TOWARDS A FUZZY GEOMETRIC THEORY OF LINEAR MULTIVARIABLE CONTROL

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Abstract This article develops a generalization of Wohnam's Geometric Theory of Linear Multivariable Control (Wohnam, 1979) by introducing a *fuzzy subspace* formulation into the conventional theory. This generalization allows for treatment, within a convenient mathematical framework, of inescapable issues in systems analysis and design such as : vagueness or imprecision in control restrictions, disturbance description and design objectives. We furnish an appropiate fuzzy geometric basis for the restatement and solution development of problems treated by the conventional Geometric theory for the control of linear multivariable systems.

Keywords. Linear systems ; multivariable control systems, fuzzy set theory, linear algebra, modelling, system theory, decoupling.

I INTRODUCTION

The purpose of this article is to introduce and develop the fuzzy set theoretic context within the Geometric theory of linear multivariable control introduced by Wohnam (Wohnam, 1979).

At the very heart of this generalization lies the concept of fuzzy subspace of a real vector space. Its introduction allows for the convenient and mathematically tractable modeling of: subjective or vague control restrictions, imprecise disturbance knowledge and fuzzy design objectives. Statements and requirements expressing a subjective knowledge of systems variables and design specifications can be adequately modeled by this technique in a way that enlarges the solution possibilities even from a practical viewpoint. This power is not sheared by the conventional "crisp" theory.

It is noteworthy that fuzzy set theory, invented by a control theoretician (Zadeh, 1965),has rapidly evolved in directions out of the control systems discipline (Cognitive Processes, Grammars, Graph theory, Risk analysis, etc. See: Zadeh, Fu, Tanaka, and Shimura, 1975) while questions related to utilizing fuzzy sets or fuzzy subspaces as a modelling device for describing poorly defined variables, signals and objectives, have only been partially explored (Dubois and Parade, 1980; Sira, 1979,1980).

Geometric theory,on the other hand, has reached a state of maturity and rapid growth bridging, day after day, with old and recent areas within the control theory arsenal. Recent contributions, especially by Willems (1980,1981,1982), point out to the necessity of relaxing some of the basic concepts in Geometric theory so that some degree of freedom and approximative characteristics are introduced in the formulation of problems succesfully treated by the Geometric theory of Wohnam. Our viewpoint touches, tangentially, on this phylosophical necessity and coincides, although in a nontechnical manner, with the theory developed by Willems (*op. cit.*) of " Almost controlled invariant and almost conditionally invariant subspaces".

This article presents in its section II all the basic definitions and results related to the proposed generalization. We introduce there the definitions of: fuzzification of a linear vector space (subspace), fuzzy subspace, fuzzy images and fuzzy kernels, fuzzy invariant subspaces and fuzzy factor spaces. These definitions allow, in turn, for the corresponding generalization, to the fuzzy set theory context, of concepts such as : reachable subspaces,(A,B)-invariant (controlled invariant,or A-mod B invariant) subspaces, (A,C)-invariant (conditionally invariant, or A|Ker C-invariant)subspaces, unobservable subspaces and controllability subspaces. All these definitions, though, are based in the elementary fuzzy-set operations of vector addition of fuzzy subspaces, intersection of fuzzy subspaces etc. We also present formulae needed in the algorithms that produce fuzzy maximal invariant subspaces, maximal reachability and maximal unobservable fuzzy subspaces lying within a given fuzzy subspace.

In essence, all the variety of problems treated by the Geometric theory of linear multivariable control could be restated in terms of the fuzzy-geometric generalization. Nevertheless, we shall only formulate one such problem; the Fuzzy Disturbance Rejection Problem (FDRP), in section III of this work, as an example of the possibilities of the proposed generalization.

All the background results and material about fuzzy sets, as needed in this work, are collected at the end of the article in an appendix. The article also presents some conclusions and suggestions for further research in this area.

II NOTATION AND BASIC DEFINITIONS

In this section we introduce the basic definitions needed throughout the entire article. These definitions are centered around the idea of *fuzzy* subspaces. We introduce this concept by simply adscribing a *membership function* (m.f.) defined on the *support subspace*. This concept joined to all known elementary operations on fuzzy sets permits the "reconstruction" of the Geometric theory within a fuzzy set theoretic context.

