ON THE ROBUST CONTROL OF SYSTEMS WITH RESPECT TO SET VALUED OBJECTIVES

by

Hebertt Sira-Ramirez
Control Systems Department, Systems Engineering School
Universidad de Los Andes
Merida-Venezuela

ABSTRACT

This paper considers an extension of the Target Tube Reachability Problem (TTRP) for discrete-time, linear systems with set-constrained disturbance inputs and observation uncertainties to include set-bounded parametric uncertainty in the state transition matrix. A backwards recursive algorithm, similar in spirit to that developed by Bertsekas & Rohdes [1], is proposed for the determination of Robust Attainability Regions in state space. These guarantee the existence of admissible control actions that mantain possible state trajectories inside a pre-specified Target Tube defined over a finite planning horizon. The basic concepts for this extension are robust direct and inverse images of closed, convex, bounded (CCB) sets through linear maps with uncertain but bounded parameters. Formulae and simple examples are furnished in detail for the ellipsoidal case.

I. INTRODUCTION

This paper considers a slight extension of the Target Tube Reachability Problem (TTRP), treated by Bertsekas and Rohdes [1], Glover and Schweppe [2], and others [3], to include unstructured set-bounded knowledge of parametric uncertainty in the state transition matrix.

Our results parallel those of [1], in the sense that a similar backwards recursive algorithm for the determination of "robust" attainability regions is proposed. These regions, in state space, guarantee the existence of an admissible control sequence which maintains the possible state trajectories, starting from an initial state uncertainty set, inside a prespecified Target Tube defined on a finite planning horizon. Targer accesibility, whenever possible, is guaranteed in spite of all involved uncertainties: initial state uncertainty, additive disturbance inputs, structural (parametric) plant uncertainty in the state transition matrix, and corruptive signals affecting the measurement program in an additive fashion. A set inclusion involving the initial state uncertainty set and the robust attainability set at initial time, has to be verified for the existence of a "robust" control action that induces target accesibility to the systems state trajectory.

The basic results that allows for our extension are the concepts of robust inverse image and robust direct image of a closed, convex, bounded (CCB) set through a linear map with set-constrained uncertain parameters. These operations allows the proper handling of the parametric uncertainty in the backwards recursive algorithm, developed in [1], and represents the main difference of our work with that performed by researchers in the late sixties and early seventies [1], [2], [6].

In Section II of this paper, we develop the two basic concepts we just mentioned above: robust direct images and robust inverse images of CCB set through linear maps with set-constrained uncertain parameters. We present, in this section, specific formulae and simple examples for the case of ellipsoidal bounds and sets. We develop approximation schemes and procedures to inner-bound robust inverse images and outer-bound robust direct images of CCB sets. The nature of the TTRP justifies our search for such bounds and not others (i.e. external bounds to robust inverse images etc.).

Section III considers the Robust TTRP for discrete time, finite dimensional linear systems. We formulate this problem and present the backwards recursive algorithm for target reachability ascertainment.

In Section IV we discuss computational issues and basic difficulties related to intermediate sets generated by the proposed algorithm. We also point out possible research directions for future developments in this area.

II. BASIC DEFINITIONS AND RESULTS

In this section we present some basic definitions about robust direct and inverse images of CCB sets under linear maps with uncertain but bounded parameters. These definitions constitute extensions of well known set operations such as direct and inverse images of CCB sets under linear maps. We assume the reader is also familiar with set-theoretic operations such as vector sums, Pontryaguin difference of sets, etc. For a detailed account of these operations the reader is refered to Schweppe [5], Sira [3], [4]. All these concepts are intimately related to the formulae presented in next section.

Definition 1: Let A and A_j $(j=1,2,..,q)$ be linear maps in R^n, and $\underline{p} \in R^q$ a vector of parameters with values in the CCB set P. We define the robust inverse image of a CCB set X in R^n, the set:

$$\{\underline{x}\in R^n : (A+\Sigma_{j=1}^{q} p_j A_j)\underline{x}\in X \ \forall\ p\in P\} \qquad (2.1)$$

This set is denoted by $(A+\Sigma\, p_j A_j)^{-R}X$.

