# A BILINEAR OBSERVER APPROACH TO A CLASS OF NONLINEAR STATE RECONSTRUCTION PROBLEMS

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Abstract This paper considers an observer design problem for linear, time-invariant plants with nonlinear output equations. The nonlinearity of the output map is describable by a linear transformation of a " family of tensor powers " of the state vector ( Brockett,1973 ; Sira, 1979 ). The problem is transformed into an equivalent bilinear system observer design for which a number of results already exist ( Derese and Noldus, 1980,1981 ; Funahashi, 1979 ; Hara and Furuta, 1976; Kou, Elliot and Tarn, 1975 ). We give necessary and sufficiency conditions for the existence of an asymptotic state observer reconstructing the associated family of tensor powers trajectory. This observer easily allows the estimation of the original systems state. A simple example is presented to illustrate our method. Suggestions for further research are included at the end of the article.

Keywords. Nonlinear Systems; observers; bilinear observers; state estimation; Tensor powers; Lie Algebra.

## INTRODUCTION

A complete body of theory and applications of deterministic state observers has been structured over the years following Luenberger's original work in the sixties ( Luenberger, 1964, 1966 ). The state reconstruction problems proposed and solved, by a multitude of authors, include continuous or discrete, time varying or invariant linear dynamic systems, delay differential systems, bilinear and nonlinear dynamics and infinite dimensional systems, just to mention but a few. The techniques underlying these studies range in a wide spectrum including, among others, frequency domain methods, state space, geometric theory and optimization. For a detailed review of the vast amount of literature and results that exist in this important area, see the recent monograph ( O'Reilly, 1983 ).

An important class of problems arise when one considers linear dynamic plants with output maps which exert a nonlinear transformation on the state vector and a state reconstruction is to be performed on the basis of perfect output knowledge. This problem is frequently present in practical situations where output measurements include saturation and other physical limitations of a nonlinear nature. A state reconstruction procedure which takes this nonlinearity into account leads, generally, to a nonlinear state observer design problem ( Kou, Elliot and Tarn , 1975 ). Such an approach, aside from theoretical difficulties involved, presents a major drawback due to hardware and digital computer routines limitations ( Zeits, 1979 ).

For a large class of nonlinear output maps and linear state dynamics, there exists the possibility of building a bilinear state observer which asymptotically reconstructs the systems trajectory. The bilinear observer dimension is directly related to the nonlinearity of the output map through the degree of an associated "homogeneous tensor power" of the state vector which describes, after a linear transformation, the output map in an exact fashion. We thus define a high-dimensional equivalent bilinear state reconstruction problem for which a significant amount of results already exist.

In section II we present a number of results related to homogeneous tensor powers, and their associated "families", defined on vectors and matrices. We closely follow Brockett (1973) and Sira (1979) for this part. We also show how to obtain a bilinear system with linear outputs which, from the input-output viewpoint, is equivalent to the original linear system with nonlinear output equations. This equivalence is achieved by describing the dynamics obeyed by an appropiate family of tensor powers of the linear systems state. The bilinear system thus obtained is actually a tensor power of the, so called, "Myhill machine" associated with the original linear system. The structure matrices of this Myhill machine form a Myhill algebra ( Krishnaprasad, 1980 ) with a very special structure. We study and conclude the solvability of the Myhill algebra associated with the structure matrices of the machine. The relations of this algebra to the possibility of upper-triangularization of the structural matrices describing the family of tensor powers evolution equations, is shown to be of particular importance for the implementation of our method. The proof of the solvability of the Myhill algebra is a straightforward application of Schur's theorem (Bellman, 1970).

In section III we formulate our main problem, and following Kou, Elliot and Tarn (1975) we take a Lyapunov approach to establish necessary and sufficiency conditions for the asymptotic stability of the bilinear system describing the error dynamics. Some of these conditions translate into observability requirements on the part of the system generating the family of tensor powers of the state vector and directly involving the original output sub-matrices.

In section IV we present a pair of simple examples. We take the harmonic oscillator with nonlinear output equations and test the validity of the proposed conditions.

Section V presents some suggestions for further research in this area.

## II NOTATION DEFINITIONS AND BACKGROUND RESULTS

In this section we give some definitions, closely following Brockett (1973) and Sira (1979), about tensor powers of vectors and matrices. Also, some facts about the infinitesimal versions of these maps.