Definition 1 A *fuzzy vector space* (subspace) is defined as a pair $(\mathcal{U},\mu)$ usually denoted by $\mathcal{U}_\mu$, where $\mathcal{U}$ is a vector space (subspace) and μ is a membership function (m.f) (i.e, a map $U \to [0,1]$) defined on U. We shall refer to U as the *support space* (subspace) of U_μ. Simbolically, we express this fuzzy space (subspace) by means of Zadeh's notation as:

$$U_\mu = \int_U \mu(\underline{v})/\underline{v} \tag{1}$$

a ν-*fuzzification of* X is understood as the adscription of a m.f. ν to a subspace (or space) X. When confussion could develop, we denote a m.f. ν adscribed to a subspace X as ν_X

Definition 2 Let B be a linear map B: $U \to X$ and let U_μ be a fuzzy subspace defined by a μ-fuzzification of U. The *fuzzy image of* B or the μ-image of B in X is determined by :

$$B_\mu = \{\underline{x} \in X: \underline{x} = B\underline{u} \text{ for some } \underline{u} \in_f U\} \tag{2}$$

B_μ is then a fuzzy subspace whose support subspace is B and its m.f is inherited from the fuzzification of U. According to the *Extension Principle* (Zadeh, 1965) we have :

$$B_\mu = \int_B \mu(B^{-1}\underline{x})/\underline{x} \tag{3}$$

We shall often use the composition symbol to express the m.f inhertitance, through certain linear map, adscribed to an original space. In the preceeding case we would write :

$$\mu_B = \mu \circ B^{-1} \tag{4}$$

Definition 3 Let V_δ denote a δ-fuzzification of V by means of the m.f δ representing the qualitative statement: " V_δ is the set of vectors *very close* to zero in V " i.e. $\delta(\underline{0}) = 1$ while $\delta(y) = 0$ $\forall \underline{y} \neq \underline{0}$. We call V_δ a *fuzzy zero* and it will be usually denoted by $\underline{0}_\delta$.

Definition 4 Let C be a linear map C : $X \to V$, then the *fuzzy kernel of* C, denoted by $\mathrm{Ker}_\delta\, C$ or δ-ker C, is the fuzzy set :

$$\mathrm{Ker}_\delta\, C = \{\underline{x} : C\underline{x} \in_f \underline{0}_\delta\} \tag{5}$$

The m.f. of this fuzzy set is inherited from that of the fuzzy zero. By virtue, again of the extension principle (*loc.cit.*) we have:

$$\delta_{\mathrm{KerC}} = \delta \circ C \tag{6}$$

Definition 5 Let V be a fuzzy subspace with m.f. μ, we say V_μ is an A-invariant fuzzy subspace if:

$$A\, V_\mu \subset_f V_\mu \tag{7}$$

This condition implies two relations, one involving the support subspace and the other involving the m.f. :

$$AV \subset V \quad \text{and} \quad \mu_{AV} \leq \mu_V \text{ or } \mu_V \circ A^{-1} \leq \mu_V \tag{8}$$

Definition 6 Let R_μ and S_ν be fuzzy subspaces, we denote the fuzzy addition of thses subspaces by $R_\mu + S_\nu$ and define it as:

$$R_\mu + S_\nu = \{\underline{x}+\underline{y} : \underline{x} \in_f R_\mu \text{ and } \underline{y} \in_f S_\nu\} \tag{9}$$

where the sum is understood in a fuzzy set-theoretic way. Adopting the definition of fuzzy adition given in Sira-Ramirez (1979)(See also the Appendix) we have:

$$R_\mu + S_\nu = \int_{R+S} \sup_{\underline{v}}[\mu(\underline{u}-\underline{v}) \wedge \nu(\underline{v})]/\underline{u} \tag{10}$$

we shall express shortly the m.f. of this sum as:

$$\zeta_{R+S} = \mu_R * \nu_S \quad \text{or} \quad R_\mu + S_\nu = (R+S)_{\mu * \nu} \tag{11}$$

where the symbol * stands for *convolution* of the involved m.f's according to the formula (10) above.

