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NONLINEAR SLIDING MANIFOLDS FOR LINEAR AND BILINEAR SYSTEMS

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ABSTRACT

This article generalizes, in a mathematically tractable fashion, the concept of sliding planes associated with the variable structure control [1] of linear and bilinear dynamic systems. Explicit conditions for the reachability of the non-linear manifold and the stable sliding operation are achieved by using linear restrictions on families of p-tensor powers [2] of the state vector as surface models.

I. INTRODUCTION

Over the last twenty years, a complete body of theory and applications of Variable Structure Systems (VSS) and their "sliding mode" control has been developed by many researchers and institutions among them Barbashin [3],Gerashnenko [4],Emelyanov [5], Utkin [6],Itkis [7] etc.A detailed survey of the vast amount of literature, and an excellent account of the status of this important branch of control systems theory, is available in Utkin's work [1],[8],[9].

The class of VSS have found over the years an increasing number of practical applications. Among these, we have : the Overspeed Protection Control for large steam turbines [10], Automatic Generation Control of steam electric power plants [11],[12],Aircraft Control Systems design [13],Automation of Hydraulic Power plants [14], Controller design for a manipulator [15] etc.

The main issue in the design of sliding mode operation for VSS is to provide a convenient switching surface in state space which can be reached by the state trajectory thanks to appropriate structural changes of the controlled system. Once the surface conditions are satisfied by the state trajectory an active switching of the variable structure controller, or compensator, mantain the state within the switching surface in what is known as the "sliding regime " . This motion is designed to force the state trajectory of the controlled system towards its stable equilibrium. The design methodology is thus confined to finding a suitabl‹ 'iding surface usually of linear nature. (See Utkin and Yang [16]) The use of more general switching surfaces has been restricted due to a lack of a mathematically tractable model capable of producing explicit structural switching conditions which ensure surface reachability. Also, the generation of a closed form expression for the sliding dynamics and the, so called, "equivalent control" [8] did not seem to be readily available for the nonlinear case.

In this paper we develope an approach for the treatment of sliding operating mode in VSS with nonlinear switching surface expressible as scalar homogeneous multinomial forms of the state space variables. Using affine restrictions on "families of p-th tensor powers" of the state space coordinates (See Brockett [2], Sira [17],[18]), a mathematically tractable model of a rather general non linear surface is achieved. We take as a case study bilinear dynamic structures obtaining as a byproduct the relevant results pertaining linear dynamic systems.

Section II presents the characterization of sliding mode regime, for single input bilinear dynamic systems, on a nonlinear switching surface. In this section we also outline a design procedure for obtaining a stable sliding motion based on Lyapunov's Second Method. The results of this section are particularized for the linear dynamic case in Section III.

Section IV contains the conclusions and suggestions for further research. In the Appendix, we collect a number of basic results related to homogeneous tensor powers of n-dimensional vectors.(See [2],[17],[18])

II PROBLEM FORMULATION AND MAIN RESULTS

We are given an autonomous bilinear dynamic system :

$$\dot{\underline{x}}(t) = A\,\underline{x}(t) + u\,B\,\underline{x}(t) \qquad (2.1)$$

with $\underline{x}(t)$ in R^n , A and B are nxn matrices, u is a scalar control input function of the form :

$$u = G_F(\underline{x}) + G_V(\underline{x},k) \qquad (2.2)$$

where $G_F(\underline{x})$ is a <u>fixed feedback gain</u> and $G_V(\underline{x},\underline{k})$ is a <u>variable structure</u> compensating control signal given by $G_V(\underline{x},\underline{k}) = \underline{k}^T\underline{x}$. Where $\underline{k}$ is a feedback gain vector whose component values switches from one limit to another in certain regions of the state space to be deined latter. i.e $\beta_i \le k_i \le \alpha_i$

Let " s " denote a nonlinear switching manifold represented by $s = \underline{m}^T \underline{x}^{[p]} = 0$ where $\underline{m}^T$ is an N*(n,p)-dimensional constant row vector with zero first entry and $\underline{x}^{[p]}$ is the "pth family of tensor powers of the state vector $\underline{x}$ "(See the appendix for notation,definitions and assumptions on $\underline{m}$ and $\underline{x}^{[P]}$).