If $\underline{x}$ is an n vector with components $x_1, x_2, \ldots, x_n$ we denote $\underline{x}^{[p]}$ the $\binom{n+p-1}{p}$-dimensional vector of homogeneous p-forms in the components of $\underline{x}$. By convention we set $\underline{x}^{[0]} = 1$. The elements of the vector $\underline{x}^{[p]}$ are of the form : $\alpha_p \prod_{i=1}^{m} x_i^{p_i}$ with $\Sigma p_i = p$, $p_i \geq 0$. We define:

$$\alpha_p^2 = \binom{p}{p_1}\binom{p-p_1}{p_2} \ldots \binom{p-p_1-p_2-\ldots-p_{n-1}}{p_n} \quad (1)$$

and $N(n,p) \triangleq \binom{n+p-1}{p}$. We shall often refer to this "power" of the vector $\underline{x}$ as the "p-th tensor power of $\underline{x}$".

If $\underline{y} = A\,\underline{x}$ then $\underline{y}^{[p]} = A^{[p]}\,\underline{x}^{[p]}$ is verified and $A^{[p]}$ is then properly called " the p-th tensor power of the matrix A ". We denote by $A_{[p]}$ the infinitesimal version of the above power, i.e if $\underline{x}$ satisfies the differential equation $d/dt\,\underline{x} = A\,\underline{x}$ then $d/dt\,\underline{x}^{[p]} = A_{[p]}\,\underline{x}^{[p]}$.

Some useful properties of tensor powers for matrices and for its infinitesimal versions are :

$$\begin{array}{ll} 1)\ (AB)^{[p]} = A^{[p]}B^{[p]} & 4)\ (A+B)_{[p]} = A_{[p]}+B_{[p]} \\ 2)\ (A^q)^{[p]} = (A^{[p]})^q & 5)\ (qA)_{[p]} = q\,A_{[p]} \\ 3)\ (A')^{[p]} = (A^{[p]})' & 6)\ (A')_{[p]} = (A_{[p]})' \end{array} \quad (2)$$

We extend the definitions of p-th tensor powers for vectors and matrices, by considering vectors wich are constituted by an ordered arrangement of increasing tensor powers of the vector $\underline{x}$. For this we take the p-th power of an augmented vector whose first component is 1 and the rest of the components are those of $\underline{x}$. We denote this vector by $\tilde{\underline{x}} = \begin{bmatrix}1\\ x\end{bmatrix}$. then $\tilde{\underline{x}}^{[p]} = [1, \underline{x}', (\underline{x}^{[2]})', \ldots, (\underline{x}^{[p]})']'$. We shall call this vector the "p-th family of powers of $\underline{x}$". The dimension of $\tilde{\underline{x}}^{[p]}$ is $\binom{n+p}{p} \triangleq \tilde{N}(n,p)$

By direct use of the previous definitions, it follows easily that if $\underline{y} = A\,\underline{x}$ then $\tilde{\underline{y}} = \tilde{A}\,\tilde{\underline{x}}$ and also $\tilde{\underline{y}}^{[p]} = \tilde{A}^{[p]}\,\tilde{\underline{x}}^{[p]}$, where $\tilde{A}^{[p]}$ is a block diagonal matrix of the form $\tilde{A}^{[p]} = \mathrm{diag}[1, A, A^{[2]}, \ldots, A^{[p]}]$. The infinitesimal version of $\tilde{A}^{[p]}$ is denoted by $\tilde{A}_{[p]} = \mathrm{diag}[0, A, A_{[2]}, \ldots, A_{[p]}]$. The properties given in (2) above for tensor powers of matrices easily extend to families of tensor powers of matrices and the same holds true for the infinitesimal versions of such powers.

Lemma 1 . If A is uppertriangular, then $A_{[p]}$ and $\tilde{A}_{[p]}$ are also uppertriangular.

Proof This fact is a direct consequence of the lexicographical order imposed on the vector components, and each component set-up, of a tensor power of an n-dimensional vector. The details are left for the reader.

Lemma 2 The eigenvalues of $A_{[p]}$ are constituted by the set of all N(n,p) formally distinct sums of p eigenvalues of A. Thus, if A is Hurwitz (i.e all its eigenvalues have negative real parts ) then $A_{[p]}$ and $\tilde{A}_{[p]}$ are also Hurwitz.

Proof The first part of the lemma is easily established either by diagonalization or Jordan form reduction of the base matrix A. The second part of the lemma is a straightforward consequence of the first part.