Definition 7 Let V_μ and R_ν be two fuzzy subspaces, then the intersection of these subspaces is a fuzzy subspace defined as :

$$V_\mu \cap R_\nu = (V \cap R)_{\mu \wedge \nu} \tag{12}$$

Definition 8 Let A: $X \to X$, be a linear map, not necessarily invertible, and let S_μ be a fuzzy subspace of X . The μ-*inverse image of* S *under* A is defined as the fuzzy subspace given by:

$$A^{-1}S_\mu = \{\underline{x} : A\,\underline{x} \in_f S_\mu\} \tag{13}$$

According to the extension principle we have :

$$A^{-1}S_\mu = (A^{-1}S)_{\mu \circ A} \tag{14}$$

Definition 9 Let X_μ be a fuzzy subspace and $S_\mu \subset_f X_\mu$ We denote by X_μ/S_μ the factor space (or subspace) of X_μ modulo S_μ. (also denoted by $X_\mu(\mathrm{mod}\, S_\mu)$. The support subspace is X/S and it constitutes a vector subspace with elements denoted by either $[\underline{x}]$ or $\underline{x}(\mathrm{mod}\, S)$.

Using the definition of equivalence classes as the sum of a singleton and a subspace, we can, after using the formula for *fuzzy translation of a fuzzy set* (in this case of a fuzzy subspace), determine the fuzzy factor space as the fuzzy subspace :

$$X_\mu/S_\mu = \int_{X/S} \sup_{\underline{u}}\{ \bigvee_{\underline{y} \in \underline{x}+S} [(\mu_X|S)(\underline{u}-\underline{y}) \wedge \mu_X(\underline{y})]\}/[\underline{x}] \tag{15}$$

In conventional Geometric control theory, when one defines the reachable or controllable subspace $\langle A|B\rangle$, use is made of the fact that $\underline{u}(t)$, the control vector, can take *any* value in the set of allowable control functions over the infinite time interval. It is not difficult to see that a μ-fuzzification of the control space U yields a fuzzification of the reachable subspace through the μ-image of the control channel map B and the systems dynamics. It is left to the reader show that actually the subspace of fuzzy reachable states in this case is given by :

$$R_\mu \triangleq \langle A|B_\mu\rangle = B_\mu + AB_\mu + A^2B_\mu + \ldots + A^{n-1}B_\mu \tag{16}$$

and then :

$$\mu_{\langle A|B_\mu\rangle} = \mu_B * \mu_{AB} * \cdots * \mu_{A^{n-1}B} = \mathop{*}_{i=0}^{n-1} \mu_B \circ A^{-i} \tag{17}$$

Definition 10 A fuzzy subspace X in R^n is said to be μ-controllable or μ-reachable if and only if : $\langle A|B_\mu\rangle \supset_f X_\nu$ i.e it is simultaneously satisfied :

$$\langle A|B\rangle \supset X \quad \text{and} \quad \mu_{\langle A|B\rangle} \geq \nu_X \tag{18}$$

Definition 11 V_ν is $(A,B)_\mu$-invariant (also called $A(\mathrm{mod}B_\mu)$-invariant or μ-Controlled invariant subspace) if there exists a map F_μ such that V_ν is $(A+BF_\mu)$-invariant, i.e. $(A+BF_\mu)V_\nu \subset_f V_\nu$.

As a counterpart to Wohnam's theorem on (A,B)-invariant subspaces, we also have:

Theorem 1 V_ν is $(A,B)_\mu$ -invariant subspace if and only if :

$$A\, V_\nu \subset_f V_\nu + B_\mu \tag{19}$$

In terms of the support subspace and the m.f. this means:

$$AV \subset V + B \quad \text{and} \quad \nu_V \circ A^{-1} \leq \nu_V * \mu_B \tag{20}$$

Definition 12 A fuzzy subspace S_ν is said to be $(C,A)_\delta$-invariant (also called $A|\mathrm{Ker}_\delta C$-invariant or δ-conditionally invariant subspace) if and only if :

$$A(S_\nu \cap \mathrm{Ker}_\delta C) \subset_f S_\nu \tag{21}$$

This condition implies the simultaneous verification of :

$$A(S \cap \mathrm{Ker}\, C) \subset S \text{ and } [\nu_S \wedge (\delta \circ C)] \circ A^{-1} \leq \nu_S \tag{22}$$

We now make use of the fuzzy kernels to define fuzzy unobservable subspaces.