We must specify a manifold s such that :

1) $s = 0$ is reached by appropriate feedback control action excercised through $G_r(\underline{x})$ and $G_v(\underline{x},\underline{k})$.

2) Sliding operation conditions of stable nature are ensured once the surface $s = 0$ is reached by the state trajectory.

We will use Utkin's reachability conditions in the vecinity of the sliding (switching) surface [8] and obtain a variable structure gain specification which guarantees reachability of the non linear manifold. Secondly, the equivalent control method will be used for the specification of a stable sliding operation regime. A Lyapunov design squeme is presented for the appropriate choosing of a nonlinear stable sliding manifold.

Sliding Manifold Reachability

The sliding manifold reachability is achieved by an appropriate selection of the feedback control action that guarantees the necessary and sufficiency conditions [7] for sliding manifold reachability. This condition is simply : $s\, ds/dt < 0$ and as a result of it we obtain the following proposition.

Proposition 2.1

The reachability condition is satisfied whenever $G_r(\underline{x}) = -(\underline{m}^T A_{[P]}\underline{x}^{[P]})/(\underline{m}^T B_{[P]}\underline{x}^{[P]})$ and $k_i = \alpha_i > 0$ for $s\, x_i\, w(s) < 0$ and $k_i = \beta_i < 0$ for $s\, x_i\, w(s) > 0$ where $w(s)$ is defined as $w(s) = (\underline{m}^T B_{[P]}\underline{x}^{[P]})$.

Moreover, $G_r(\underline{x})$ represents the so called "equivalent control" (Utkin [1]).

Proof.

The proof of this proposition easily follows by considering the product $s\, ds/dt$ and using the results of the appendix on dynamic systems associated with derivatives of tensor powers and enforcing the equivalent control condition ($ds/dt = 0$, see Utkin [1]) on $G_v(\underline{x})$.

Remark 2.1

An additional switching condition, represented by $w(s) = 0$,appears for the case of nonlinear sliding manifolds which is not generally present in the linear surface case. An intersection of $w(s) = 0$ with the sliding manifold causes an undefined right hand side in the ideal sliding dynamic equations. Particularly, in the equivalent control specification. It is therfore mandatory to avoid such intersections at least in the operating portion of the state space where the sliding regime is taking place. It should be pointed out, however, that the flexibility generated by a nonlinear model in sliding surface design allows one to circumvent, to a rather large extent, any undesired intersections. This statement, certainly, can not be made in the linear surface case.

Sliding Dynamics

In order to obtain the equations for the sliding mode, it is necessary to solve the system of equations given by $s = 0$ and $ds/dt = 0$, for the control and the n-th state vector component x_n. The solution for u constitutes the "equivalent control problem" as defined by Utkin [8].

It is easy to see from lemma A.2 in the Appendix, that the bilinear dynamic system represented by $d/dt\ \underline{z} = A_{[P]}\ \underline{z} + u\ B_{[P]}\ \underline{z}$ where $\underline{z}^T = (1, z_1, z_2, \ldots, z_n, z_{n+1}, \ldots, z_{N(n,P)})$ is equivalent to the bilinear dynamic system $d/dt\ \underline{x}^{[P]} = A_{[P]}\underline{x}^{[P]} + u\ B_{[P]}\underline{x}^{[P]}$ provided the initial conditions for $\underline{z}$ are properly chosen (i.e whenever $\underline{z}(0) = \underline{x}^{[P]}(0)$, $\underline{z}(t) = \underline{x}^{[P]}$). Also, in terms of $\underline{z}$, the sliding surface is simply expressed as $\underline{m}^T\underline{z} = 0$.

Notice that above only the first n+1 equations are truly independent. The rest of the equations , i.e those defining z_{n+1} through $z_{N(n,P)}$, are <u>redundant</u> .