Lemma 3 Let A be an nxn matrix and T a non-singular matrix, then :

$$(TAT^{-1})_{[p]} = (T^{[p]})\,A_{[p]}\,(T^{[p]})^{-1} \quad (3)$$

Proof Let $d/dt\,\underline{x} = A\,\underline{x}$, then $d/dt\,\underline{x}^{[p]} = A_{[p]}\underline{x}^{[p]}$ if $\underline{z} = T\,\underline{x}$, then $d/dt\,\underline{z} = TAT^{-1}\,\underline{z}$ and $\underline{z}^{[p]} = T^{[p]}\underline{x}^{[p]}$. Therefore $d/dt\,\underline{z}^{[p]} = (TAT^{-1})_{[p]}\,\underline{z}^{[p]}$. On the other hand we have: $d/dt\,\underline{z}^{[p]} = T^{[p]}d/dt\,\underline{x}^{[p]} = T^{[p]}A_{[p]}\underline{x}^{[p]} = T^{[p]}A_{[p]}(T^{[p]})^{-1}\,\underline{z}^{[p]}$. The result follows.

As a consequence of this lemma we obtain a corresponding formula for the family of tensor powers

$$(T\,\tilde{A}\,T^{-1})_{[p]} = (\tilde{T}^{[p]})\,\tilde{A}_{[p]}(\tilde{T}^{[p]})^{-1} \quad (4)$$

The proof of this fact is left for the reader.

Lemma 4 Let $\underline{b}$ be an n-dimensional column vector. Denote by $\bar{B}$ the matrix :

$$\bar{B} = \begin{vmatrix} 0 & \underline{0}_{1xn} \\ \underline{b} & 0_{nxn} \end{vmatrix} \quad (4)$$

Then the matrix $\bar{B}_{[p]}$ has its nonzero entries in blocks immediately below the main zero diagonal blocks according with the following structure:

$$\bar{B}_{[p]} = \begin{vmatrix} 0 & \underline{0}_{1xn} & \underline{0}_{1xN(n,2)} & \cdots & \underline{0}_{1xN(n,p)} \\ \tilde{\underline{b}}_{(1)} & [0_{nxn}]_{[1]} & \underline{0}_{nxN(n,2)} & \cdots & \underline{0}_{nxN(n,p)} \\ \underline{0}_{nx1} & \tilde{\underline{b}}_{(2)} & [0_{nxn}]_{[2]} & \cdots & \vdots \\ \vdots & \vdots & \vdots & & \\ \underline{0}_{N(n,p-1)x1} & \cdots & & \cdots [0_{nxn}]_{[p-1]} & \vdots \\ \underline{0}_{N(n,p)x1} & \cdots & & \cdots\ \tilde{\underline{b}}_{(p)} & [0_{nxn}]_{[p]} \end{vmatrix}$$

where $\tilde{\underline{b}}_{(k)}$ is a N(n,k)xN(n,k-1) matrix with $\tilde{\underline{b}}_{(1)} = \underline{b}$. The map $\underline{b} \to \tilde{\underline{b}}_{(k)}$ is linear and $[0_{nxn}]_{[k]}$ is a N(n,k) x N(n,k) matrix.

Lemma 5 Let $B = [\underline{b}_1, \underline{b}_2, \ldots, \underline{b}_m]$ be an nxm matrix. Suppose $\underline{x}$ satisfies the linear vector differential equation :

$$\frac{d}{dt}\,\underline{x} = A\,\underline{x} + B\,\underline{u} \quad (6)$$

then $\tilde{\underline{x}}^{[p]}$ evolves according to the bilinear dynamics :

$$\frac{d}{dt}\,\tilde{\underline{x}}^{[p]} = (\tilde{A}_{[p]} + \textstyle\sum_{i=1}^{m} u_i\,\bar{B}_{i[p]})\,\tilde{\underline{x}}^{[p]} \quad (7)$$

where the $u_i$'s are the components of $\underline{u} \in R^m$ ; $\tilde{A}_{[p]}$ is as defined before and $\bar{B}_{i[p]}$ has the structure given in Lemma 4.

Proof The proof is based on the simple fact that (6) can be written as a bilinear system with a state given by the composite vector $\tilde{\underline{x}}$. This vector evolves subject to :

$$\frac{d}{dt}\,\tilde{\underline{x}} = [\,\tilde{A} + \textstyle\sum_{i=1}^{m} u_i\bar{B}_i\,]\,\tilde{\underline{x}} \quad (8)$$

The result follows immediately after using property 4) of (2).

As a corollary to the previous lemma, it follows for k=0,1,..., etc :

$$\frac{d}{dt} \underline{x}^{[k]} = A_{[k]}\underline{x}^{[k]} + \sum_{i=1}^{m} u_i \underline{\tilde{b}}_{i(k)} \underline{x}^{[k-1]} \quad (9)$$

We now present a result related to the solvability of the Lie algebra generated by the structural matrices of system (8). By a well known result ( Sagle, 1973 ) this also proves the solvability of the Lie algebra generated by the structural matrices of the system (7).