Definition 13 A subspace Z_ζ is δ-unobservable if and only if:

$$Z_\zeta \subset_f \bigcap_{i=0}^{n-1} A^{-i} \,\mathrm{Ker}_\delta\, C \qquad (23)$$

This condition implies :

$$Z \subset \bigcap_{i=0}^{n-1} A^{-i}\mathrm{Ker}C \;\; \underline{\text{and}} \;\; \bigwedge_{i=0}^{n-1} [\,\delta \circ (CA^i)] \geq \zeta \qquad (24)$$

In conventional Geometric theory of linear systems, it is shown that the class of (A,B)-invariant subspaces of a vector space N , denoted by $I(A,B,N)$ or simply $I(N)$, is an upper semilattice relative to subspace inclusion and addition. As such, if $I(N)$ is non-empty, then it contains a unique supremal (A,B)-invariant subspace V^*. This subspace can be computed in a finite number of steps according to the following recursive algorithm:

$$V^* = V^n;\; V^i = N \cap A^{-1}(B + V^{i-1}),\; i \in \underline{n}\;;\; V^0 = N \quad (25)$$

Let N_δ be a fuzzy subspace and R^m_μ a μ-fuzzification of R^m. The supremal (or largest) $(A,B)_\mu$-invariant subspace contained in N_δ is obtained recursively by means of :

$$V^*_{\psi^*} = V^n_{\lim_{j\to\infty}\psi_j}\;;\; V^i_{\psi_j} = N \cap A^{-1}(B_\mu + V^{i-1}_{\psi_{j-1}});\; i=j \text{ in } \underline{n} \;\; i=n,\, j\geq n+1$$

$$\text{with} \quad V^0_{\psi_0} = N_\delta \qquad (26)$$

The above algorithm splits in two algorithmic procedures; one defined on the support subspaces and the other defined on the m.f.'s. The first one converges, necessarily, in n steps at the most. The second algorithm needs, in general, a larger number of iterations. For this reason, the subindices are equal in $\underline{n}$ but from the nth step on, the subindex i stops at $\overline{n}$ while the subindex j continues growing until convergence of the m.f. The support subspace algorithm is the same given by (25) above, while the m.f. algorithm is represented by :

$$\psi^* = \lim_{j\to\infty}\psi_j;\quad \psi_j = \delta \wedge [(\mu_B * \psi_{j-1})\circ A]\;;\; j=1,2\ldots$$

$$\text{with} \quad \psi_0 = \delta \qquad (27)$$

The dual notion of V^*, the smallest A|KerC-invariant subspace (or (C,A)-invariant subspace) containing a given subspace K, is denoted by J_* and the algorithm that produces this subspace (Wohnam, 1979) is :

$$J_* = J^n;\; J^i = K + A[\,J^{i-1} \cap \mathrm{Ker}C\,],\; i\in\underline{n}\,;\; J^0 = 0. \quad (28)$$

The fuzzy version of this algorithmic procedure is stated as follows: Let K_μ be a fuzzy subspace and 0_δ a fuzzy zero in R^n. Let us denote by ∂(u) the Dirac unit impulse function centered at $\underline{u}$, i.e ∂($\underline{u}$) = 1 and ∂($\underline{v}$) = 0 ∀ $\underline{v} \neq \underline{u}$. The infimal (smallest) $(C,A)_\delta$-invariant subspace containing K_μ is obtained recursively by means of :

$$J_{*\nu_*} = J_{\lim_{j\to\infty}\nu_j}\;;\; J^i_{\nu_j} = K_\mu + A[J^{i-1}_{\nu_{j-1}} \cap \mathrm{Ker}\; C] \;\; i=j \text{ in } \underline{n} \;\; i=n,\, j\geq n+1$$

$$J^0_{\nu_0} = \{0\}_{\partial(0)} \qquad (29)$$

i.e the m.f. algorithm would be given by :

$$\nu_* = \lim_{j\to\infty}\nu_j;\; \nu_j = \mu * \{[\nu_{j-1} \wedge (\delta \circ C)]\circ A^{-1}\};$$

$$j=1,2,\ldots \text{ with } \nu_0 = \partial(0) \qquad (30)$$

One of the crucial concepts in Geometric control theory is that of a Controllability Subspace (C.S). The fuzzy version of these important subspaces is characterized by the statement given in the following definition.