Assuming m_n is different from zero, z_n can be solved in terms of all the components of $\underline{z}$ from $\underline{m}^T\ \underline{z} = 0$. Discarding the (n+1)-st equation in the system defining $\underline{z}$ (i.e the one originally defining z_n) and substituting z_n in the remaining equations results in :

$$d/dt\ \underline{z}^* = A^*_{[P]}\underline{z}^* + u\ B^*_{[P]}\underline{z}^* \qquad (2.3)$$

where $\underline{z}^*$ contains all the components of $\underline{z}$ except for z_n and $A^*_{[P]}$, $B^*_{[P]}$ are thus implcitly defined by (2.3).

Notice that this substitution process entails an <u>implicit</u> substitution of x_n by means of a <u>formally explicit</u> substitution of $\underline{z}_n$. The actual <u>explicit</u> substitution of x_n would require the consideration of the several possible real solutions of a highly involved algebraic equation represented by the surface condition, $\underline{m}^T\underline{x}^{[P]} = 0$, thus complicating matters unnecesarily. Below, we opt for redundancy in order to circumvent this problem .

Thus, we <u>formally</u> regard the components of $\underline{z}$ as independent and adscribe their actual interdependency to the appropriate choosing of the initial state. After substitution of $\underline{z}_n$ in the system defining $\underline{z}$ only the n-first differential equations are properly independent while the rest of the equations are to be regarded as <u>auxiliary</u> equations. This process is carried out for the sake of "closedness" in the description of the sliding dynamics.

It should be pointed out that some of the redundant differential equations can be discarded while preserving closedness in the description of the sliding dynamics.

Moreover, if u is a function of $\underline{x}^{[P]}$ (i.e of $\underline{z}$),the equations definig $\underline{z}_{n-1} = (1, z_1, \ldots, z_{n-1}) = \underline{x}_{n-1}$ are written as :

$$d\underline{z}_{n-1}/dt = A^*_{n-1}\underline{z}_{n-1} + u^*(\underline{z}_{n-1})\, B^*_{n-1}\underline{z}_{n-1} \qquad (2.4)$$

where $u^*(\underline{z}_{n-1}) = u(\underline{z})$ with z_n obtained from $s = 0$ while A^*_{n-1} and B^*_{n-1} are matrix blocks of appropriate dimensions, extracted from $A^*_{[p]}$ and $B^*_{[p]}$ respectively.

In terms of $\underline{x}$, equation (2.4) is simply : $d\underline{x}_{n-1}/dt = A^*_{n-1}\underline{x}_{n-1} + u^*((\underline{x}^{[p]})^*)B^*_{n-1}\underline{x}_{n-1}$ where evidently $(\underline{x}^{[p]})^* \neq (\underline{x}^*)^{[p]}$. This differential equation does not conform a "closed" system we defined it previously.

Proposition 2.2

The ideal sliding dynamics governing the systems operation on the nonlinear manifold $s = 0$ satisfy the following system of nonlinear differential equations:

$$d/dt\ \underline{x}_{n-1} = A^*_{n-1}\underline{x}_{n-1} + (\underline{m}^T A^*_{[p]}([\underline{x}^{[p]}]^*/\underline{m}^T B^*_{[p]}[\underline{x}^{[p]}]^*)B^*_{n-1}\underline{x}_{n-1} \qquad (2.5)$$

Remark 2.2

Notice that the possibility of undefined right hand side in equation (2.5) can be avoided by the fact that the dynamic behaviour described by this equation is necessarily confined to the sliding surface $s = 0$. Consequently, the quantity in the denominator of the equivalent control function is generally non-zero on the sliding surface, except at the intersection with the surface $w(s) = 0$. As pointed out before,those intersections must be avoided in the design process. On the other hand, such a quantity is identically zero in the event of having $B = I$ (the identity operator in the n-dimensional vector space). Aside from the unnatural condition possed by such an event, the class of systems arising in this case has already been studied in Sira [18]. It was shown there, that systems described by these differential equations possess an explicit solution . It is therefore perfectly possible to precompute any finite escape time. The designer would be aware of the imminence of a singular behaviour in the sliding trajectory. In such an event, a small but significative change in the vector $\underline{m}$ would define a new sliding surface where this behaviour no longer subsists.