In (8) it is easily verified that $[\bar{B}_i,\bar{B}_j] = 0$ (the bracket operation [ , ] stands for Lie bracket operation ) for all i's and j's. On the other hand we have :

$$[\tilde{A},\bar{B}_i] = \tilde{A}\,\bar{B}_i - \bar{B}_i\,\tilde{A} = \begin{vmatrix} 0 & \underline{0}_{1xn} \\ A^k\underline{b}_i & 0_{nxn} \end{vmatrix} \quad (10)$$

If we denote by $\mathcal{B} = \{\underline{b}_1,\underline{b}_2,\dots,\underline{b}_m\}_{LS}$ ( the linear span of the $\underline{b}_i$'s ) then the Lie algebra generated by the matrices $\{\tilde{A},\bar{B}_1,\dots,\bar{B}_m\}$ and denoted by $\{\tilde{A},\bar{B}_1,\dots,\bar{B}_m\}_{LA} = \tilde{L}$ , is characterized by :

$$\tilde{L} = \{\begin{vmatrix} 0 & \underline{0}_{1xn} \\ A^k\,\underline{v} & A \end{vmatrix} ;\ \underline{v} \in \mathcal{B},\ \alpha \in R ;\ k=0,1,\dots,n-1\} \quad (11)$$

Proposition 1 The Lie algebra $\tilde{L}$ is solvable.

We can prove this proposition by either computing the derived series ( See Sagle *op. cit.* ) or else using a well known result about the possibility of upper-triangularizing simultaneously all the elements in the Lie algebra by use of a single similarity transformation ( Willsky, 1975 ). Schur's theorem ( Bellman, 1970 ) ensures , for this case, the existence of such transformation thus proving $\tilde{L}$ is solvable. The first route leads one to the conclusion that the second derivative of the algebra is zero thus proving solvability. We present in detail the second approach.

Let $\tilde{\underline{z}}^{-} = \tilde{T}\,\tilde{\underline{x}}$ with $\tilde{T}$ being :

$$\tilde{T} = \begin{vmatrix} 0_{nx1} & T \\ 1 & 0_{1xn} \end{vmatrix} \quad \text{and} \quad \tilde{\underline{z}}^{-} = [\begin{smallmatrix} \underline{z} \\ 1 \end{smallmatrix}] \quad (12)$$

and T a unitary matrix such that $TAT^{-1}$ is upper-triangular. Schur's theorem guarantees that T always exists. Applying the similarity transformation $\tilde{T}$ on $\tilde{L}$ we have :

$$\tilde{T}\tilde{L}\tilde{T}^{-1} = \{\begin{vmatrix} \alpha TAT^{-1} & TA^k\underline{v} \\ \underline{0}_{1xn} & 0 \end{vmatrix} ;\ v \in \mathcal{B},\ \alpha \in R ;\ k = 0,1,2,\dots, n-1\} \quad (13)$$

In particular, the matrices $\tilde{A}$ and $\bar{B}_i$ $\forall i$ , are upper-triangularized by $\tilde{T}$.

If we denote by $\tilde{L}_{[p]}$ the Lie algebra generated by $\{\tilde{A}_{[p]},\bar{B}_{1[p]},\dots,\bar{B}_{m[p]}\}$ then it is a well known fact that $\tilde{L}_{[p]}$ is also solvable. This result easily follows by taking the p-th tensor power in equation (12). Without loss of generality, one can say that the system (7) is constituted by structural matrices of uppertriangular nature. This has two main implications 1) the differential equations (7) and (8) have a global solution ( as it was expected ) expressible in terms of products of exponentials ( See: Wei and Norman ; 1963,1964 ) and 2) the solution to both (7) and (8) can be written in terms of integrals.

Remark The Lie algebra $\tilde{L}$ is also known as the Myhill algebra of the linear system $\dot{\underline{x}} = A\underline{x} + B\underline{u}$ (See : Krishnaprasad; 1980 ). This algebra is a matrix Lie sub-algebra of the general linear group $g\ell(n+1,R)$, spanned by the type of matrices in (11).

