ROBUST CONTROL DESIGN TECHNIQUES FOR A CLASS OF NONLINEAR SYSTEMS

Mark W. Spong

Hebertt Sira-Ramirez[1]

Coordinated Science Laboratory
University of Illinois at Urbana-Champaign
Urbana, Illinois, 61801
U.S.A

We consider the design of robust stabilizing control laws for nonlinear systems which are equivalent under C^∞-state space coordinate transformations and nonlinear feedback to controllable linear systems. We are motivated by the problem of nonlinear control given simplified or uncertain system models. Assuming that certain *structure matching conditions* are satisfied between the plant and the model of the plant, we reduce the problem to that of stabilizing a *perturbed linear system* and discuss several design schemes that can be used to guarantee stability. An example of robust tracking for a robot manipulator is given.

1. INTRODUCTION

In this paper we consider the robust control for a class of nonlinear systems which are feedback equivalent to controllable linear systems. The basic idea behind the notion of feedback linearization of nonlinear systems is that there exists a suitable coordinate system in which the nonlinearities in the system may be cancelled and replaced by linear terms. Such cancellation of nonlinearities however leaves open many issues of sensitivity and robustness. In addition, the required nonlinear feedback may be computationally difficult to perform in real-time and it is desirable therefore to consider feedback linearization based on simplified or uncertain models.

In this paper we discuss several design schemes that can be used to guarantee robust stabilization subject to norm bounds on the extent of model uncertainty. We first discuss a Variable Structure Controller (VSC) using the hierarchy of controls method. For comparison we then discuss two other techniques, one based on Lyapunov's second method, and the third a linear dynamic compensator designed using the method of stable factorization.

2. MODELING

We consider a class of nonlinear systems of the form:

$$\begin{aligned}\dot{x} &= f(x) + \sum_{i=1}^{m} g_i(x)u_i(t) \\ &= f(x) + G(x)u\end{aligned} \tag{2.1}$$

The vector fields $f(x), g_1(x), \cdots, g_m(x)$ are assumed to be C^∞ on a dense submanifold M of R^n, with $f(0)=0$. We assume that the plant is globally equivalent under C^∞- state space change of coordinates and nonlinear feedback to a controllable linear system. Necessary and sufficient conditions for this global equivalence are known[2], and in particular, our assumption implies that there exists a diffeomorphism $T(x)$ on M and nonlinear functions $\alpha(x)$, $\beta(x)$ of appropriate dimensions, with $\beta(x)$ invertible, such that

$$f(x) + G(x)\alpha(x) = T_x^{-1}AT(x) \tag{2.2}$$

$$G(x)\beta(x) = T_x^{-1}B \tag{2.3}$$

where T_x is the Jacobian of T at x. If such a transformation exists then with change of coordinates

$$z = T(x) \tag{2.4}$$

and nonlinear feedback

$$u = \alpha(x) + \beta(x)v \tag{2.5}$$

the plant becomes

$$\dot{z} = Az + Bv . \tag{2.6}$$

The pair (A, B) is taken as a controllable linear system in Brunovsky canonical form with controllability indices $\kappa_1, \cdots, \kappa_m$. v is an additional input which is designed to control the linear system (2.6).

Given the plant (2.1), we therefore assume an *available model* of the form

$$\begin{aligned}\dot{x} &= \hat{f}(x) + \sum_{i=1}^{m} \hat{g}_i(x)u_i(t) \\ &= \hat{f}(x) + \hat{G}(x)u .\end{aligned} \tag{2.7}$$

In general $\hat{f}, \hat{G}$ are simplified and/or nominal versions of f, G respectively. We assume the following

Structure Matching Assumption 2.1 : The plant (2.1) and the model (2.7) are in the same orbit under the action of the nonlinear "feedback group" and this orbit contains the linear system (2.6) with Kronecker indices $(\kappa_1, \cdots, \kappa_m)$.

Remarks 2.2 : Roughly speaking, assumption (2.1) says that we can choose our model of the plant with the same controllability indices as the plant. This is a reasonable assumption. Even if certain parameters are unknown one usually has some knowledge about the system structure. For example, many physical systems such as mechanical systems have a natural Lagrangian or Hamiltonian structure and it is natural to assume that the model has the same structure.

