ON THE CONVEX HULL OF REACHABLE SETS FOR BI-LINEAR SYSTEMS

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ABSTRACT

The convex hull of reachable sets for bi-linear systems is examined through a support functional type of characterization. An optimal control problem is formulated for such characterization. Computational algorithms and some approximating scheme are also presented. Approximations relate to polyhedral characterization of the convex hull of reachable sets for this class of systems.

I INTRODUCTION

Reachable sets play an important role in both : optimal control theory and set-theoretic estimation and control problems [1],[2],[3]. A good amount of results are available in connection with these sets, for the linear case [4],[5],[6]. The most important features of reachable sets for linear dynamic systems are , among others, convexity, closedness, boundedness and connectedness (inputs are assumed to be bounded).

Recently effort has been placed in the study of reachable sets for bi-linear systems. The works of Brockett [7],[8],[9] and others [10],[11], have shown that desirable properties of reachable sets, present in the linear case, are not retained in the bi-linear case. For instance, even if the reachable set is closed whenever the control set is closed, bounded and convex , it is not true that the reachable set is necessarily closed when the control set is only compact. For bi-linear systems the reachable set is tipically non-convex and non-simply connected [9]. Rather stringent conditions and limitations have to be imposed on the bi-linear structure to retain convexity and compactness of the reachable set (RS).

In this paper we shall treat the problem of characterizing the convex hull (CH) of the RS for bi-linear systems. This problem is handled through the introduction of the functional support description of the RS. This description naturally convexifies the set making it mathematically tractable. The functional support description leads to a terminal time optimization problem defined on the bi-linear structure with bounded control signals. For reasons of space, only the amplitude bounded case will be treated , the results are easily extended to the energy constrained case.

The CH of the RS is characterized by the solution of a Matrix differential equation with split boundary conditions which are only partially known . Two equivalent algorithms are proposed for the computational solution of the problem. A sampling process of the support vectors yield a polyhedral approximation of the CH of the RS for our bi-linear structure. Some suggestions for further research are included at the end of the paper.

II PROBLEM FORMULATION AND MAIN RESULT

Given the bi-linear system:

$$\frac{d}{dt}\underline{x}(t) = (A + \textstyle\sum_{i=1}^{m} u_i(t)B_i)\underline{x}(t) \;;\quad \underline{x}(t_0)=\underline{x}_0 \qquad (1)$$

with $\underline{u}_i$ measurable inputs in $[t_0,T]$ constrained by the relation: $|u_i| \leq \beta_i$, find the CH of the RS in R^n at some fixed time $T < \infty$.

The support functional description of the CH of the RS is given by the expression:

$$\text{Co } R(T) \triangleq \{\underline{x} \in R^n : <\underline{x}, \underline{y}> \leq \sigma(\underline{y}) \quad \forall \ \|\underline{y}\| = 1\} \qquad (2)$$

where R(T) denotes the RS at time T, and $\sigma(y)$ is the support functional defined as the solution of the following static optimization problem :

$$\sigma(\underline{y}) = \sup_{\underline{x} \in R(T)} <\underline{x}, \underline{y}> \qquad (3)$$

In our problem the sup operation is a maximization due to the compactness of the RS .

Proposition 1

The CH of R(T) is characterized by:

$$\text{Co } R(T) \triangleq \{\underline{x} \in R^n : <\underline{x},\underline{y}> \leq \operatorname{Tr} K_T(\underline{y}) \quad \forall \ \|\underline{y}\| = 1\} \qquad (4)$$

where $K_T(\underline{y})$ is the unique solution at time T, of the non-linear matrix differential equation:

$$\frac{d}{dt}K(t) = [A,K(t)] + \textstyle\sum_{i=1}^{m} \beta_i \operatorname{sgn}(\operatorname{Tr} B_i K(t))\,[B_i, K(t)] \qquad (5)$$

with split boundary conditions:

$$K(t_0) = \underline{x}_0\, p_0'^{\infty} \quad ; \quad K(T) = K_T(\underline{y}) = \underline{x}^{\infty}(T)\, \underline{y}' \qquad (6)$$

where ' denotes transpose and the symbol $[A, B] = AB-BA$ is the commutator product of the involved matrices. The vectors $\underline{x}^{\infty}(T)$ and p_0^{∞} are not initially known but they are found via the following algorithm:

1) Select or guess a vector $p_0 = p_0^1$
2) Set $k = 1$
3) Compute the matrix $K^k(t_0) = \underline{x}_0 p_0'^{k}$ and integrate forward in time equation (5) with $K(t) \equiv K^k(t)$
4) Choose
$$u_i^k(t) = \beta_i \operatorname{sgn}(\operatorname{Tr}(B_i K^k(t))) \qquad (7)$$
5) Integrate backwards in time the vector differential equation:

$$\frac{d}{dt}\underline{p}^{k+1}(t) = -\,(A + \textstyle\sum_{i=1}^{m} u_i^k(t)\, B_i)'\, \underline{p}^{k+1}(t) \quad \text{with} \quad \underline{p}^{k+1}(T) = -\underline{y} \qquad (8)$$