## III PROBLEM FORMULATION AND MAIN RESULTS

In this section we shall consider the state reconstruction problem defined on a linear system :

$$\frac{d}{dt}\underline{x} = A\,\underline{x} + B\,\underline{u} \quad (14)$$

$\underline{x} \in R^n$, $\underline{u} \in R^m$, $B = [\underline{b}_1,\underline{b}_2,\dots,\underline{b}_m]$ and a nonlinear output equation of the form :

$$\underline{y} = g(\underline{x}) = \sum_{i=0}^{p} G_i\underline{x}^{[i]} = [G_0,G_1,\dots,G_p]\tilde{\underline{x}}^{[p]} \triangleq \tilde{G}_p\,\tilde{\underline{x}}^{[p]} \quad (15)$$

$\tilde{G}_p$ is a q x $\tilde{N}(n,p)$ matrix of rank q > n. $G_k$ is a q x N(n,k) matrix. We wish to build an observer for the system (14)-(15) which asymptotically reconstructs the state $\underline{x}$ .

Using the results of the previous section, we can reduce the above nonlinear problem to an equivalent bilinear observer design problem defined on the dynamic system describing the evolution of the vector $\tilde{\underline{x}}^{[p]}$.

It is easy to see that from an input-output viewpoint system (14)-(15) is totally equivalent to the system defined by (7) with output equation given by (15). Therefore, the problem of estimating $\underline{x}$ in (14)-(15) is equivalent to that of estimating $\tilde{\underline{x}}^{[p]}$ from the system (7)-(15). Regardless of the highly nonlinear interdependency among the components of $\tilde{\underline{x}}^{[p]}$, the state reconstruction problem of our original system with nonlinear output equations is also equivalent to an observer design problem for the following bilinear system in which the initial state is a p-th family tensor power of some vector in $R^n$.

$$\frac{d}{dt}\underline{n} = ( \tilde{A}_{[p]} + \sum_{i=1}^{m} u_i\bar{B}_{i[p]} )\underline{n} \quad (16)$$

$$\underline{y} = \tilde{G}_p\,\underline{n} \quad (17)$$

where $\underline{n}$ is an $\tilde{N}(n,p)$-dimensional state vector which would reproduce $\tilde{\underline{x}}^{[p]}$ if the initial states were chosen to be equal. If we build an asymptotic observer for system (16)-(17),then, obviously, the second to the (n+1)st component of the estimate of $\underline{n}$ ( which we denote as $\hat{\underline{n}}$ ) will asymtotically reconstruct $\underline{x}$. We remark that an initial state for (16) can not be free since its components have to be those of a family of tensor powers of some n-dimensional vector. Also, a similarity transformation on (16) is necessarily restricted to matrices of the form diag$[1,T,\dots,T^{[p]}]$ with $\underline{n} = \tilde{T}^{[p]}\,\underline{\zeta}$ . This fact impeeds one to assume that $\tilde{G}_p$ could be of the form $[0,I_{qxq},0,\dots,0]$ i.e the results commonly found in the literature cannot be directly applied. ( See Derese and Noldus, 1980,1981; Funahashi, 1979; Hara and Furuta, 1976 ; Kou, Elliot and Tarn, 1975 ).

Let $H_0,H_1,\dots,H_m$ be design matrices and define an estimate $\hat{\underline{n}}$ of $\underline{n}$ by means of :

$$\frac{d}{dt}\hat{\underline{n}} = (\tilde{A}_{[p]} + \sum_{i=1}^{m}u_i\bar{B}_{i[p]})\hat{\underline{n}} + H_0(\underline{y}-\hat{\underline{y}}) + \sum_{i=1}^{m}u_iH_i(\underline{y}-\hat{\underline{y}}) \quad (18)$$

$$\underline{\hat{y}} = \tilde{G}_p \, \underline{\hat{n}} \tag{19}$$

Defining the estimation error as : $e = \underline{n} - \underline{\hat{n}}$, the error dynamics is, then, governed by the bilinear system :

$$\frac{d}{dt}\underline{e} = \{[\tilde{A}_{[p]} - H_0\tilde{G}_p] + \sum_{i=1}^{m} u_i [\overline{B}_{i[p]} - H_i\tilde{G}_p]\}\underline{e} \tag{20}$$

As it is easily accepted, the estimation error should be independent of inputs and initial states. Following Derese and Noldus (1980), we take a Lyapunov approach which allows for an observer design whose error signal stability is independent of the input signals and initial states. We summarize the result in the following proposition:

Proposition 2 (Derese and Noldus, 1980). The state observer (18)-(19) is an asymptotic observer for (16)-(17) ( i.e (14)-(15)) if and only if there exists, for any positive definite matrix $Q$ ($\tilde{N}(n,p) \times \tilde{N}(n,p)$), a positive definite matrix $P$ ($\tilde{N}(n,p) \times \tilde{N}(n,p)$) such that :

$$\mathcal{P}\,(\mathcal{A} - \mathcal{H}\mathcal{G}) + (\mathcal{A} - \mathcal{H}\mathcal{G})'\,\mathcal{P} = -\mathcal{Q} \tag{21}$$

with:

$$\mathcal{Q} = \mathrm{diag}[Q,0,0,\ldots,0],\; Q=Q' \;;\quad \mathcal{H} = \mathrm{diag}[H_0,H_1,\ldots,H_m]$$
$$\mathcal{A} = \mathrm{diag}[\tilde{A}_{[p]},\overline{B}_{1[p]},\ldots,\overline{B}_{m[p]}];\; \mathcal{G} = \mathrm{diag}[\tilde{G}_p,\tilde{G}_p,\ldots,\tilde{G}_p]$$
$$\mathcal{P} = \mathrm{diag}[P,P,\ldots,P] \;;\; P = P' \tag{22}$$

Proof The proof amounts to investigate the stability of the estimation error trajectory by choosing a quadratic Lyapunov function $V(\underline{e}) = \underline{e}'P\underline{e}$ with $P=P' > 0$. Differentiating one gets:

$$\frac{d}{dt}V(\underline{e}) = \underline{e}'[P(\tilde{A}_{[p]} - H_0\tilde{G}_p) + (\tilde{A}_{[p]} - H_0\tilde{G}_p)'P]\underline{e} +$$
$$\underline{e}'\{\sum_{i=1}^{m} u_i [P(\overline{B}_{i[p]} - H_i\tilde{G}_p) + (\overline{B}_{i[p]} - H_i\tilde{G}_p)'P]\}\underline{e}$$

The asymptotic stability of the error dynamics is guaranteed if and only if there exists a $P = P' > 0$ such that for any $Q = Q' > 0$ one has :

$$P(\tilde{A}_{[p]} - H_0\tilde{G}_p) + (\tilde{A}_{[p]} - H_0\tilde{G}_p)'P = -Q \tag{23}$$

and

$$P(\overline{B}_{i[p]} - H_i\tilde{G}_p) + (\overline{B}_{i[p]} - H_i\tilde{G}_p)'P = 0 \;; \quad i=1,2,\ldots,m \tag{24}$$

Using the definitions in (22) this set of conditions is sinthesized in (21).

Remark If the pair $(\tilde{A}_{[p]},\tilde{G}_p)$ is observable, then condition (23) is automatically satisfied. This requirement can also be weakened to have $(\tilde{A}_{[p]},\tilde{G}_p)$ detectable ( See: Wohnam (1979) ).

Proposition 3 $(A_{[k]},G_k)$ is observable for all k if the pair $(\tilde{A}_{[p]},\tilde{G}_p)$ is observable.( For k=0, the condition means $G_0 \neq 0$ ).

Proof Suppose $(\tilde{A}_{[p]},\tilde{G}_p)$ is observable. Then, the matrix given in (25) is full rank.

This matrix has dimensions $\tilde{N}(n,p) \times q\tilde{N}(n,p)$, thus its rank is necessarily $\tilde{N}(n,p)$. Each block of columns representing the observability matrices for the pairs $(A_{[k]},G_k)$ must therefore be also full rank. This proves the sufficiency of the proposition. We note,however, that the condition is not necessary, i.e. each and every pair $(A_{[k]},G_k)$ could be observable and yet the overall pair $(\tilde{A}_{[p]},\tilde{G}_p)$ may not be observable. In the next section we present a simple example where this fact is verified.

$$\tilde{\mathcal{O}} = \begin{vmatrix} G_0 & G_1 & G_2 & \cdots & G_p \\ 0 & G_1A & G_2A_{[2]} & & G_pA_{[p]} \\ 0 & G_1A^2 & G_2A^2_{[2]} & & G_pA^2_{[p]} \\ \vdots & \vdots & \vdots & & \vdots \\ \vdots & G_1A^{n-1} & \vdots & & \vdots \\ \vdots & * & G_2A^{N(n,2)-1}_{[2]} & & \vdots \\ \vdots & * & * & & G_pA^{N(n,p)-1}_{[p]} \\ \vdots & * & * & & * \end{vmatrix} \tag{25}$$

Thus, if the overall bilinear system (7) is observable then each subsystem constituting it is also observable. The observability of each subsystem represented by (9) with output equation given by : $\underline{y}_k = G_k\,\underline{x}^{[k]}$ does not imply full observability of the system (7).