Let

$$\Delta f(x) = f(x) - \hat{f}(x) \tag{2.8}$$

[1]On leave from the Control Systems Department, Systems Engineering School, Universidad de Los Andes, Merida, Venezuela.

$$\Delta G(x) = G(x) - \hat{G}(x) \tag{2.9}$$

denote the mismatch between the plant and model, so that the plant can now be written as

$$\dot{x} = \hat{f}(x) + \Delta f(x) + (\hat{G}(x) + \Delta G(x))u \tag{2.10}$$

Since (2.1) and (2.7) are transformable to the same linear system, they are themselves feedback equivalents by transitivity. This implies that

$$\begin{aligned} \Delta f(x) &\in \mathcal{G} \\ \Delta G(x) &\in \mathcal{G} \end{aligned} \tag{2.11}$$

where $\mathcal{G}$ denotes the distribution on M defined by $\hat{g}_1, \cdots, \hat{g}_m$. It follows (after a posible coordinate transformation in the plant) that there exists smooth functions $D(x)$, $E(x)$ on M such that

$$\begin{aligned} \Delta f(x) &= \hat{G}(x)D(x) \\ \Delta G(x) &= \hat{G}(x)E(x) \,. \end{aligned} \tag{2.12}$$

Note from (2.12) that

$$G = \hat{G}(I + E) := \hat{G}(x)\delta(x) \tag{2.13}$$

It follows that (2.7) is of the form

$$\dot{x} = \hat{f}(x) + \hat{G}(x)\{u(t) + D(x) + E(x)u(t)\} \tag{2.14}$$

Remark 2.3: It is important to note that the conditions (2.12), which are similar to the *matching conditions* in the literature on uncertain dynamical systems,[5],[7],[9] are **not** a priori assumptions on $\Delta f(x)$ and $\Delta G(x)$, but follow from the assumption that the plant and model share the same controllability indices.

We next choose the control u to linearize the model (2.7). That is we find a diffeomorphism $T(x)$ and functions $\hat{\alpha}(x)$, $\hat{\beta}(x)$ such that the input

$$u = \hat{\alpha}(x) + \hat{\beta}(x)v \tag{2.15}$$

and change of coordinates

$$z = T(x) \tag{2.16}$$

transforms the model (2.7) into the linear system (2.6). Now, applying the control (2.15) to the plant (2.14) yields after a short computation

$$\dot{z} = Az + B\{v + \eta(z, v)\} \tag{2.17}$$

where

$$\begin{aligned} \eta(z, v) &= \hat{\beta}^{-1}(D + E(\hat{\alpha} + \hat{\beta}v))\,|_{x = T^{-1}(z)} \\ &= \hat{\beta}^{-1}D + \hat{\beta}^{-1}E\hat{\alpha} + \hat{\beta}^{-1}E\hat{\beta}v \end{aligned} \tag{2.18}$$

which we write as

$$\eta(z, v) = \Phi(z) + \Psi(z)v \,. \tag{2.19}$$

with obvious definitions of Φ and Ψ. Note that the system (2.17) is still a nonlinear system since η is a function of both z and v. Thus the new control input v must be designed not to stabilize the Brunovosky form (A, B) which is trivial, but to guarantee stability of the nonlinear system (2.17). However the form of (2.17) is that of a nonlinear perturbation of a linear system. Moreover the nonlinearities lie in the range space of B, and so several design schemes can now be employed to stabilize the system (2.17) by suitable choice of v.

We make the following assumptions on the function η in (2.19)

1) There exists a positive constant $\alpha < 1$ such that for $z \in R^n$

$$||\Psi(z)|| \leqslant \alpha \tag{2.20}$$

2) There exist positive constants a and b such that for $z \in R^n$

$$||\Phi(z)|| \leqslant a + b\,||z|| \tag{2.21}$$

3. OUTER LOOP DESIGN TECHNIQUES

In this section we discuss the so-called *outer loop design* for the system (2.17), that is the design of the control input v for robust control of (2.17). We shall survey several schemes which may be used to guarantee stability under the assumptions (2.20),(2.21). The approaches that we consider in turn are variable structure control, the second method of Lyapunov, and a linear design based on the method of stable factorization.