Stability Conditions in the Sliding Mode

Only in very special circumstances one may desire to provoke an unstable sliding dynamic regime. No reason,other than reaching a second stable sliding manifold , seems to justify such a practise (See Johnson [20]).

The design techniques in sliding motion control are devised to produce stable sliding operation by finding a sliding plane as the outcome of a quadratic functional minimization problem defined in terms of the sliding state trajectory or else by a parametric optimization procedure leading to a minimum equivalent control effort (See Utkin and Yang [16]. Some other approaches use Lyapunov functions for the design of stable sliding motion or simply resort to the traditional pole placement design procedure.

We have found particularly suited for the design of nonlinear sliding manifolds, the use of Lyapunov stability theory and the associated methods for generating appropriate Lyapunov functions. One such method is the " Variable Gradient method" [21].

The simplest way to proceed in designing VSS with non-linear sliding manifolds is:

1) Obtain the ideal sliding dynamic equations in terms of the surface parameter $\underline{m}$.

2) Propose a suitable Liapunov function for the stability assesment of the ideal sliding dynamics and compute its time derivative in terms of $\underline{m}$

3) Find a region for $\underline{m}$ in which the design is stable by forcing the time derivative to be negative definite or semidefinite. Choose any value that succeeds in avoiding undesired intersections, in the operating range, with surface $w(s) = 0$.

III LINEAR DYNAMIC SYSTEMS AND NONLINEAR SLIDING MANIFOLDS

In this section we particularize the results obtained thus far to the case of linear dynamic systems.

We are given a single input linear dynamic system described by:

$$d/dt\ \underline{x} = A\ \underline{x} + \underline{b}\ u \qquad (3.1)$$

and a nonlinear manifold $s = \underline{m}^T\underline{x} = 0$ where $\underline{m}$ satisfies all the conditions and remarks included in the Appendix .

The next proposition follows directly from Lemma A.2 and Corollary A.1

Proposition 3.1

The reachability condition for $s = 0$ is satisfied whenever the control action on the system (3.1) is specified as :

$$G_F(\underline{x}) = -\Big(\sum_{i=1}^{p} \underline{m}^T_{<i>} A_{<i>} \underline{x}^{<i>}\Big) \Big/ \sum_{i=1}^{p} \underline{m}^T_{<i>} \underline{\tilde{b}}_{<i>} \underline{x}^{<i-1>} \qquad (3.2)$$

$k_i = \alpha_i > 0$ if $s\, x_i\, w(s) > 0$ and $k_i = \beta_i < 0$ if $s\, x_i\, w(s) < 0$ where we define

$$w(s) = \Big(\sum_{i=1}^{p} \underline{m}^T_{<i>} \tilde{b}_{<i>} \underline{x}^{<i-1>}\Big)$$

and $\underline{x}^{<i>}$ is the i-th tensor power of $\underline{x}$, $\underline{m}_{<i>}$ and $b_{<i>}$ are defined in the Appendix.

Proof

Immediate from Proposition 2.1 and Lemmae A.1, A.2 of the Appendix.

Proposition 3.2

The ideal dynamics governing the systems operation on the nonlinear sliding manifold $s = 0$ satisfies the following system of nonlinear differential equations ($\underline{x}_{n-1} = \underline{z}_{n-1}$) :

$$d/dt\,\underline{x}_{(n-1)} = A^*_{n-1}\,\underline{x}_{n-1} -$$

$$\Big(\sum_{i=1}^{p} \underline{m}^T A^*_{n-1}(\underline{x}^{(i)})^* \Big/ \sum_{i=1}^{p} \underline{m}^T \tilde{\underline{b}}^*_{(i-1)}(\underline{x}^{(i-1)})^*\Big)\underline{b} \qquad (3.3)$$

where the expressions containing the symbol " $*$ " imply a substitution of x_n as it was defined in the previous section (See also example 3.1).