## IV EXAMPLES

Example 1 We consider now a simple example constituted by a controlled harmonic oscillator with state equations:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 + u \tag{26}$$

and nonlinear output equation of the form:

$$\underline{y} = 1 + x_1 + a\,x_2^2 \tag{27}$$

As we have shown, an observer design for this nonlinear problem is equivalent to designing an observer for the system:

$$\frac{d}{dt}\begin{vmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \\ n_5 \\ n_6 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & -\sqrt{2} & 0 & \sqrt{2} \\ 0 & 0 & 0 & 0 & -\sqrt{2} & 0 \end{vmatrix} \begin{vmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \\ n_5 \\ n_6 \end{vmatrix}$$

$$+ \; u \begin{vmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \end{vmatrix} \begin{vmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \\ n_5 \\ n_6 \end{vmatrix} \tag{28}$$

$$\underline{y} = [\,1 \;\; 1 \;\; 0 \;\; 0 \;\; 0 \;\; a\,]\,\underline{n} \tag{29}$$

The observability matrix for the pair $(\tilde{A}_{[2]},\tilde{G}_2)$ is given by:

$$\tilde{\mathcal{O}}_a = \begin{vmatrix} 1 & 1 & 0 & 0 & 0 & a \\ 0 & 0 & 1 & 0 & -\sqrt{2} & 0 \\ 0 & -1 & 0 & 2a & 0 & -2a \\ 0 & 0 & -1 & 0 & 4\sqrt{2}a & 0 \\ 0 & 1 & 0 & -8a & 0 & 8a \\ 0 & 0 & 1 & 0 & -16\sqrt{2}a & 0 \end{vmatrix} \tag{30}$$

In this matrix, every pair $(A_{[k]},G_k)$ ; k=0,1,2. is clearly observable and yet the overall matrix is not full rank. Thus, the overall pair $(\tilde{A}_{[2]},\tilde{G}_2)$ is not even stabilizable i.e one of the necessary conditions for the solution to the problem to exist is not fulfilled and an asymptotic observer does not exist. Furthermore, it can be shown that, in this example, for any single output full second order multinomial in $x_1,x_2$ , the resulting pair $(\tilde{A}_{[2]},\tilde{G}_2)$ is unobservable. If we addition, nevertheless, a linearly independent output to (29) we may obtain an observable system as the following example shows.

Example 2 Consider the previous example with output equations :

$$y_1 = 1 + x_1 + x_2^2 \tag{31}$$

$$y_2 = 1 + \sqrt{2}\, x_1 x_2 \tag{32}$$

i.e in this case we have :

$$\tilde{G}_2 = \begin{vmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{vmatrix} \tag{33}$$

The observability matrix for the pair $(A_{[2]},G_2)$ is, in this case, full rank as it can be easily verified.

$$\tilde{O} = \begin{vmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 & 0 & 2 \\ 0 & -1 & 0 & 2 & 0 & -2 \\ 0 & 0 & 0 & 0 & -4 & 0 \\ * & * & * & * & * & * \\ * & * & * & * & * & * \end{vmatrix} \tag{34}$$

In the full observability matrix (34), we have isolated the blocks that correspond to each of the subsystems generating the tensor powers that constitute the family of tensor powers of the state, of appropiate dimension, as to generate the equivalent bilinear system defined above in (28). It is also immediate to verify that each subsystem is observable in the sense that the pairs $(A_{[k]},G_k)$ are observable for all k.

The asterisks in the observability matrix correspond to elements, not shown, which complete the matrix to its 12th row.

According to well established results, there exists a matrix $H_0$ such that $\tilde{A}_{[2]}-H_0\tilde{G}_2$ is stable and all its eigenvalues can be chosen at will within the complex left half plane, modulo symmetry with respect to the real axis. This means that there exists a unique solution for equation (23) which is also positive definite for each positive definite matrix Q.

It is easy to see that a necessary condition for the existence of a non trivial solution to equation (24) is that $(\bar{B}_{i[2]}-H_i\tilde{G}_2)$ ( i=1 ) has either a zero eigenvalue or at least a pair of opossing eigenvalues. In our example the matrix in question has a zero eigenvalue for any value of $H_1$ thus fulfilling this requirement.

## CONCLUSIONS AND SUGGESTIONS FOR FURTHER RESEARCH

In this article we have examined a class of nonlinear state reconstruction problems defined on a linear dynamic plant with nonlinear multinomial outputs describable by appropiate state tensor power equations. An observer is proposed by viewing the plant and measurement device set-up as an equivalent bilinear dynamic system describing the state tensor family of powers evolution and a linear output equation defined on such "state vector". This problem is solved by already well known results on bilinear observer theory.