6) Obtain a solution vector $\underline{p}_0^{k+1} = \underline{p}^{k+1}(t_0)$
7) Set $k = k+1$ and repeat the process from step 3 on until convergence.
8) $\underline{x}^{\infty}(T)$ is computed by solving (1) with $u_i = u_i^{\infty}$

Proof

The proposition is an immediate consequence of application of Pontryaguin's Maximum Principle to the constrained terminal cost optimal control problem. The Hamiltonian for the problem is:

$$H(\underline{x},\underline{p},\underline{u},t)= \langle \underline{p} , (A + \sum_{i=1}^{m} u_i B_i)\ \underline{x} \rangle \qquad (9)$$

with $|u_i| \leq \beta_i$ and the canonical equations are given by:

$$\frac{d}{dt}\underline{x}(t) = (A + \sum_{i=1}^{m} u_i(t)B_i)\ \underline{x}(t) \qquad (10)$$
$$\underline{x}(t_0) = \underline{x}_0$$

$$\frac{d}{dt}\underline{p}(t) = -(A + \sum_{i=1}^{m} u_i(t)B_i)'\underline{p}(t) \qquad (11)$$
$$\underline{p}(T) = -\underline{y}$$

Hamiltonian minimization subject to the control restrictions yields:

$$u_i(t) = \beta_i \operatorname{sgn} \underline{p}'(t)\ B_i\ \underline{x}(t) \quad \forall\ i \qquad (12)$$

If we define the matrix $K(t)= \underline{x}(t)\underline{p}'(t)$ then $\underline{x}'(t)\underline{p}(t)=$ Tr $K(t)=$ constant, due to the fact that $d/dt\ \underline{x}'(t)\underline{p}(t) = 0$. The control vector components are seen to equal $u_i = \beta_i \operatorname{sgn} Tr(B_i\underline{x}(t)p'(t)) = \beta_i \operatorname{sgn}Tr(B_iK(t))$ Taking derivatives on $K(t)$ and using the canonical equations associated with the optimization problem results in the differential equation for $K(t)$ as stated in (5). The split boundary conditions arise from the definition of $K(t)$ in a trivial manner.

Uniqueness of solutions for the matrix differential system are immediate from well established Lipschitz conditions of the right hand side of (5). The proposed algorithm constitutes a solution to the two point boundary value problem that arises by the coupling of (10),(11) and (12). The matrix form of the necessary conditions produces an ill-defined two point boundary value problem due to the partial knowledge of these conditions on either extreme of the time interval.

Since it is possible to have measurable intervals of time where sign $B_iK(t)$ could be undefined, singular solutions could arise. It is easy to verify that this can happen only if the smallest Lie Algebta generated by $A,B_1,..B_m$ is Abelian. (Frick and Wei [12]) .

III AN ALTERNATIVE COMPUTATIONAL ALGORITHM

The following is a computational algorithm that establishes an interplay among the matrix differential equation and the canonical equations for the solution of the TPBV problem which characterizes the CH of the RS.

1) Select or guess a vector $X(T) = x^1(T)$
2) Set $k = 1$
3) Compute the matrix $K^k(T) = -\underline{x}^k(T)y$ and integrate backwards in time equation (5) with $K(t) \equiv K^k(t)$.
4) Choose $u_i^k(t) = \beta_i \operatorname{sgn}(Tr\ B_iK^k(t))$
5) Integrate forward in time the vector differential equation (1) with $\underline{x}(t) \equiv \underline{x}^{k+1}(t)$ and boundary condition $x^{k+1}(t_0) = \underline{x}_0$.
6) Obtain $\underline{x}^{k+1}(T)$ and use this solution to repeat the process from step 3 on setting now $k = k+1$, until convergence.

Once the support functional has been determined as Tr $K_T(y)$ = Tr $K^{\infty}(T)$ for a particular $\underline{y}$, then a sampling of the possible values of $\underline{y}$ over the unit sphere is necessary to obtain a convex polyhedron approximation for the CH of the RS. A bounding hyperbox is readily obtained by taking the unit coordinate vectors $\underline{e}_i=(0,..,1,...0)$. Two and even three dimensional examples can benefit of a greater number of "sampling points" in the unit sphere to yield a better approximation of the CH for the RS.

IV CONCLUSIONS AND SUGGESTIONS FOR FURTHER RESEARCH

A characterization of the CH of the RS for bi-linear systems has been presented. A straightforward application of Pontryaguin's principle yields the answer to the optimization problem inherent in the support functional description of such CH. The TPBV problem that arises in conection with the optimal control problem can be solved via two equivalent computational algorithms that we have presented. The energy constrained case for the structural controls is treated similarly and it will be the subject of a forthcoming publication. The inhomogeneous structure case has not been treated here and deserves some attention. Numerical experience is needed in this area.

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