Definition 2: Let A, A_j $(j=1,2,..,q)$, $\underline{p}$ and P be as above. We define the robust direct image of a CCB set X in R^n, the set:

$$\{\underline{y}\in R^n : \underline{y}=(A+\Sigma_{j=1}^{q} p_j A_j)\underline{x} \text{ for some } \underline{x}\in X \text{ and some } \underline{p}\in P\} \qquad (2.2)$$

This set is denoted by $(A+\Sigma p_j A_j)^{R}X$.

In terms of inverse and direct images, for each $\underline{p}$, the above set operations are easily seen to be equivalent to:

$$(A+\Sigma p_j A_j)^{-R}X = \bigcap_{p\in P} (A+\Sigma_{j=1}^{q} p_j A_j)^{-1} X \qquad (2.3)$$

$$(A+\Sigma p_j A_j)^{R}X = \bigcup_{p\in P} (A+\Sigma_{j=1}^{q} p_j A_j) X \qquad (2.4)$$

CH1788-9/82/0000-1295$00.75

The robust inverse image of a CCB set is closed and convex although not necessarily bounded. On the other hand, robust direct images of a CCB set is, generally, non-convex, although closed and bounded.

Next, we particularize the above definitions for the case of ellipsoids. This class of sets is commonly used to describe unstructured knowledge of bounded uncertainty [1]-[5].

Let X and P be ellipsoids described by:

$$X = \{\underline{x}\epsilon R^n:\ \underline{x}'Q\ \underline{x} \le 1\} \tag{2.5}$$

$$P = \{\underline{p}\epsilon R^q:\ \underline{p}'S\ \underline{p} \le 1\} \tag{2.6}$$

Then, the sets $(A+\Sigma p_j A_j)^{-R}X$ and $(A+\Sigma p_j A_j)^R X$ are not, in general, ellipsoids. We must, therefore, device procedures to obtain tight inner and outer bounds to these sets respectively. It will be clear from the mini-max nature of the solution to the TTRP that it is precisely these bounds what we need, to guarantee "strong" solutions to our problem [1].

Inner bound ellipsoid to a Robust Inverse Image of X

Formula (2.3) entails the intersection of an infinite number of ellipsoids. We are required to find an inner-bound ellipsoid to such intersection set. We shall present a procedure which reduces the problem to finding an inner-bound to the intersection of a finite number of ellipsoids which we call "extremal". The inner bound we find, is the tightest inner bound ellipsoid for the infinite intersection. (i.e. is the "greatest" ellipsoid inside the infinite intersection set).

Consider, for some $\underline{p}\epsilon P$, the inverse image of X under $(A+\Sigma p_j A_j)$

$$(A+\Sigma p_j A_j)^{-1}X=\{\underline{x}\epsilon R^n:\underline{x}'(A+\Sigma_{j=1}^q p_j A_j)'Q(A+\Sigma_{j=1}^q p_j A_j)\underline{x} \le 1\} \tag{2.7}$$

It is easy to see that the Trace (Tr) of the dispersion matrix defining this ellipsoid equals the sum of the square inverses of the distances from the origin to the intersection points of the ellipsoid with the coordinate axis. From here it follows that maximizing the trace of the dispersion matrix, over all possible values of the parameter vector $\underline{p}$, one finds the values of $\underline{p}$ in P which "identify" those ellipsoids ("extermal" for us) which produce the closest intersection points to the origin. It is not difficult to see that there is only a finite number of such extremal ellipsoids. Each one of these will correspond to a solution $\underline{p}$ of the static optimization problem defined on the trace of the dispersion matrix, subject to the parameter vector constraint.