<u>Definition 14</u> A fuzzy subspace R_ν is a μ-controllability subspace if there exists a matrix F_μ such that:

$$R_\nu = \langle A + BF_\mu | B_\mu \cap R_\nu \rangle \qquad (31)$$

According to this, the m.f. satisfies :

$$\nu = \bigstar_{i=0}^{n-1}[(\mu \wedge \nu)\circ (A+BF)^{-i}] \qquad (32)$$

The fundamental theorem which allows characterization of fuzzy C.S.'s is the following:

<u>Theorem 2</u> Let A, B, R_ν be fixed; $R \subset X$, R_ν is a fuzzy C.S. of (A,B) if and only if R_ν is $(A,B)_F$-invariant and $R_\nu = S_{*\nu_*}$ where $S_{*\nu_*}$ is the infimal subspace such that :

$$S_{*\nu_*} = R_\nu \cap (AS_{*\nu_*} + B_\mu) \qquad (33)$$

furthermore:

$$S_{*\nu_*} = S_{\lim_{j\to\infty}\nu_j}\;;\; S^i_{\nu_j} = R_\nu \cap (A S^{i-1}_{\nu_{j-1}} + B_\mu)\;;\; i=j \text{ in } \underline{n} \;\; i=n,\, j\geq n+1$$

$$\text{with} \quad S^0_{\nu_0} = \{0\}_{\partial(0)} \qquad (34)$$

i.e the support subspace satisfies the algorithm :

$$R = S_*\,;\; S_* = R \cap (AS_* + B)\;;\; S^i = R \cap (A\,S^{i-1} + B)\;;\; S^0 = 0 \qquad (35)$$

while the m.f. satisfies:

$$\nu_* = \lim_{j\to\infty}\nu_j\;;\; \nu_j = \nu \wedge [(\nu_{j-1}\circ A^{-1}) * \mu_B]\;;\; j=1,2,\ldots$$

$$\nu_0 = \partial(0) \qquad (36)$$

This procedure allows one to check whether a fuzzy subspace R_ν is a fuzzy C.S.

Let R^m_μ be a μ-fuzzification of R^m and let $V^*_{\psi^*}$ be the largest $(A,B)_\mu$-invariant subspace contained in K_δ . Then, the supremal fuzzy controllability subspace algorithm (FCSA) which renders $V^*_{\psi^*}$ is given by:

$$R^*_{\xi^*} = R^n_{\lim_{j\to\infty}\xi_j}\;;\; R^i_{\xi_j} = V^*_{\xi^*} \cap (B_\mu + A\,R^{i-1}_{\xi_{j-1}});\; j=1 \text{ in } \underline{n} \;\; i=n,\, j\geq n+1$$

$$\text{with} \quad R^0_{\xi_0} = \{0\}_{\partial(0)} \qquad (37)$$

The support subspace is generated recursively via:

$$R^* = R\;;\; R^i = V^* \cap (\,B + A\,R^{i-1})\;;\; i\in\underline{n}\;;\; R^0 = 0 \quad (38)$$

while the m.f. is computed by means of:

$$\xi_* = \lim_{j\to\infty}\xi_j;\; \xi_j = \psi_* \wedge [\,\mu_B * (\xi_{j-1}\circ A^{-1})];\; j=1,2,\ldots$$

$$\text{with} \quad \xi_0 = \partial(0) \qquad (38)$$

Another important concept in Geometric control theory of linear systems is that of *detectability* which turns out to be a property weaker than that of observability and closely related to it (Wohnam, 1979). This property is established when it is found that, for a particular system, the subspace of unstable modes is observable.

Let the minimal polynomial (m.p) α(λ) of A be factored as $\alpha(\lambda) = \alpha^+(\lambda)\,\alpha^-(\lambda)$. where the zeros of $\alpha^+(\lambda)$ (respectively $\alpha^-(\lambda)$) in the complex plane lay in the closed right (respectively: open left) half plane. Let :

$$X^+(A) = \mathrm{Ker}\,\alpha^+(A)\;;\quad X^-(A) = \mathrm{Ker}\,\alpha^-(A) \qquad (39)$$

a pair (C,A) is *detectable* if :

$$\bigcap_{i=0}^{n-1} A^{-i}\,\mathrm{Ker}\,C \subset X^-(A) \qquad (40)$$

i.e. A is stable on the unobservable subspace of (C,A).

