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VARIABLE STRUCTURE CONTROL OF NONLINEAR SYSTEMS THROUGH SIMPLIFIED UNCERTAIN MODELS[1]

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Abstract

A Variable Structure Control (VSC) approach is proposed for the outer loop robust stabilization of feedback equivalent systems whose available model lies in the same structural orbit of a linear system in Brunovsky's canonical form. A simple illustrative example is presented.

I INTRODUCTION

The robust stabilization of exactly linearizable systems through uncertain models is studied in [1] under the assumption that a simplified available model of the plant is in the same orbit, specified by the action of the nonlinear "feedback group", containing the linear Brunovsky form of the equivalent plant [2]. As a consequence of this, the model and the plant have, roughly speaking, the same Kronecker controllability indices. This is physically appealing and philosophically sound in realistic control oriented modeling strategies. But, probably, the nicest consequence lies in the fact that matching conditions, similar to those connected with controller design for uncertain systems [3],[4],[5], emerge as a natural consequence of structural equivalence among the model and the plant, rather than as a priori assumption. Both Lyapunov-based and high-gain controllers are developed in [1] for the stabilization of a nonlinearly perturbed Brunovsky model resulting from model-based linearization.

Here, we shall propose an alternative design scheme using VSC design. For extensive surveys on VSC, the reader is referred to Utkin [6],[7]. As most recommended books, the reader is referred to Utkin [8], and also Itkis [9]. For closely related developments to the particular application of this article is devoted to, see [10],[11],[12].

Section II of this article develops a variable structure controller to stabilize the nonlinear perturbed model whose linear part is in Brunovsky's canonical form. The original nonlinear equivalent plant is thus provided with a nonlinear VSC compensator which locally stabilizes its motion towards the origin of coordinates. The linear sliding surface, expressed in linearizing coordinates, actually represents a nonlinear smooth switching manifold in the original state space coordinates where the sliding motions take place.

Section III is devoted to the VSC feedback stabilization of a single link robot manipulator with a flexible joint coupling the DC motor actuator and the link [13]. A robust controller is specified using the results developed in section II.

Section IV contains the Conclusions and Suggestions for further research in this area.

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II MAIN RESULTS

2.1. Simplified Uncertain Models of Feedback Linearizable Systems

Consider the nonlinear system:

$$\frac{dx}{dt} = f(x)+g(x)u \tag{2.1}$$

where $f(x), g(x)$ are smooth (i,e either C^∞ or analytic) representations of local vector fields on a smooth, Hausdorff real n-dimensional manifold M, which we usually take as R^n.

The **available model** of (2.1) consists in a simplified version of the local fields and it is written as :

$$\frac{dx}{dt} = \hat{f}(x)+\hat{g}(x)u \tag{2.2}$$

The main assumption about the plant (2.1) and its simplified model (2.2) is the structure matching assumption [1] by which the plant and the model are both feedback equivalents of a linear system :

$$\frac{dz}{dt} = Az+bu \tag{2.3}$$

with (A,b) controllable in Brunovsky canonical form. This is more explicitly stated by saying that : using $u = \alpha(x)+\beta(x)v$ and $u=\hat{\alpha}(x)+\beta(x)v$ in (2.1) and (2.2) respectively, with $\beta(x)$ and $\hat{\beta}(x)$ nonzero, followed by the same diffeomorphic coordinate transformation $z=T(x)$ one obtains, in both cases, exactly the same linear model (2.3). This assumption implies that the mismatch or error fields :

$$\delta f(x) = f(x)-\hat{f}(x) \ , \ \delta g(x) = g(x)-\hat{g}(x) \tag{2.4}$$

lie in the span of the model input field $g(x)$, i.e there exists scalar functions $d(x) = \hat{\alpha}(x)-\alpha(x)-(\beta(x)-\beta(x))\beta^{-1}$ and $e(x)=\beta(x)\beta^{-1}(x)-1$ for which $\hat{g}(x)d(x) = \delta f(x)$ and $\hat{g}(x)e(x) = \delta g(x)$. As a consequence of this, the plant equations can be reformulated in terms of the model equations as :