The maximization of the trace of the dispersion matrix subject to the parameter constraints is a standard algebraic optimization problem solvable by Lagrange multipliers. The involved formulae are:

Let p_j^{*i} be the j-th component of the vector $\underline{p}^{*i}$ given by:

$$\underline{p}^{*i} = -(\lambda_i S+K_B)^{-1}\ \underline{b} \tag{2.8}$$

with λ_i being the i-th real solution of the algebraic equation:

$$\underline{b}'(\lambda^2 S+2\lambda K_B + K_B S^{-1}K_B)^{-1}\ \underline{b}=1 \tag{2.9}$$

where the vector $\underline{b}$ has components $b_j = Tr(A_j'Q\ A)$; $j=1,2,\ldots,q$, while K_B is a qxq symmetric matrix whose i,j-th entry is $K_B(i,j) = Tr(A_i'Q\ A_j)$. Then the i-th "extreme" ellipsoid is given by:

$$\{\underline{x}\epsilon R^n:\underline{x}'(A+\Sigma_{j=1}^q p_j^{*i}A_j)'Q(A+\Sigma_{j=1}^q p_j^{*i}A_j)\underline{x} \le 1\} \tag{2.10}$$

Formula (2.8) is the consequence of taking derivatives with respect to $\underline{p}$ in the Lagrangian of the optimization problem posed above. Formula (2.9) is the outcome of substituting (2.8) into the parameter restriction equations (ellipsoid P). The algebraic equation (2.9) is 2q-th order and therefore has 2q solutions which are necessarily real according to the same equation. We thus have 2q possible solutions for $\underline{p}^*$. This finite set of ellipsoids produces, in all the range of $\underline{p}$, the closest distances from the origin of coordinates to the intersection points with particular coordinate axis. There can not be, in the infinite collection of ellipsoids, one which intersects any axis closer than the corresponding one in the finite set. This assertion can be made because of the optimal character of the solution, the symmetry of the restriction set P, the convexity of the ellipsoids, the quadratic nature of the objective function, and the boundedness of P and the ellipsoids of the form in (2.10) above. In summary the procedure would be as follows:

1) Compute the trace of the dispersion matrix of the ellipsoid (2.10).
2) Maximize this expression subject to the restriction $\underline{p}\epsilon P$.
3) Consider all the "extreme" solution ellipsoids found in the previous steps and intersect these solutions (This set is not, in general an ellipsoid).
4) Find an inner bound ellipsoid to this intersection set. (This can be done applying the ideas of Schweppe [5] or Glover and Schweppe [2] by taking intersections of two ellipsoids at a time and finding an inner bound for this intersection).

Example 1

Let A=1 and $A_1 = \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}$, $p^2 \le 1$ and $X = \{\underline{x}\epsilon R^2:x'x \le 1\}$. The robust inverse image of X under $A+p\ A_1$ is given by:

$$\bigcap_{p\epsilon P}\{\underline{x}\epsilon R^2:\ \underline{x}'\begin{vmatrix} 1 & p \\ p & 1+p^2 \end{vmatrix}\underline{x} \le 1\}$$

Thus, the Lagrangian optimization problem would entail solving:

$$\max(2+p^2)\ \text{s.t.}\ \ p^2-1 \le 0$$

There are two solutions to this problem p=+1 and p=-1 corresponding to the "extreme" elliptic regions:

$$x_1^2 + 2x_1x_2 + 2x_2^2 \le 1 \text{ and } x_1^2 - 2x_1x_2 + 2x_2^2 \le 1$$

An inner bound to the intersection of these two elliptic regions is given by the disk: $x_1^2 + x_1^2 \le 0.381964$.

All these sets are represented in Fig. 1.

Outer bound ellipsoid to the Robust Direct Image of X

As before, formula (2.4) entails the union of an infinite number of ellipsoids and we are required to find an outer-bound ellipsoid to such union set. A procedure almost identical to the preceeding one allows one to deal only with a finite number of ellipsoids and then use standard procedures for finding the outer bound of their union. The procedure allows to find the "smallest" outer-bound ellipsoid for the infinite union.