The equivalent problem is, generally, of a much higher dimension than that of the original plant. This dimension is precisely determined by the "degree" of the nonlinearity present in the output map.

The advantage of our approach lies in the fact that a bilinear state reconstruction problem requires more readily available hardware for analog computer implementation, than the required by a nonlinear problem.

Conditions for the existence of an asymptotic observer are of the algebraic type, represented by Lyapunov equations whose solution can be obtained by standard computer routines extensively available.

The ideas and results developed by Loparo and Blankenship (1978) allows one to extend the results of this article to more general situations. In particular, output maps of the analytic type can be conveniently approximated to any desired degree of precision by use of series expansions in terms of a sufficiently "large" family of tensor powers. This would permit approximate asymptotic state reconstruction for rather general classes of nonlinear output maps. The unavoidable estimation error for these kind of problems could even be precomputed and trade-offs established with the dimension factor present in our approach.

Following the ideas in Loparo and Blankenship (1978), for an unrelated problem it is straightforward to extend our results to the case of, so called, linear analytic systems ( i.e systems of the form :

$$\dot{\underline{x}} = \underline{f}(\underline{x}) + \sum_{i=1}^{m} u_i \underline{g}_i(\underline{x})$$

$$\underline{y} = \underline{h}(\underline{x})$$

with $\underline{f}$, $\underline{g}_i$ ( i=1,2,...,m ) and $\underline{h}$ analytic nonlinear maps ). The defree of approximation for the model whose state is to be reconstructed has been fully considered by those authors.

## REFERENCES


Bellman, R. (1970). *Introduction to Matrix Analysis*. McGraw Hill, New York.

Brockett, R. W. (1973). Lie algebras and Lie groups in control theory. In R.W. Brockett and D.Q. Mayne (Eds.) *Geometric Methods in Systems Theory*. Reidel, The Netherlands.

Brockett, R. W. (1976). Nonlinear systems and differential geometry. *Proc. IEEE.*, 64,61-72.

Derese, J., and E. J. Noldus (1980). Nonlinear control of bilinear systems. *Proc. IEE.*, 127, Part D, No.4.34-49.

Derese, J., and E. J. Noldus (1981). Existence of bilinear state observers for bilinear systems. *IEEE Trans. on Auto. Contr.* AC-26, 590-592.

Funahashi, Y. (1979) Stable state estimator for bilinear systems. *Int. J. on Contr.* 29,181-188.

Hara,S., and K. Furuta (1976). Minimal order state observers for bilinear systems. *Int. J. on Contr.* 24, 705-718.

Kou, S. R., D.L. Elliot, and T.J. Tarn (1975).Exponential observers for nonlinear dynamic systems. *Info. and Contr.* 29, 204-216.

Krishnaprasad, P.S. (1980). Deformation of Lie algebras and the Wei-Norman equations. *Proc. 19th IEEE C.D.C.* 1, 661-662.
Loparo, K., and G. Blankenship (1978). Estimating the domain of attraction of nonlinear feedback systems. *IEEE Trans. on Auto. Contr.* AC-23, 602-608.
Luenberger, D. G. (1964). Observing the state of a linear system. *IEEE Trans. on Mil. Electr.* MIL-8, 74-80.
Luenberger, D. G. (1966). Observers for multivariable systems. *IEEE Trans. on Auto. Contr.* AC-11, 190-197.
O'Reilly, J. (1983). *Observers for Linear Systems.* Academic Press, London.
Sagle, A. A., and R.E. Walde (1973). *Introduction to Lie Groups and Lie Algebras.* Academic Press, New York.
Sira-Ramirez, H. (1979). Determining the feasible region for non-stationary nonlinear feedback systems. *Proc. XVII Allerton Conf.* 1, 571-577.
Wei, J., and E. Norman (1963). Lie algebraic solution to linear differential equations. *J. of Math. Phys.* 4, 575-581.
Wei, J., and E. Norman (1964). On global representations of the solutions of linear differential equations as product of exponentials. *Proc. Am. Math. Soc.* 15, 327-334.
Willsky, S.A., S.I. Marcus and D.N. Martin (1975). On the stochastic stability of linear systems containing colored multiplicative noise. *IEEE Trans. on Auto. Contr.* AC-20, 711-713.
Wohnam, M.W. (1979). *Linear Multivariable Control.* 2nd ed. Springer Verlag, New York.
Wong, W.S. (1980). Nonlinear systems invariants with applications to identification. *Proc. 19th. IEEE C.D.C.* 1, 654-660.
Zeits, M. (1979) Nichtlineare Beobachter. *Regelunstech.* 27, 241-249.