$$\frac{dx}{dt}=\hat{f}(x)+\hat{g}(x)[u(t)+d(x)+e(x)u(t)] \tag{2.5}$$

Applying to the actual system the linearizing feedback control derived for the model : $u=\hat{\alpha}(x)+\hat{\beta}(x)v$ is equivalent to using it on (2.5), thus obtaining the nonlinearly perturbed model :

$$\frac{dz}{dt} = Az+b[v+\phi(z)+\psi(z)v] \tag{2.6}$$

with

$$\phi(z) = \frac{d(x)+e(x)\hat{\alpha}(x)}{\hat{\beta}(x)}\Big|_{x=T^{-1}(z)}$$
$$= \hat{\alpha}(x)\beta^{-1}(x)-\alpha(x)\hat{\beta}^{-1}(x)+\hat{\beta}^{-1}(x)-\beta^{-1}(x)\Big|_{x=T^{-1}(z)}$$

$$\psi(z) = e(x)\big|_{x = T^{-1}(z)}$$

$$= \hat{\beta}(x)\beta^{-1}(x) - 1\big|_{x = T^{-1}(z)} \qquad (2.7)$$

The control input v must be designed to stabilize (2.6) in spite of the modeling errors $\phi(z)$ and $\psi(z)$. These errors are often expressed in terms of appropriate interval or absolute value bounds. We shall assume that such bounds exist in the form of point-wise compact sets in R, given by :

$$\phi(z) \in \Omega_\phi(z) := \{\phi \in R^{n-1} : \|\phi(z)\| \leq h(z)\} \subset R^{n-1}$$

$$\psi(z) \in \Omega_\psi(z) := \{\psi \in R : |\psi(z)| \leq \hat{\epsilon} < 1\} \subset R \qquad (2.8)$$

2.2. Equations for a Variable Structure Controller

Define a partition of the state vector z as :

$$z =: [\underline{z}_1{}^T, z_n]^T \; ; \underline{z}_1 := (z_1, z_2, \ldots, z_{n-1})^T \qquad (2.9)$$

The switching function for the sliding dynamics is defined as :

$$s = (m^T, 1)z = m^T \underline{z}_1 + z_n = \sum_{j=1}^{n-1} m_j z_j + z_n \qquad (2.10)$$

The set $S = \{z \in R^n : s(z) = 0\}$ defines the **sliding surface** . The function s is frequently taken as the " surface coordinate function ". It is useful to obtain a differential equation for its time evolution. Eq. (2.6) can be rewritten as :

$$\frac{d}{dt}\underline{z}_1 = \hat{A}\,\underline{z}_1 + \hat{b}\,s \;\; : \underline{z}_1 \in R^{n-1}$$

$$\frac{ds}{dt} = \mu\, s + \eta^T \underline{z}_1 + [1+\psi(\underline{z}_1, s)]\, v + \phi(\underline{z}_1, s) \qquad (2.11)$$

with:

$$\hat{A} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & 1 \\ -m_1 & -m_2 & \cdots & -m_{n-1} \end{bmatrix} \in R^{(n-1)\times(n-1)}$$

$$\hat{b} = [0, 0, \cdots, 1]^T \in R^{n-1} \; ; \; \mu = m_{n-1}$$

$$; \; \eta^T = (\eta_1, \eta_2, \cdots, \eta_{n-1})$$

and

$$\eta_j = m_{(j-1)} - m_{(n-1)}\, m_j \; ; \; j = 1,2,\ldots,n-1 \; ; \; m_0 = 0$$

$$\psi(\underline{z}_1, s) = \psi(z)\big|_{z_n = s - m^T \underline{z}_1} \; ; \; \phi(\underline{z}_1, s) = \phi(z)\big|_{z_n = s - m^T \underline{z}_1} \qquad (2.12)$$

The input v is synthesized as :

$$v = -f^T \underline{z}_1 - d_v \qquad (2.13)$$

$$(f)_j = \begin{cases} \alpha_j & \text{for } s z_j > 0 : \;\; 0 < \alpha_j \\ -\alpha_j & \text{for } s z_j < 0 : \; j = 1,2,\ldots,r \leq n-1 \end{cases}$$

$$d_v = \begin{cases} d(\underline{z}_1) > 0 & \text{for } s > 0 \\ -d(\underline{z}_1) & \text{for } s < 0 \end{cases} \qquad (2.14)$$

The design problem consists in specifying the gains α_j, β_j and the relay term d_v so that a sliding motion takes place on the switching surface S .