Let 1_{X^-} denote the m.f. of the crisp set $X^-(A)$ along the subspace indicated in the second formula of (39). This m.f. has the value of zero elsewhere. Using this m.f. and adscribing it to $X^-(A)$ we can propose the concept of a *fuzzy detectable pair* whenever, given a fuzzy zero characterized by the m.f. δ in the output space, it is verified that :

$$\bigwedge_{i=0}^{n-1} \delta \circ C A^i \leq 1_{X^-} \qquad (41)$$

III FUZZY DISTURBANCE DECOUPLING PROBLEM

In this section we shall formulate the Fuzzy Disturbance Decoupling Problem (FDDP). This problem is concerned with decoupling in an approximate (i. e. fuzzy) way the effect of exogenous disturbances on the output of the linear plant. We deal with two special cases; the first one assumes unrestricted control actions in R^m. This case represents the closest formulation to that of Willems' approach (Willems,1980) to geometric problems within the methodology of "almost invariant subspaces". We remark however, that our generalization <u>does not</u> englobe Willems' theory (notice that <u>a fuzzy</u> controlled invariant subspace does not generalize the concept of almost controlled invariant subspace). The second case we shall deal with, assumes fuzzy restrictions on the control space. These two problems could be termed : Fuzzy Disturbance Decoupling Problem with Unrestricted Controls (FDDPUC) and Fuzzy Disturbance Decoupling Problem with Restricted Controls (FDDPRC) respectively.

<u>Fuzzy Disturbance Decoupling Problem with Unrestricted Control</u>

Consider the linear dynamic system:

$$\begin{aligned} \dot{\underline{x}}(t) &= A\,\underline{x}(t) + B\,\underline{u}(t) + D\,\underline{q}(t) \\ \underline{y}(t) &= C\,\underline{x}(t) \end{aligned} \qquad (42)$$

where the term $\underline{q}(t)$ is an exogenous disturbance in $\mathcal{Q}= R^p$ whose structure is not precisely known except for the fact that only a qualitative statement is at our disposal about the nature of such acting signal (i.e. only a fuzzy description of the signal is available. e.g. " q(t) is a *biased* unknown signal whose value *fluctuates around* such value " or else, " q(t) is a *small* disturbance which oscillates *very rapidly* ", etc.). This disturbance in R^p is,thus,best modeled by prescribing a, let us say, γ-fuzzification of $\mathcal{Q}$. The control vector $\underline{u}$ in $\mathcal{U} = R^m$ is totaly free, while the output vector $\underline{y}(t)$ lies in $\mathcal{Y} = R^q$ for all t.

It is required to find (if possible) a state feedback gain F such that the influence of the disturbance signal $\underline{q}(t)$ is *very small* on the output $\underline{y}(t)$. (Alternatively, we could say that it is required that the zero input response of the system, i.e. the response for $\underline{u}(t) = 0\ \forall\ t$, be *very close* to zero or *almost* zero).

Formally, we are given a fuzzy zero in $\mathcal{Y}$ characterized by a m.f. δ (i.e. we are given a δ-fuzzification of $\mathcal{Y}$),and it is required to find a feedback gain F such that if $\mathcal{D}_\gamma$ is the fuzzy image of D, in the state space, of the γ-fuzzification of $\mathcal{Q}$, then:

$$<(A+BF)\,|\,\mathcal{D}_\gamma > \subset_f \mathrm{Ker}_\delta C \qquad (43)$$

This condition constitutes the *fuzzy disturbance decoupling condition* for our problem. Because we have not imposed any (fuzzy or otherwise) restriction on the input signals, this problem has an important conceptual similarity with that treated by Willems (1980). One of the differences is that Willems' constraints on the output values, due to the disturbance action on the plant, are of the "hard type", thus forcing the norm of these outputs to become smaller than a preassigned quantity. On the other hand we impose a "soft" type of constraint on the zero input response allowing it to stay around the zero value so as to be able, at least from a qualitative viewpoint, to asses our control effectiveness, by a mere subjective evaluation of the disturbance influence on the observed output. The other main difference stems from the fact that our definitions of fuzzy invariant subspaces and fuzzy controllability subspaces <u>do not</u> generalize those of almost invariant and almost controllability subspaces. Our definitions relate to properties (whether qualitative or vaguely known) of elements which <u>do</u> belong to certain subspaces while Willems' <u>essential</u> concepts relate to properties of subspaces with respect to points of the space which do not belong to these subspaces and which together conform a systems trajectory very close to the subspace.