Consider the direct image, for some $\underline{p}$ in P, of X under the map $A+\Sigma p_j A_j$. The support functional description for this set is (Schweppe [5]):

$$(A+\Sigma p_j A_j)X=\{\underline{x}\epsilon R^n:\underline{x}'\underline{y} \le [y'(A+\Sigma_{j=1}^q p_j A_j)Q^{-1}(A+\Sigma_{j=1}^q p_j A_j)'\underline{y}]^{1/2} \text{ for all } y'y=1\} \tag{2.11}$$

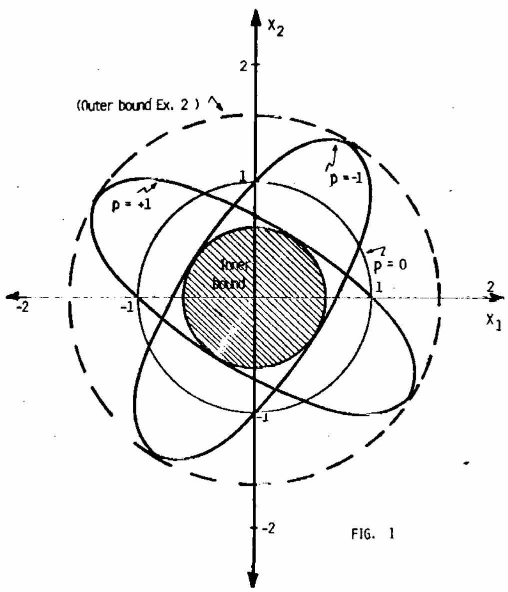

FIG. 1

It is easy to see that the trace of the matrix in the support functional above, represents the sum of the squares of the distances from the origin to the intersection of the support hyperplane, tangent to the ellipsoid, with each coordinate axis. This support hyperplane is, in each case, perpendicular to the coordinate axis. From here it follows that maximizing the trace of this "dispersion" matrix we can find the ellipsoids which possess points far away from the origin. Those "extreme" ellipsoids would, again, be parametrized by $\underline{p}$ and as before there is only a finite number of such "extreme" objects.

The static optimization problem we are suggesting is very similar to the one proposed before. We only have to notice the similarity of the objective functions and note that it is only required to replace Q by Q^{-1} the matrix A by A' and each A_j by A_j' $(j=1,2,..,q)$. i.e. we would have:

$$\underline{p}^{*i} = -(\bar{\lambda}_i S + \bar{K}_B)\ \bar{\underline{b}} \tag{2.12}$$

with $\bar{\lambda}_i$ being the i-th real solution of the algebraic equation:

$$\bar{\underline{b}}'(\bar{\lambda}^2 S + 2\bar{\lambda}\ \bar{K}_B + \bar{K}_B S^{-1} \bar{K}_B)^{-1}\ \bar{\underline{b}} = 1 \tag{2.13}$$

where the vector $\bar{\underline{b}}$ has components $\bar{b}_j = Tr(A_j Q^{-1} A')$; $j=1,2,..,q$, while $\bar{K}_B$ is a qxq symmetric matrix with $\bar{K}_B(i,j) = Tr(A_i Q^{-1} A_j')$. Then the i-th "extreme" ellipsoid is given by (2.11) replacing p_j by p_j^{*i}. The procedure would then be very similar, in spirit, to the one proposed before:

1) Compute the trace of the dispersion matrix in (2.11).
2) Maximize this quantity over all possible values of $\underline{p}$ in P.
3) Consider all the "extreme" solution ellipsoids found in the previous step and consider the union of this finite set of ellipsoids.
4) Find an outer bound ellipsoid to this union set. A sphere of radius equal to the square root of the maximum eigenvalue of the inverse of the dispersion matrix in (2.11) always provides a tight outer-bound to our problem.

Example 2:

Consider the same matrices and ellipsoids in R^2 of the previous example. The robust direct image of X under the map $A+pA_1$ is given by:

$$\bigcup_{p\varepsilon P}\{\underline{x}\varepsilon R^2 : \underline{x}'\underline{y} \leq [\underline{y}' \begin{vmatrix} 1+p^2 & p \\ p & 1 \end{vmatrix} \underline{y}]^{1/2} \text{ for all } \underline{y}'\underline{y}=1\}.$$

As before, there are two solutions to the optimization problem prescribed by the above procedure. One for p=+1 and other for p=-1. These extreme elliptic regions are given respectively by:

$$x_1^2 - 2x_1x_2 + 2x_2^2 \leq 1 \text{ and } x_1^2 + 2x_1x_2 + 2x_2^2 \leq 1$$

Since they coincide with the extreme solution ellipsoids of the previous example (although for oposite values of p) we show also in Fig. 1 the outer bound for the union of the above elliptic regions. This bound is a disk with equation: $x_1^2 + x_2^2 \leq 2.618034$.