2.3. The ideal Sliding Mode and the Set of Equivalent Gains

The feedback control action of the VSC is to be designed in such a manner that reachability of the sliding surface is achieved, possibly in finite time. Secondly, the controller is to maintain the motions of the system on the sliding surface so that its static characteristics prevail as the defining field for the dynamic behavior of the reduced order controlled plant.

The ideal sliding conditions are simply :

1) $s = 0$

2) $\frac{ds}{dt} = 0$

The first condition leads to the determination of the ideal reduced dynamics:

$$\frac{d}{dt}\underline{z}_1 = \hat{A}\,\underline{z}_1 \qquad (2.15)$$

i.e. the system dynamics is totally governed by the design coefficients of the switching surface S. As a result, the controlled system enjoys a robust stable motion with preassigned rapidity of convergence towards the origin.

In the above equations, full feedback of the systems state variables is necessary in order to synthesize the surface values although the controller function (2.13) may require limited state feedback. In the appendix, a procedure is developed for the appropriate stabilization of the system based in this particular conception of "limited state variable feedback". The constraints that this fact imposes on the ideal sliding mode stability specification are also shown explicitly.

The second condition is responsible for producing the **Equivalent Control** [6] and represents the average control action that takes place by effect of the "infinite frequency" switchings exercised by the VSC. The equivalent control represents the smooth feedback control that would keep the state trajectory on the sliding surface, ideally satisfying the invariance conditions. However, due to the lack of knowledge about the scalar quantity $\psi(z)$ and the disturbance $\phi(z)$, the equivalent control can not be uniquely specified. Substituting (2.13) in (2.11) one obtains :

$$\dot{s} = \mu s + [\eta - (1+\psi(\underline{z}_1, s))f\,]^T \underline{z}_1 + [\phi(\underline{z}_1, s) - (1+\psi(\underline{z}_1, s))d\,(\underline{z}_1)] \qquad (2.16)$$

Define the sets :

$$F = \{f \in R^{n-1} : \;\; \psi(\underline{z}_1, 0) \in \Omega_\psi(\underline{z}_1, 0) \; s.t. \; \eta - [1+\psi(\underline{z}_1, 0)]f = 0$$

$$= \{f \in R^{n-1} : |f_j| \leq \frac{|\eta_j|}{1-\hat{\epsilon}} \;\;\; j = 1,2,\ldots,n-1\} \qquad (2.17)$$

$$D(\underline{z}_1) = \{d \in R^{n-1} : \;\; \phi(\underline{z}_1, 0) \in \Omega_\phi(\underline{z}_1, 0) \text{ and } \psi(\underline{z}_1, 0) \in \Omega_\psi(\underline{z}_1, 0)$$

$$s.t. \; \phi(\underline{z}_1, 0) - [1+\psi(\underline{z}_1, 0)]d = 0\}$$

$$= \{d \in R^{n-1} : |d_j(\underline{z}_1)| \leq \frac{h_j(\underline{z}_1, 0)}{1-\hat{\epsilon}} \; ; \; j = 1,2,\ldots,n-1\} \qquad (2.18)$$

The **set of equivalent gains** is defined as :

$$\Omega_{EQ}(\underline{z}_1) = F \times D(\underline{z}_1) \qquad (2.19)$$

Notice that the elements in the set of equivalent gains represent those values of the feedback gains and relay terms for which the invariance conditions $s = 0$; $\dot{s} = 0$ might be satisfied. In specifying the actual gains and relay term values that achieve sliding surface reachability such set of values must be avoided.