In terms of all redundant equations, (3.3) is expressed as :

$$d/dt(\underline{x}^{[P]})^* = A^*_{[P]}(\underline{x}^{[P]})^*$$

$$- [\underline{m}^T A^*_{[P]}(\underline{x}^{[P]})^* / \underline{m}^T \tilde{B}^*_{[P]}(\underline{x}^{[P]})^*](\underline{x}^{[P]})^* \qquad (3.4)$$

Remark 3.1

Notice that for $\underline{x} = 0$ the right hand side of equation (3.4) is well defined. According to Lemma A.1 the identity $\underline{m}^T \tilde{B}_{[P]} \underline{x}^{[P]} = \underline{m}^T \underline{x}^{[P]}$ can never be satisfied for all $\underline{x}$ due to the structure exhibited by $\tilde{B}_{[P]}$.

The restorative action of $G_v(\underline{x},\underline{k})$ would force the trajectories back to the sliding surface if small deviations from the manifold should occur. Ideally, on the sliding manifold only the "equivalent control " takes place while the variable structure feedback is standing by.

Example 3.1

Consider the doble integrating plant :

$dx_1/dt = x_2$; $dx_2/dt = u$ with $u = u_F(\underline{x}) + k\,x_1$. k being a variable feedback gain. taking one of two possible values ; $k = +1$ or $k = -1$.

A nonlinear switching curve of the form : $s = x_1 - m_2 x_2 - m_9 x_2^3 = 0$; $m_2 \neq 0$ is proposed. i.e : $s = \underline{m}^T \underline{x}^{[3]} = 0$ with $\underline{m}^T = (0,1,-m_2,0,0,0,0,0,0,-m_9)$.

The control actions implied by the reachability conditions will induce trajectories leading to the switching curve when the state is either above or below it . A straightforward application of the formulae leads to: $u_F(\underline{x}) = x_2/(m_2 + 3\,m_9\,x_2)$ (for $s = 0$). $k = +1$ for $s\,x_1\,w(s) < 0$ and $k = -1$ for $s\,x_1\,w(s) > 0$ where $w(s) = m_2 + 3\,m_9\,x_2^2$ (for $s \neq 0$).

The ideal sliding motion is thus described by : $dx_1/dt = x_2$; $dx_2/dt = x_2/(m_2+3m_9x_2^2)$ while the sliding curve equation yields the implicit representation for x_2 as $x_2 = (x_1 - m_9\,x_2^3)/m_2$

Therefore the ideal sliding dynamics expressed in terms of the minimum number of differential equations that conform a closed " system is given by :

$$dz_1/dt = (1/m_2)z_1 - (m_9/m_2)z_9$$

$$dz_5/dt = 2\,z_5/(m_2 + 3\,m_9\,z_5)$$

$$dz_9/dt = 3\,z_9/(m_2 + 3\,m_9\,z_5)$$

where $z_1 = x_1$, $z_5 = x_2^2$ and $z_9 = x_2^3$.

Notice that ,upon integration, the equations for z_1 and z_5 yield back the switching manifold equation $s = 0$ while the equations for z_5 and z_9 simply represent the algebraic relation connecting the second and third powers of x_2.

Consider the positive definite Liapunov function defined on the redundant system of differential equations :

$$V(z_1,z_5) = z_1^2 + z_5^2$$

where there is no need for consideration of z_9 by virtue of its algebraic relation with z_5. The time derivative of $V(z_1,z_5)$ is simply given by : $dV/dt = (2/m_2)z_1^2 - (2m_9/m_2)z_1z_9 + 2z_5^2/(m_2 + 3m_9\,z_5)$. where, as it is evident from the definitions, z_5 and z_1 z_9 are positive quantities. Therefore, having m_2 and $m_9 < 0$ yields a negative definite Liapunov function and thus a stable design. Notice that choosing m_2 and m_9 negative yields no surface at all which may cause an undesired intersection with the sliding curve and thus the switchings occur at the x_2 axis and the surface $s = 0$ itself. i.e $k = -\mathrm{sgn}(x_1\,s)$.