Condition (43) signifies a pair of mathematical requirements: one on the support subspace and the other on the m.f.'s. These are :

$$<(A+BF)\,|\,\mathcal{D}> \subset \mathrm{Ker}\,C \qquad (44)$$

$$\mathop{*}_{i=0}^{n-1} \gamma \circ D^{-1}(A+BF)^{-i} \leq \delta \circ C \qquad (45)$$

The solution to the FDDPUC is solvable if and only if :

$$\mathcal{V}^*_{\psi^*} \supset_f \mathcal{D}_\gamma \qquad (45)$$

where $\mathcal{V}^*$is the supremal (A,B)-invariant subspace contained in the Ker C . This subspace is given by the algorithm in (25) with $N = \mathrm{Ker}\,C$, while the corresponding m.f. is generated recursively by means of (27). According to (45) this m.f. must satisfy:

$$\psi^* \geq \gamma \qquad (46)$$

<u>Fuzzy Disturbance Decoupling Problem with Restricted Controls.</u>

This case, slightly generalizes the preceeding one by adscribing a μ-fuzzification to the input space $\mathcal{U}$. We now make use of the $(A,B)_\mu$-invariant subspace concept, in order to propose a solution to our problem. In the preceeding problem only (A,B)-invariance had to be invoked due to the "crispness" associated with the input space.

As before, the solution to the problem exists if and only if :

$$\mathcal{V}^*_{\psi^*} \supset_f \mathcal{D}_\gamma \qquad (47)$$

where $\mathcal{V}^*_{\psi^*}$ is the supremal $(A,B)_\mu$-invariant subspace contained in the fuzzy kernel of C ; $\mathrm{Ker}_\delta C$. ψ^* is given by (27).

IV CONCLUSIONS AND SUGGESTIONS FOR FURTHER RESEARCH

In this article we have given initial steps towards the establishment of a "Fuzzy Geometric Theory of Linear Multivariable Control". We have provided basic definitions that range from fuzzy subspaces and fuzzification of linear vector spaces to fuzzy controlled invariant subspaces and fuzzy controllability subspaces. This framework allows for the adequate treatement, within the fuzzy geometric technique, of whichever problem has already been formulated and solved by means of Wohnams' theory. We can therefore parallel the developments of the Geometric theory in a fuzzy-theoretic frame.

The results, thus far obtained, generalize those of the existing Geometric theory of Wohnam. This generalization actually represents a needed relaxation of the concepts that researchers have been using both in the geometric approach and in fuzzy systems theory (Britov, 1981). This relaxation has a significance, not only from the phylosophical viewpoint, but even from a practical standpoint. This is demonstrated by the closely related line of work initiated by Willems with his "almost invariant subspace theory". Willems methodology, which is rapidly expanding into a new and original approach to a number of classical and modern control problems, could have important inter-relations with the approach we are proposing in this article. Eventhough we did not pay much attention to the similarities and parallelisms that could be exploited and explored with both theories at hand, we are convinced that an area for further research could be constituted by the bridging of both approaches.

As mentioned before, a wide variety of problems already treated by the conventional Geometric theory remain unexplored within the Fuzzy Geometric context presented here. Specific examples remain to be worked outand the variety of solutions obtained, analyzed from both a theoretical and practical viewpoint. Problems such as the Decoupling Problem, the Output Regulation Problem, the disturbance decoupling with measurement feedback and stability problem, the disturbance decoupling with measurement and pole placement problem, the unknown input observer problem (also known as the disturbance decoupled estimation problem), etc. deserve attention from a fuzzy geometric viewpoint.

Finally, a natural particularization for the m.f's appearing in this work, is the case of "Gaussian Fuzzy m.f.'s " defined over the entire space or subspace (See Sira-Ramirez, 1979,1980 for more details). Interesting connections already exist among Fuzzy Sets or variables and random variables and Probability theory. The exploitation of these bridges could shed some light on the developments of "Stochastic Geometric Theory" mentioned in Wohnam (1979).