3.1 VARIABLE STRUCTURE CONTROL

In this section we propose a design scheme for the robust stabilization of (2.17) using a Variable Structure Controller (VSC) design based on the "hierarchy of controls method" [25]. For extensive surveys on VSC the reader is referred to Utkin [21],[22] and, for developments closely related to the present work, to [23],[24].

Our aim is to stabilize the nonlinear perturbed model whose linear part is in Brunovsky's canonical form. Upon inversion of the diffeomorphic transformation $T(x)$, and using (2.15), the original nonlinear plant is provided with a nonlinear VSC compensator which locally stabilizes the motion toward the local origin of coordinates.

Define the following partitions of the state variable z according to the values of the Kronecker indices: $z = (z_1^T, z_2^T, \cdots, z_m^T)^T$ with $z_i \in R^{k_i}$ for $i = 1, 2, \cdots, m$. Let $\bar{z}_i = (z_{i1}, \cdots z_{ik_i - 1})$ and $\bar{z} = (\bar{z}_1^T, \cdots, \bar{z}_m^T)^T$). The i-th switching surface, corresponding to the i-th subsystem (i-th string of integrators) of (2.17) is defined as:

$$\begin{aligned} s_i &= [m_i^T, 1]z_i = m_i^T\bar{z}_i + z_{ik_i} \\ &= \sum_{j=1}^{k_i - 1} m_{ij} z_{ij} + z_{ik_i} \end{aligned} \tag{3.1}$$

Then (2.17) can be written as

$$\begin{aligned} \frac{d}{dt}\bar{z} &= \bar{A}\bar{z} + \bar{B}s \qquad \bar{z} \in R^{n-m} \\ \frac{d}{dt}s &= Ms + N\bar{z} + [I + \Psi(\bar{z}, s)]v + \Phi(\bar{z}, s) \end{aligned} \tag{3.2}$$

with $s = col(s_1, \cdots, s_m)$ and

$$\bar{A} = \begin{vmatrix} \bar{A}_1 & 0 & \cdots & 0 \\ 0 & \bar{A}_2 & \cdots & 0 \\ & & & \\ 0 & 0 & \cdots & \bar{A}_m \end{vmatrix}$$

$$\bar{A}_i = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ -m_{i1} & -m_{i2} & \cdots & -m_{i\kappa_i-1} \end{bmatrix} \in R^{(\kappa_i-1)\times(\kappa_i-1)}$$

$$\bar{B} = diag(\bar{b}_1, \cdots \bar{b}_m) \; ; \; \bar{b}_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \in R^{\kappa_i-1}$$

$$M = diag(m_{1\kappa_1-1}, \cdots, m_{m\kappa_m-1})$$

$$N = \begin{bmatrix} c_1^T & 0 & \cdots & 0 \\ 0 & c_2^T & \cdots & 0 \\ & & & \\ 0 & 0 & \cdots & c_m^T \end{bmatrix}$$

$$(c_i^T)_j = m_{i(j-1)} - m_{i(\kappa_i-1)} m_{ij} \quad i=1,...,m \quad i=1,...\kappa_i-1$$

$$m_{i0} := 0$$

$$\Psi(\bar{z},s) = \Psi(z)|_{z_{i\kappa_i} = s_i - m_i^T z_i}$$
$$\Phi(\bar{z},s) = \Phi(z)|_{z_{i\kappa_i} = s_i - m_i^T z_i} \tag{3.3}$$

In correspondence with the i-th entry position in the vector v we shall also use the following partitions for v and the matrix $I+\Psi$.

$$v^T = [(v^{i-1})^T, v_i, (\tilde{v}^{i+1})^T]$$
$$(I+\Psi)_i = [\Psi_i^{i-1}, 1+\Psi_{ii}, \tilde{\Psi}_i^{i+1}]$$

with

$$v^{i-1} = [v_1, \cdots v_{i-1}] : \tilde{v}^{i+1} = [v_{i+1}, \cdots, v_m]$$
$$\Psi^{i-1} = [\Psi_{i1}, \cdots \Psi_{i\,i-1}] \tag{3.4}$$
$$\tilde{\Psi}_i^{i+1} = [\Psi_{i,i+1}, \Psi_{i,i+2}, \cdots \Psi_{i,m}]$$