2.4. Reachability Condition

Accessibility of the switching surface, on the part of the state trajectories, is a crucial task in designing sliding modes for Variable Structure Systems (VSS). The necessary and sufficient conditions for a sliding regime to exist are: [6]

$$\lim_{s\to 0^+} \frac{ds}{dt} < 0 \; ; \; \lim_{s\to 0^-} \frac{ds}{dt} > 0 \qquad (2.20)$$

which are equivalent to the more compact form [9] :

$$\lim_{s\to 0} s \frac{ds}{dt} < 0 \qquad (2.21)$$

If this condition is satisfied, the state trajectories of the system move towards the sliding surface S and once it is reached, the VSC will maintain the trajectories in the vicinity of this surface in a chattering motion that eventually leads to a stable equilibrium at the origin of coordinates. It follows from (2.16) and (2.21) that the necessary gains for the linear controller (2.13) are given by :

$$\min_{\Omega_\psi} [1+\psi(\underline{z}_1,0)]\, \alpha_j = [1-\hat{\epsilon}]\alpha_j > |m_{j-1}-m_{n-1}m_j|$$

$$j=1,2,\ldots,r < n-1 \qquad (2.22)$$

and the relay term is given by :

$$\min_{\Omega_\psi} [1 + \psi(\underline{z}_1,0)]\, d_j(\underline{z}_1) = [1-\hat{\epsilon}]\, d_j(\underline{z}_1) >$$

$$\max_{\Omega_\phi(\underline{z}_1,0)} \phi_j(\underline{z}_1,0) = h_j(\underline{z}_1,0)$$

$$(2.23)$$

The set of gains and relay terms in (2.22),(2.23) delimits a region in the gain space which contains the "set of equivalent gains" (2.19). This set inclusion constitutes a necessary and sufficient condition for the existence of a sliding motion on S (See also [6]). Once a synthesis for the VSC has been achieved by means of (2.22)-(2.23), a nonlinear Variable Structure feedback compensator is obtained as an outer loop controller in combination with the inner loop feedback linearizing control law :

$$u = \hat{\alpha}(x) + \hat{\beta}(x)\nu(x) \qquad (2.24)$$

with $\nu(x)$ given by (2.13),(2.14) and (2.22),(2.23). The switchings occur on the surface S represented by (2.10) with $z = T(x)$.

III EXAMPLE

Consider a single revolute joint with elastic gears coupling a DC-motor actuator and a rigid link with inertia J_1 about the rotation axis, such as the one shown in Fig. 1.(Marino and Spong [13], Spong [14]). The elasticity associated with the gears is modeled as a torsional spring with stiffness coefficient k . Let x_1, x_3 denote the angular position of the link and the motor shaft respectively. Defining $x_2 = \dot{x}_1$ and $x_4 = \dot{x}_3$ the state space form of the model is :

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\frac{Mgl}{J_1}\sin x_1 - \frac{k}{J_1}(x_1 - x_3)$$

$$\dot{x}_3 = x_4$$

$$\dot{x}_4 = \frac{k}{J_m}(x_1 - x_3) + \frac{1}{J_m}u \qquad (3.1)$$

where M is the total mass of the link, l is the length measured from the link center of mass to the axes of rotation, g is the gravitational constant, J_m denotes the rotor inertia and u is the torque applied to the motor shaft by the actuator. (See Fig.1)

The global state coordinate transformation represented by the diffeomorphism :

$$z_1 = x_1$$

$$z_2 = x_2$$

$$z_3 = -\frac{Mgl}{J_1}\sin x_1 - \frac{k}{J_1}(x_1 - x_3)$$

$$z_4 = -\frac{Mgl}{J_1}x_2\cos x_1 - \frac{k}{J_1}(x_2 - x_4) \qquad (3.2)$$

and the associated inverse transformation :

$$x_1 = z_1$$

$$x_2 = z_2$$

$$x_3 = \frac{J_1}{k}z_3 + z_1 + \frac{Mgl}{k}\sin z_1$$

$$x_4 = \frac{J_1}{k}z_4 + z_2 + \frac{Mgl}{k}z_2\cos z_1 \qquad (3.3)$$

lead to the **open loop** transformed system :

$$\dot{z}_i = z_{i+1} \;\; ; \;\; i = 1,2,3. \qquad (3.4)$$

$$\dot{z}_4 = p(z) + q(z)u \equiv \nu$$

with :

$$p(z) = \frac{Mgl}{J_1}[(z_2^2 - \frac{k}{J_m})\sin z_1 - z_3\cos z_1] - \frac{k}{J_1}z_3(1 + \frac{J_1}{J_m}) \; ;$$

$$q(z) = \frac{k}{J_1 J_m} \qquad (3.5)$$

It is clear from (3.4) that the feedback linearizing control law is simply :

$$u = \alpha(z) + \beta(z)\nu := [-q^{-1}(z)p(z)] + [q^{-1}(z)]\nu \qquad (3.6)$$

Remark.