III. PROBLEM FORMULATION AND MAIN RESULTS

Consider the discrete-time linear dynamic system:

$$\underline{x}(k+1) = (A_0 + \Sigma_{j=1}^{q} p_j(k) A_j)\underline{x}(k) + B\ \underline{u}(k) + G\ \underline{w}(k) \tag{3.1}$$

where $\underline{x}(k)\varepsilon R^n$ is a vector called the state of the system at time k, $\underline{u}(k)\varepsilon R^m$ is the control vector, $\underline{w}(k)$ is a perturbation input signal, $p_j(k)$ is an uncertain parameter vector of the system transition matrix. The matrices A_0, A_j, B and G are real valued matrices of the appropriate dimensions according to: (the time index $k=0,1,\ldots,N-1$)

$\underline{x}(0)\varepsilon X_0 \subset R^n$, X_0 is a CCB set

$\underline{w}(k)\varepsilon W_k\ \forall\ k$, $W_k \subset R^r$ and is a CCB set

$\underline{p}(k)\varepsilon P_k\ \forall k$, $P_k \subset R^q$ is a CCB set. The element of $\underline{p}(k)$ are $p_j(k)$.

$\underline{u}(k)\varepsilon U_k\ \forall\ k$, $U_k \subset R^m$ is a CCB set of Admissible Controls.

At each instant of time the controller performs measurements on the state of the system according to the rule (measurement program):

$$\underline{z}(k) = H\underline{x}(k) + \underline{v}(k) \tag{3.2}$$

where $\underline{z}(k)\varepsilon R^p$ is the measurement vector and $\underline{v}(k)\varepsilon R^p$ is a vector quantity of unknown nature, known only to be an element of a prescribed CCB set of R^p called the measurement disturbance set denoted by V_k. The matrix H has the appropriate dimensions.

It is required to find an admissible control sequence $\{\underline{u}(k),\ k=0,1,..,N-1,\ \underline{u}(k)\varepsilon U_k \forall\ k\}$ such that the state of the system $\underline{x}(k)$ is found, at each instant of time k, within CCB set X^k (Target Tube) in spite of the values that the variables $\underline{x}_0$, $\underline{w}(k)$, $\underline{v}(k)$, and the vector $\underline{p}(k)$ may take within their respective restriction (uncertainty sets).

Formally stated our problem would be possed as:

Problem: Given the discrete time linear system (3.1) and the measurement program (3.2) find (if it exists) an admissible control sequence $\underline{u}(k)$, k=0, 1,..,N-1, with the property that at each instant of time the state of the system is contained in a prespecified target set $\{X^k, k=1,2,..,N\}$ for all possible disturbance sequences $\underline{w}(k)$, $\underline{v}(k)$, all possible initial states $\underline{x}(0)$ in X_0 and all possible values of the parameters $\underline{p}(k)$.

We say that the target tube is reachable in a robust sense whenever a solution to the above problem exists. (see Bertsekas [1]).

Proposition 1

The Target Tube $\{X^k\}$ is reachable in a robust sense from the initial state uncertainty set X_0 if and only if $X_R^0 \subset X_0$. The set X_R^0 is computed (off-line) by means of the following backwards recursive algorithm:

$$X_M^{k+1} = X_R^{k+1} - GW^k \tag{3.3}$$

$$X_A^k = X_M^{k+1} + (-BU_k) \tag{3.4}$$

$$X_{RA}^k = (A_0+\Sigma p_j(k)A_j)^{-R} X_A^k \tag{3.5}$$

$$X_R^k = X_{RA}^k \cap X^k \tag{3.6}$$

with the "initial condition"

$$X_R^N = X^N \tag{3.7}$$

The set X_M^k is called the Modified Target Set at time k, and conforms a Modified Target Tube when all k's are considered. This set represents the region of the state space for which no value of the plant perturbation input can force the state, at time k, out of the Reduced Target Set X_R^k. The Modified Target Set is thus a robust set with respect to the additive action of the plant uncertain input signal on the present value of the propagated state through the state transition matrix.