$(\cdot)_i$ denotes the i-th row vector of the enclosed matrix. Each input v_i is synthesized as

$$v_i = -f_i^T \bar{z}_i - d_{v_i} \; ; i=1,...,m$$

with

$$(f_i^T)_j = \begin{cases} \alpha_{ij} & \text{for } s_i z_{ij} > 0 \\ \beta_{ij} & \text{for } s_i z_{ij} < 0 \end{cases} \quad d_{v_i} = \begin{cases} d_i > 0 \text{ for } s_i > 0 \\ -d_i < 0 \text{ for } s_i < 0 \end{cases} \tag{3.5}$$

where $\alpha_{ij} > 0$, $\beta_{ij} < 0$; $i=1,...,m$; $j=1,...r_i \leqslant \kappa_i - 1$.

The VSC design problem consists in specifying the gains α_{ij}, β_{ij} and the relay terms d_i, so that a sliding motion takes place on the intersection of the m switching surfaces : $s_i=0$ for all i. Notice that if a sliding motion occurs, then the ideal trajectories (equivalently, the ideal dynamics) are governed by $\dot{\bar{z}} = \bar{A}\bar{z}$, which represents m-uncoupled subsystems of order $\kappa_i - 1$ given by:

$$\frac{d}{dt}\bar{z}_i = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \\ -m_{i1} & -m_{i2} & \cdots & \cdots & -m_{i\kappa_i-1} \end{bmatrix} \bar{z}_i \tag{3.6}$$

i.e., each subsystem dynamics is totally governed by the design coefficients specifying the i-th switching surface, and the nonlinear interactions are eliminated.

Conditions $s_i=0$; $\frac{ds_i}{dt}=0$ for all i are known as "ideal sliding conditions". The control function obtained from enforcing these conditions constitutes the "equivalent control" [21]. Due to our lack of knowledge about the matrix $\Psi(\bar{z},s)$ and the vector $\Phi(\bar{z},s)$ the equivalent control can only be specified as a *set*. The assumptions (2.20), (2.21) can be expressed as set containments $\Psi(z) \in \Omega_\Psi$ and $\Phi(z) \in \Omega_\Phi$, where Ω_Ψ, Ω_Φ are compact subsets of R^n defined implicitly by the constraints (2.20), (2.21). Note in this design it is not necessary for (2.21) to be defined by an affine function of $||z||$. Any compact set will suffice. We now define the *equivalent control set* as:

$$\bar{\Omega}_{EQ}(\bar{z}) = \{v(\bar{z}) \in R^m \mid [I+\Psi(\bar{z},0)]v(\bar{z}) + \Phi(\bar{z},0) + N\bar{z} = 0$$
$$\text{for some } \Psi(\bar{z},0) \in \Omega_\Psi \text{ and } \Phi(\bar{z},0) \in \Omega_\Phi(\bar{z})\}.$$

Although this set is, in general, non-convex even for convex Ω_Ψ and pointwise convex $\Omega_\Phi(\bar{z},0)$, it is always possible to produce a bounding convex set sufficiently tight containing $\bar{\Omega}_{EQ}(\bar{z})$. We denote this set by $\Omega_{EQ}(\bar{z})$. For the computation of the appropriate gains of the VSC a *bounding hyperbox* is needed for the sets $\Omega_{EQ}(\bar{z})$, $\Omega_\Phi(\bar{z},0)$ and for the set associated with the uncertain i-th row vector Ω_{Ψ_i} of the matrix $[I+\Psi(\bar{z},s)]$.

Let the symbol "$*$" stand for any of the subscripts EQ, Φ or Ψ_i and let e_i denote the i-th unit vector of R^m. Then

$$w_{i_*}(\cdot) = \left\{ v_i \in R \; : \; \min_{v \in \Omega_*} e_i^T v \leqslant v_i \leqslant \max_{v \in \Omega_*} e_i^T v \right\} \tag{3.8}$$

defines an interval on the real line. The *set product* of all intervals determines a bounding hyperbox for these sets ;

$$\prod_{i=1}^m w_{i_*}(\cdot) \supseteq \Omega_*(\cdot).$$

We define the set $\{\tilde{v}^{i+1}\}$ as the *finite set of values* which can be taken by the components of

$$\tilde{v}^{i+1} = [v_{i+1}(\bar{z},s), ..., v_m(\bar{z},s)]$$

according to (3.5), for a given pair of vectors $(\bar{z},s)$.