System (3.4) is to be regarded as the **plant**. Simplifications leading to the **available model** are to be carried on $p(z)$ and $q(z)$ to obtain $\hat{p}(z)$ and $\hat{q}(z)$. These values lead to the simplified linearizing control law :

$$u = \hat{\alpha}(z) + \hat{\beta}(z)\nu = [-\hat{q}^{-1}(z)\hat{p}(z)] + [\hat{q}^{-1}(z)]\nu$$

Use of this feedback control law on the plant yields the perturbed Brunovsky model :

$$\dot{z}_i = z_{i+1} \;\; ; \;\; i = 1,2,3 \; ;$$

$$\dot{z}_4 = \nu + \phi(z) + \psi(z)\nu \qquad (3.7)$$

with

$$\phi(z) = \hat{q}(z)p(z)q^{-1}(z) - \hat{q}^{-1}(z)\hat{p}(z)q(z) + \hat{q}^{-1}(z)q(z)$$

$$\psi(z) = \hat{q}^{-1}(z)q(z) - 1$$

A sliding surface is proposed which renders asymptotically stable closed loop trajectories for the reduced order sliding dynamics.

$$s = \sum_{i=1}^{3} m_i z_i + z_4 =: m^T \underline{z}_1 + z_4 \qquad (3.8)$$

A variable structure controller is prescribed as :

$$\nu = -\sum_{i=1}^{3} f_i z_i - d_\nu(\underline{z}_1)$$

with d_v being a relay term and the gains f_i (i = 1,2,3) are assumed to take only one of two values $\{-\alpha_j, \alpha_j\}$. The equations for a sliding motion result, in this case, in :

$$\dot{z}_i = z_{i+1} \; ; \; i=1,2$$

$$\dot{z}_3 = -m_1 z_1 - m_2 z_2 - m_3 z_3 + s$$

$$\dot{s} = m_3 s + [-m_1 m_3 - (1+\psi)k_1]z_1 + [m_1 - m_2 m_3 - (1+\psi)k_2]z_2$$

$$+ [m_2 - m_3^2 - (1+\psi)k_3]z_3 + [\phi(z) - (1+\psi)d_v] \qquad (3.9)$$

The ideal sliding dynamics is readily obtained from (3.9) by enforcing the invariance conditions $s = 0$ and $\dot{s} = 0$:

$$\dot{z}_i = z_{i+1} \; : \; i=1,2$$

$$\dot{z}_3 = -m_1 z_1 - m_2 z_2 - m_3 z_3 \qquad (3.10)$$

where the constant coefficients m_1, m_2, m_3 are chosen as $m_1 > 0$, $m_3 > 0$, $m_2 m_3 > m_1$ to guarantee asymptotically stable behavior of (3.11).

The reachability condition of the sliding surface $s\dot{s} < 0$ results in the following gain and relay term prescription :

$$k_j = \begin{cases} \alpha_j > \dfrac{|m_{j-1} - m_3 m_j|}{1-\hat{\epsilon}} & \text{for } sz_j > 0 \\ -\alpha_j & \text{for } sz_j < 0 \end{cases} \quad : j = 1,2,3$$

$$d_v = \begin{cases} d^+ > \dfrac{h(z)}{1-\hat{\epsilon}} & \text{for } s > 0 \\ d^- = -d^+ & \text{for } s < 0 \end{cases} \qquad (3.11)$$