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APPENDIX

In this appendix we present the basic notation and definitions we use in this article. These definitions were given in Sira-Ramirez (1979) with minor modifications.

Definition A1 We denote a fuzzy set A with m.f. $\mu_A(\ )$, defined over the universe of discourse U in the n-dimensional euclidean space R^n, as :

$$A = \int_U \mu_A(\underline{u})/\underline{u} \qquad (A.1)$$

A non-fuzzy set will be termed "crisp". Its m.f. has the value 1 over the entire domain of the set. We usually denote this by means of 1_A indicating that outside A the m.f. has value zero. We use the simbol ε_f to indicate fuzzy membership.

A straightforward application of the *Extension Principle* (Zadeh, 1965) allows us to define linear transformations, crisp translations, sums, etc. of fuzzy sets in R^n.

Definition A2 We define a *crisp translation* of a fuzzy set A, in the direction of the vector $\underline{v}$, the fuzzy set given by:

$$A + \{1/\underline{v}\} = \int_{A+\underline{v}} \mu_A(\underline{u}-\underline{v})/\underline{u} \qquad (A.2)$$

where A stands for the *support* of the fuzzy set A (i.e. the set of points where the m.f. of A is not zero) " + " denotes the vector sum of the involved sets.

As a generalization of the preceeding definition, consider the *fuzzy singleton* $\underline{v}$ with m.f. $\mu_B(\underline{v})$, i.e $B = \{\mu_B(\underline{v})/\underline{v}\}$.

Definition A3 A fuzzy translation of the fuzzy set A in the direction and extent of the fuzzy singleton B , defined above, is a fuzzy set described by:

$$A + \{\mu_B(\underline{v})/\underline{v}\} = \int_{A+\underline{v}} [\mu_A(\underline{u}-\underline{v}) \wedge \mu_B(\underline{v})]/\underline{u} \qquad (A.3)$$

where the symbol " $\wedge$ " stands for the infimum function of the two function values specified to its sides.

As a generalization of the previous definitions, we introduce now the *direct sum* or *vector sum* definition of two fuzzy sets. This definition generalizes that of a vector sum of two crisp sets.

Definition A4 Let A be a fuzzy set with m.f. μ_A , and similarly let B be a fuzzy set characterized by μ_B. We define the vector sum of A and B as the fuzzy set specified by:

$$A+B = \int_{A+B} \sup_{\underline{v}} [\mu_A(\underline{u}-\underline{v}) \wedge \mu_B(\underline{v})]/\underline{u} \qquad (A.4)$$

where A + B denotes that the universe of discourse, where the fuzzy sum is to be defined, is the vector sum of the crisp sets that serve as supports for the fuzzy sets A and B respectively. The "sup" operation is necessary to eliminate the possibility of having ill-defined membership values for those elements that can be expressed in a non-unique fashion as sum of elements in A and B. This supremum operation usually results in a maximization operation.

The above formula (A.4) constitutes a natural "convolution operation" on the m.f.'s of the fuzzy summands. Naturaly, this definition includes the case where A and B are both crisp sets. The formula (A.4) is, not surprisingly, reminiscent of that which establishes the probability density function of the sum of two independent random variables. Note also that the roles of $\underline{u}$ and $\underline{v}$ can be interchanged.

Remark It is a basic principle that the result can never exceed the exactitude of the data. Equally true, the sum of two fuzzy sets can not be a crisp set. This fact is established by simple inspection of (A.4). For this reason the difference of two fuzzy sets is not always defined.

As a simple application of the extension principle we define now the direct and inverse images of a fuzzy set A in R^n under linear transformations.

Definition A5 Given a non-singular linear transformation P, and a fuzzy set A with m.f. μ_A, we define the *direct image of* A *under* P, the fuzzy set given by:

$$P(A) = \int_{P(A)} \mu_A(P^{-1} \underline{u})/\underline{u} \qquad (A.5)$$

Definition A6 The *inverse image* of a fuzzy set A with m.f. μ_A under the linear map Q, not necessarily invertible, is the fuzzy set described by means of :

$$Q^{-1}(A) = \int_{Q^{-1}(A)} \mu_A(Q\underline{u}) / \underline{u} \qquad (A.6)$$