Several procedures may be followed for choosing m_2 and m_9 ; One may consist in optimizing a cuadratic cost penalizing the equivalent control effort and the state deviation from the origin [16].A second avenue is simply try to guarantee the existence of the sliding regime in all of the switching surface. In our particular case this is easy to achieve as long as we avoid intersection with the stable eigenvalue line of the variable structure plant ($k = 1$).

After few experiments, we have chosen as parameters defining the sliding curve the values: $m_2 = -1.5$; $m_9 = -4.0$. Figure 3.1 depicts the behaviour of the state variables x_1 and x_2 in the phase plane. Fig 3.2 represents typical trajectories in the time domain for both x_1 and x_2. Fig. 3.3 shows the time evolution of the variable structure controller both outside and on the sliding curve .

IV CONCLUSIONS

In this article we have investigated the possibility of using nonlinear sliding manifolds in the variable structure feedback control of linear and bilinear systems. A mathematically tractable method has been presented for the modeling of a large class of nonlinear surfaces. These models are exclusively based in considering affine restrictions on a multinomial homogeneous p form of the state vector (also known as a p-th family of tensor powers). Explicit conditions were also found for the reachability of the sliding surface in its immediate vecinity. The ideal sliding equations are shown to substitute the linear or bilinear nature of the original controlled system by a nonlinear set of differential equations. Stability conditions on the ideal sliding regime can be established in terms of Lyapunov functions. For this class of problems, the structure of the sliding dynamics make it particularly suitable the use of the Variable Gradient Method for the determination of an appropriate Lyapunov function.

In this instance we have not treated the multiple input case and it is left as a research suggestion.

For the non-linear case the assignement of an arbitrary fast response to the sliding dynamics is far from solved . Only the initial steps have been taken in this article.

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VI APPENDIX

If $\underline{x}$ is an n-vector with components x_1, $x_2, \ldots, x_n$ we denote $\underline{x}^{<p>}$ the $\binom{n+p-1}{p}$-dimensional vector whose elements are homogeneous p-forms in the components of $\underline{x}$. By convention we set $\underline{x}^{<0>} = 1$. We define $N(n,p) = \binom{n+p-1}{p}$. We shall often refer to this "power" of $\underline{x}$ as the "p-th tensor power of the vector $\underline{x}$".

We denote by $A_{<p>}$ the infinitesimal version of the above defined power, i.e if $\underline{x}$ satisfies the differential equation $d\underline{x}/dt = A\underline{x}$ then $d/dt\,\underline{x}^{<p>} = A_{<p>}\,\underline{x}^{<p>}$. Some properties of these tensor powers are :

1) $(AB)^{\langle p \rangle} = A^{\langle p \rangle} B^{\langle p \rangle}$
2) $(A + B)_{\langle p \rangle} = A_{\langle p \rangle} + B_{\langle p \rangle}$
3) $(A^q)^{\langle p \rangle} = (A^{\langle p \rangle})^q$
4) $(q\,A)_{\langle p \rangle} = q\,A_{\langle p \rangle}$
5) $(A^T)^{\langle p \rangle} = (A^{\langle p \rangle})^T$
6) $(A^T)_{\langle p \rangle} = (A_{\langle p \rangle})^T$

We extend these definitions by considering vectors which are an ordered arrangement of increasing tensor powers of a vector $\underline{x}$. We denote this vector as $\underline{x}^{[p]} = (1, \underline{x}^T, (\underline{x}^{\langle 2 \rangle})^T, \ldots, (\underline{x}^{\langle p \rangle})^T)^T$. We call this vector "the p-th order family of tensor powers of $\underline{x}$". The dimension of $\underline{x}$ is $N^*(n,p) = \binom{n+p}{p}$

Lemma_A.1

Let $\underline{b}$ be an n-dimensional column vector. Denote $\tilde{B}$ the matrix whose first column vector is $(0, \underline{b}^T)^T$ and the rest of entries are all zeroes then the matrix $\tilde{B}_{[p]}$ has its non-zero entries in blocks immediately below the main zero diagonal blocks (See Sira [17] for details)