Several model simplification options can be proposed on (3.4)-(3.5). We shall adopt the one suggested by the open loop Brunovsky canonical form of the linearized version of (3.1) (with $\sin x_1 \approx x_1$). We also include modeling uncertainty on the value of the link inertia J_1 to account for the range of possible load values handled by the manipulator. Thus, $\hat{p}(z)$, $\hat{q}(z)$ result in :

$$\hat{p}(z) = -\frac{k}{\hat{J}_1}[1 + \frac{\hat{J}_1}{J_m} + \frac{Mgl}{k}]z_3 - \frac{kMgl}{\hat{J}_1 J_m}$$

$$\hat{q}(z) = \frac{k}{\hat{J}_1 J_m} \qquad (3.12)$$

with these simplifications the perturbation terms in (3.7) are given by :

$$\psi(z) = \psi(\underline{z}_1) = \frac{\hat{J}_1}{J_1} - 1$$

$$\phi(z) = \phi(\underline{z}_1) = \frac{kMgl}{J_m}[\frac{1}{J_1}z_1 + \frac{1}{\hat{J}_1}\sin z_1] - \frac{Mgl}{\hat{J}_1}z_2^2 \sin z_1$$

$$+kz_3[\frac{1}{\hat{J}_1}(1+\frac{J_1}{J_m}) + \frac{1}{J_1}(1+\frac{\hat{J}_1}{J_m}+\frac{Mgl}{k})] + \frac{k}{J_m}[\frac{1}{\hat{J}_1} - \frac{1}{J_1}] \qquad (3.13)$$

An upper bound for the absolute value of the function $\phi(z)$ is readily obtained as :

$$|\phi(z)| \leq \frac{kMgl}{J_1 J_m}|z_1| + \frac{Mgl}{J_1(1-\hat{\epsilon})}z_2^2 + \frac{k(\hat{\epsilon}+Mgl)}{J_1 J_m(1-\hat{\epsilon})}$$

$$+ [\frac{k(J_1+J_m)}{J_1 J_m(1-\hat{\epsilon})} + \frac{kJ_m + kJ_1(1+\hat{\epsilon}) + J_m Mgl}{J_1 J_m}]|z_3| \equiv h(\underline{z}_1) \qquad (3.14)$$

A nonlinear variable structure control logic and an associated nonlinear sliding manifold would be obtained upon use of the inverse diffeomorphic transformation which takes the z variables back to the original variables x. However, due to the uncertainty present in the system parameters this is not possible and the control scheme is to operate by either direct measurement of the z variables or its accurate on line estimation. In this case, as pointed out in [14], these variables have physical significance and they represent position, velocity, acceleration and jerk values for the controlled link. Progress has been reported in both directions : solid state accelerometers are becoming available and results are also available for the exact nonlinear state estimation problem [15]. Combination of both of these possiblities must be considered for actual implementation.

IV CONCLUSIONS

A Variable Structure Control approach has been presented for the robust stabilization of feedback equivalent nonlinear systems whose proposed model lies in the same structural orbit of a linear system in Brunovsky's canonical form.

An attempt to exactly linearize the nonlinear plant on the basis of the feedback control law derived for the available model results in a nonlinearly perturbed canonical system with which account for the expanded class of possible equivalent control functions. Conservatism tends to grow as modeling errors become larger.

In order to preserve internal controllability structure of the plant, it has been proposed that model simplification be carried on the open loop transformed system. As an example, a controller was developed for a single link manipulator with elastic joint.

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APPENDIX

In this section we treat in detail the problem of specifying a stable dynamics on the controlled reduced system (ideal sliding dynamics) when a "limited state variable feedback" compensator of variable structure is sought. Notice that such a name is inappropriate since, at the end, all state variables are needed for the synthesis of the surface coordinate s .

Let $r = n-1$ in (2.13)-(2.14) $\dot{s}$ is written as:

$$\dot{s} = m_{n-1}\, s + \sum_{j=1}^{n-1}[m_{j-1} - m_{n-1}m_j - [1+\psi(\underline{z}_1,s)]f_j]\, z_j + \phi(\underline{z}_1,s) - (1+\psi(\underline{z}_1,s))d_v \qquad (A.1)$$

If the f_j 's and the relay term d_v are chosen in such a form that the limit condition (2.21) is satisfied, then a sliding motion exists on s . However, if only a smaller number of state variables is allowed in the synthesis of the control signal u and $r < n-1$ then, the above equation results in :

$$\dot{s} = m_{n-1}\, s + \sum_{j=1}^{r}[m_{j-1} - m_{n-1}m_j - (1+\psi(\underline{z}_1,s))f_j]z_j + \sum_{j=r+1}^{n-1}[m_{j-1} - m_{n-1}m_j]z_j + \phi(\underline{z}_1,s) - [1+\psi(\underline{z}_1,s)]d_v \qquad (A.2)$$

and the condition :

$$m_{j-1} - m_{n-1}m_j = 0 \;;\; j = r+1, r+2, \ldots, n-1 \qquad (A.3)$$

has to be enforced in order to satisfy (2.21). This has an immediate consequence on the ideal sliding dynamics modes of the controlled system. Indeed, since now the coefficients in the matrix $\hat{A}$ in (2.12) are no longer free, it is not possible, in general, to comply with an arbitrarily chosen dynamics.