The set X_A^k is called the Attainability Target Set and represents the set in state space for which an admissible control vector $\underline{u}(k)$ can be found such that the next state is found within the Modified Target Set. This Attainability Target Set thus contains everything that is possible to transfer to a secure region in the state space from which the additive disturbances do not take the state out of the Reduced Target Set at the next instant of time.

The set X_{RA}^k is called the Robust Attainability Set at time k. This set represents that portion of the Attainable Set which is immune to the parametric uncertainty multiplicative action on the states at time k, so that reachability of the Modified Target Set, at the next instant of time, can be achieved. This is the smallest set for which, no matter what parameters or perturbation signals nature chooses to apply, its elements will always be capable of transition to the Reduced Target Set.

The set X_R^k is the Reduce Target Set at time k and, as might have been already inferred, this set contains that part of the state space which has to be achieved by the state (i.e. reached) and at the same time, contains those states which can be guaranteed to posses an admissible control sequence for reachability of the rest of the Tube prescribed as a target. This is a compromise set where you have what you want to reach and what you must reach to insure a long term satisfactory behavior of the systems state trajectory.

The Dynamic Programming spirit underlying this backwards recursive process is self evident. Its off-line character allows room for computer studies and feasibility experiments.

All the basic set-theoretic operations involved in the above a priori recursive algorithm do not destroy the convexity of the original data sets.

Suppose that target reachability, in a robust sense, has been verified with the aid of the previous algorithm. We must then proceed to find a control sequence that actually produces target reachability. We shall now indicate how to compute a set in the control space, for each instant of time k, (i.e. a tube) which has the property that any of its elements produces, at the proper instant of time, state transitions that insure robust reachability of the Target Tube. We call such tube the Strategy Control Tube or Robust Strategy Control Tube. This set is given by:

$$\vec{U}^k = B^{-1}[X_M^{k+1} - (A_0+\Sigma p_j(k)A_j)^R X_{k|k}] \cap U_k \tag{3.8}$$

where $X_{k|k}$ is either a singleton, in the case of perfect measurements, or an estimate set of the state at time k, produced by a processing of the observations of the noise corrupted measurements. This process would necessarily have to be on-line.

In general, the Strategy Control Tube is constituted by a sequence of sets which are non-convex. This is easily infered from the above formula due to the presence of a robust direct image of the estimate set $X_{k|k}$. However, using the ideas of Section II, it is possible to prescribe an outer-bound for the robust direct image in Pontryaguin difference with the Modified Target Set at time k+1. We thus obtain a "strong" Strategy Control Set.

As pointed out above, the sets $X_{k|k}$ are the outcome of an on-line estimation process that is performed according to the systems dynamics and the compatibility of the set of possible states with those rendered by the measurement program. The estimation process is accomplished in the following manner:

$$X_{k|k} = X_{k|k-1} \cap H^{-1}(V^k + \{-\underline{z}(k)\}) \tag{3.9}$$

$$X_{k|k-1} = (A_0+\Sigma p_j(k)A_j)^R X_{k-1|k-1} + \{B\bar{\underline{u}}(k-1)\} + GW_{k-1} \tag{3.10}$$

with the "initial condition"

$$X_{0|0} = X_0 \tag{3.11}$$

The control vector $\bar{\underline{u}}(k-1)$ is an element of the Strategy Control Set $\vec{U}^{k-1}$.

As in the case of the Strategy Control Sets, the estimation process involves dealing with robust direct images of sets (i.e. non-convex sets, in general). In this case, it would be necessary to compute an outer-bound to the first summand in (3.10) so that certain "strength" is added to the on-line process of simultaneous estimation and control.

We conclude this section by pointing out that all the formulae in the algorithm (3.3)-(3.7) can be particularized for the case of ellipsoidal bounds. This only calls for set operations such as direct images, robust inverse images, intersections and Pontryaguin differences of the involved sets. Approximation formulae for all this operations exist in the published literature (Schweppe [5], Glover and Schweppe [2], Bertsekas and Rohdes [1], Schlaepfer and Schweppe [7], Sira [3],[4] etc.). The estimation process (3.9)-(3.11) is equally suitable for particularization in the ellipsoidal case. The formulae and procedures given in Section II for the robust direct images of

CCB sets allows such particularization. The rest of the operations in this process are widely known and has been described in full detail in the above mentioned references.