Lemma_A.2. Let $d\underline{x}/dt = A\underline{x} + u\,B\underline{x}$ be an n-dimenional bi-linear system. Then $\underline{x}^{[p]}$ evolves according to $d\underline{x}^{[p]}/dt = A_{[p]}\,\underline{x}^{[p]} + u\,B_{[p]}\,\underline{x}^{[p]}$

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A linear type restriction on a p-th family of tensor powers of the state vector $\underline{x}$ produces a nonlinear manifold ,of a rather general nature, in R^n . The equation : $s = \underline{m}^T \underline{x}^{[p]} = 0$ is by itself a nonlinear relation among the state coordinates capable of representing smooth manifolds of the most diverse type and shape. On the other hand, this surface model can also be regarded as a truncated approximation of p-th order to an analytic surface of general nature.

Remark_A.1 $\underline{m}^T$ is an $N^*(n,p)$-dimensional row vector with the following structure: $\underline{m}^T = (m_o, \underline{m}^T_{\langle 1 \rangle}, \underline{m}^T_{\langle 2 \rangle} \ldots \underline{m}^T_{\langle p \rangle})$ where $\underline{m}_{\langle k \rangle}$ is an element of $R^{N\langle n,k \rangle}$ and m_o is a scalar. For the case of surfaces containing the origin, it will be assumed that the first component of $\underline{m}$; m_o is zero.

Remark_A.3 We shall assume that the surface equation $s = 0$ allows one to define without ambiguity at least one original state variable, say x_n . This definition could even be of the implicit type i.e. in terms of the state variables and some of their respective homogeneous powers including, possibly, expressions involving x_n itself. An expression $M\underline{x}^{[p]}$ subject to the condition $s = 0$ can be written as $M^*(\underline{x}^{[p]})^*$ where we indicate by means of the symbol "*" that substitution of x_n , has been carried out.

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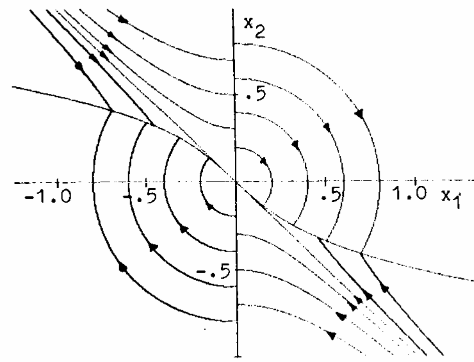

Fig. 3.1

Phase Portrait of 2nd. order Variable Structure System

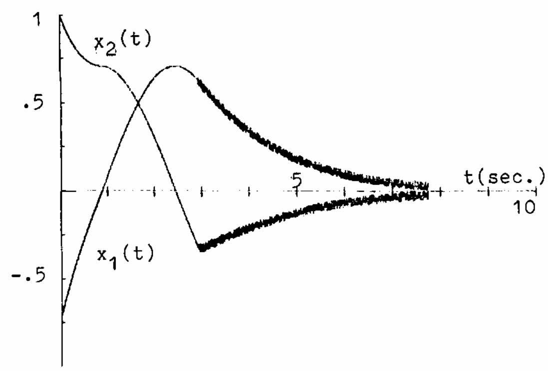

Fig. 3.2

State Trajectories in Sliding Regime

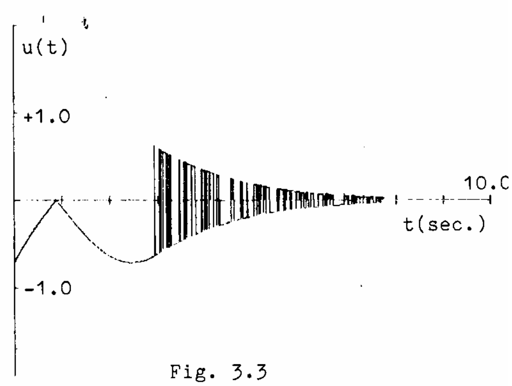

Fig. 3.3

Variable Structure Stabilizing Control