1979 IEEE Symp. Circ. & Syst. Tokio. Japan

Sira, H., (1979b) . "Evolution of Generalized Set-Theoretic Uncertainties in Linear Systems", Proc. 1979 Conf. on Info. Sci. & Syst.: 562-567. The John Hopkins Univ. MA.

Sira, H., (1979c) . "Accesibilidad de Conjuntos y Tubos de Blanco en Sistemas Continuos de Estructura Variable", II°Congreso Venezolano de Matematica. Univ. de Oriente. Cumana. Venezuela.

Sira, H., (1979d) . "Evolution of Fuzzy Sets in Linear Dynamic Systems," Proc. 1979 Intl. Conf. on Cybts. & Soc., :807-812. Denver.

Sira, H., (1977) . " Set-Theoretic Control of Large Scale Uncertain Systems", MIT. PhD Thesis. Cambridge. MASS.

Witsenhausen, H., (1966) . " Minimax Control of Uncertain Systems", MIT. PhD Thesis. Cambridge. MASS.

Witsenhausen, H., (1968) . "Set of Possible States of Linear Systems Given Perturbed Observations", IEEE Trans. Auto.Control, Vol. AC-13, No.5 : 556-558.

Yared, K., (1976) . " Average Frequency Trajectory Control: Normal Mode of Operation", MIT. M.S. Thesis. Cambridge. MASS