IV. CONCLUSIONS AND SUGGESTIONS FOR FURTHER RESEARCH

In this paper we have presented a straightforward extension of previously known results to include the parametric uncertainty case in the TTRP. A backwards recursive algorithm was proposed for Target Reachability ascertainment. The main ideas that allows for treatment of the parametric uncertain case lies in the concepts of robust inverse image and robust direct image of a CCB set under linear maps with uncertain but bounded parameters. These operations entail, basically, infinite intersections and unions of CCB sets respectively. We have shown, for a particular case, that only a finite number of the involved sets need be considered for the prescription of inner and outer bounds. We have only considered ellipsoidal bound for the uncertainty in the results concerning these robust inverse and robust direct images. Polyhedral cases are treated elsewhere [4].

One of the long-standing problems associated with the TTRP is the question of computational feasibility. The approximation schemes become somehow tirening for computer capabilities. A number of methods have been proposed in the past to handle the inescapable memory growth associated with the backwards recursive algorithm for Target reachability ascertainment. Elimination of redundant constraints, in the case of polyhedral bounds, minimal description of sets and canonical direction bases [8] are among the many incursions in the computational problems associated with the "unknown but bounded" technique for dynamic systems estimation and control problems.

Some new problems may arise from this preliminary study. Potential areas for further research are, among others, a) Computer implementation of the results. b) Development of new methods for efficient approximation (inner and outer bounds) of infinite unions and infinite intersections of ellipsoids and polytopes (bounded polyhedra). c) Connection of our problem with reachability of Target sets and tubes in bi-linear dynamic plants d) Consideration of other classes of bounds (cones, generalized polyhedra, subspaces) e) Extensions of this type of problems to the case of fuzzy sets (See also Sira [9],[10]).

ACKNOWLEDGMENTS

The comments and suggestions of the reviewers for an early version of this paper are gratefully appreciated. The publication of this paper would have been impossible without the efficient cooperation of Fifa Monserrate at the Massachusetts Institute of Technology.

REFERENCES

[1] Bertsekas, D., and Rohdes, I., "On the Mini-max Reachability of Target Sets and Tubes," Automatica, Vol. 7; pp. 233-247, 1971.

[2] Glover, D., and Schweppe, F., "Control of Linear Dynamic Systems with Set Constrained Disturbances," IEEE Trans. on Aut. Contr., Vol. AC-16, No. 2; pp. 411-423, 1971.

[3] Sira-Ramirez, H., "Set-Theoretic Control of Large Scale Uncertain Systems," MIT Ph.D. thesis, Cambridge, MA, 1977.

[4] Sira-Ramirez, H., "On The Robust Control of Systems to Set-Valued Objectives," 3d Brazilian Congress of Aut. Contr., Rio de Janeiro, Sept., 1980.

[5] Schweppe, F., Uncertain Dynamic Systems, Prentice Hall, Inc., New Jersey, 1973.

[6] Delfour, M., and Mitter, S., "Reachability of Perturbed Systems and Min-Sup Problems," SIAM Journal on Control, Vol. 7, No. 4; pp. 521-533, 1969.

[7] Schlaepfer, F., and Schweppe, F., "Continuous-Time State Estimation Under Disturbances Bounded by Convex Sets," IEEE Transactions on Auto. Control, Vol. AC-17, No. 2, pp. 197-205, 1972.

[8] Hnyilicza, E., "A Set-Theoretic Approach to State Estimation," MS. Thesis, MIT, Cambridge, Mass., 1967.

[9] Sira-Ramirez, H., "Evolution of Fuzzy Sets in Linear Dyanmic Systems," Proceedings of the International Conference on Cybernetics and Society, pp. 807-812. Denver, Colorado, October 1979.

[10] Sira-Ramirez, H., "Fuzzy State Estimation in Linear Dynamic Systems," Proceedings of the 19-th IEEE Conference on Decision and Control, Vol. 1, pp. 380-382, Albuquerque, New Mexico, December 1980.