The problem is reduced to finding the effect of (A.3) on the behavior of the ideal sliding dynamics. We may thus consider the full version of the Brunovsky canonical system, free from parametric uncertainties and nonlinearities. Consider then :

$$\frac{dz}{dt} = A\, z + bu \qquad (A.4)$$

The following design procedure generalizes some low dimensional examples presented in Utkin [8].

Procedure

1) Find a linear control function of the form :

$$v = \sum_{j=1}^{r} \gamma_j z_j \qquad (A.5)$$

with γ_j constant and $r < n-1$ such that the resulting closed loop system (A.4) has one of its eigenvalues real and say equal to λ which will be necessarily positive, while the rest of the eigenvalues have negative real parts.

2) Equate the coefficients of the "equivalent control ":

$$v_{EQ} = -\sum_{j=1}^{r}[m_{j-1} - m_{n-1}m_j]\, z_j \qquad (A.6)$$

to the corresponding coefficients γ_j of the linear control law in (A.5). Equate also λ to m_{n-1} and obtain :

$$m_j = \lambda^{n-j} \quad ; \quad j = r, r+1, \ldots, n-1$$

$$m_j = \lambda^{n-j} + \sum_{k=1}^{r-j}\gamma_{j+k}\lambda^{k-1} \;;\; j = 1,2,\ldots, r-1 \qquad (A.5)$$

3) Define now :

$$\gamma_1 = \lambda w_1 \,;\, \gamma_2 = \lambda w_2 - w_1 \,;\, \gamma_3 = \lambda w_3 - w_2 \,;\ldots;$$

$$\gamma_{r-1} = \lambda\, w_{r-1} - w_{r-2} \;;\; \gamma_r = \lambda^{n-r+1} - w_{r-1} \qquad (A.6)$$

4) The characteristic polynomial of the closed loop system can now be factored as :

$$(p-\lambda)(p^{n-1} + \lambda\, p^{n-2} + \lambda^2 p^{n-3} + \ldots + \lambda^{n-r} p^{r-1} + w_{r-1}p^{r-2} + \ldots + w_2 p + w_1) = 0 \qquad (A.7)$$

5) With $w_1 > 0 \,;\, w_2 > 0 \,;\ldots;\, \lambda^{n-r+1} > w_{r-1} \,;\ldots$ etc. all roots of the second factor of (A.6) have negative real parts. Substitution of the defined γ's in (A.5) allows one to obtain the sliding surface coefficients for the system in terms of the w_i 's. These are evidently :

$$m_1 = w_1 \;;\; m_2 = w_2 \;;\; \cdots \;;\; m_{r-1} = w_{r-1} \;;$$

$$m_r = \lambda^{n-r} \;;\; \cdots \;;\; \cdots \;;\; m_{n-2} = \lambda^2 \;;\; m_{n-1} = \lambda \qquad (A.8)$$

The design procedure for the ideal sliding dynamics should now be clear. The second polynomial factor of (A.7) contains all the stable eigenvalues of the ideal sliding dynamics for the reduced order controlled system.

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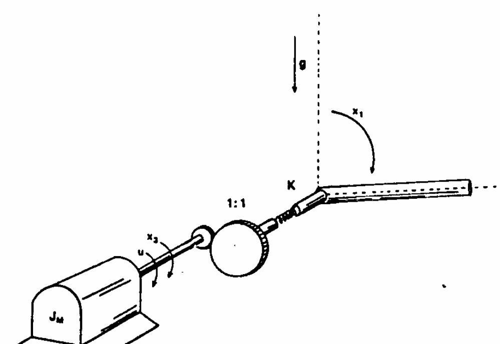

Fig. 1

FLEXIBLE JOINT MANIPULATOR