Several methods have been proposed for inducing sliding regimes in multivariable systems [26],[22]. They generally group under the "diagonalization" and "hierarchical" categories. The first class relies on either state coordinate transformations, for appropriately changing the sliding manifold equations, or else by input space transformations. The idea is to decouple surface coordinates interaction as much as possible. For our particular case, either one of the two versions of the diagonalization method is not suitable. These methods amount to being able to invert either the input channel matrix or some linear function of it. The uncertainty surrounding our knowledge of the input matrix precludes these possibilities.

The hierarchy of controls method is better suited to handle uncertainty in general while providing a "single input" approach for the creation of a sliding regime [21],[25],[26]. The collective sliding motion is obtained as a consequence of sequentially ordered individual efforts by the inputs to drive the state trajectory towards its individual sliding surface. The individual sliding conditions are enforced under the assumptions of having all members in higher positions of the hierarchy already in sliding regime, while those in lower positions use one out of a finite set of possible feedback control structures. An off-line procedure for computing the necessary gains can then be started from the bottom of the hierarchy working up towards the higher positions of the hierarchy. These computations require, in general, knowledge of the state in the form of a priori bounding estimate set defined in the vicinity of the sliding surface itself.

The following procedure outlines the steps to be considered for the computation of the gains that result in the creation of an appropriate sliding regime stabilizing the nonlinearly perturbed system (2.17).

Step 1: Assume that the hierarchy $s_1 \rightarrow s_2 \rightarrow \cdots \rightarrow s_m$ has been chosen.

Step 2: Let $i = m$

Step 3: Suppose sliding motions occur on the surfaces $s_j = 0$ for $j = 1, \ldots i-1$.

Step 4: For the surface $s_i = 0$, find α_{ij}, β_{ij}, d_i ; $j = 1, \ldots, r_i$ such that

$$s_i \frac{ds_i}{dt} < 0 \tag{3.9}$$

If (3.9) is satisfied, the state trajectories of the system move towards the surface $s_i = 0$ along the intersection of $s_j = 0$; $j = 1, \ldots k-1$. Once $s_{i=0}$ is reached, the VSC will maintain the trajectory on this new surface.

Step 5: Let $i = i-1$. If $i > 0$ go to Step 3 ,else Stop.

If condition (3.9) is satisfied, then, it follows that

$$\min_{w_{\Psi_{ii}}}[1+\Psi_{ii}(\bar{z},s)]\alpha_{ij} \geqslant m_{i(j-1)} - m_{i(\kappa_i-1)}m_{ij}$$

$$\max_{w_{\Psi_{ii}}}[1+\Psi_{ii}(\bar{z},s)]\beta_{ij} \leqslant m_{i(j-1)} - m_{i(\kappa_i-1)}m_{ij} \tag{3.10}$$

$$j = 1, \ldots r_i \;\; ; \; i = 1, \ldots m$$

and for $i = 1, \cdots, m$

$$\min_{w_{\Psi_{ii}}}[1+\Psi_{ii}(\bar{z},s)]d_i \geqslant$$

$$\max_{\zeta_i(\bar{z},s)}[\Phi_i(\bar{z},s) + \dot{\Phi}_i^{i+1}(\bar{z},s)\bar{v}^{i+1}(\bar{z},s) + \Psi_i^{i-1}(\bar{z})v_{EQ}^{i-1}(\bar{z})] \tag{3.11}$$

$$-\max_{w_{\Psi_{ii}}}[1+\Psi_{ii}(\bar{z},s)](d_i) \leqslant$$

$$\min_{\zeta_i(\bar{z},s)}[\Phi_i(\bar{z},s) + \tilde{\Psi}_i^{i+1}(\bar{z},s)\bar{v}^{i+1}(\bar{z},s) + \Psi_i^{i-1}(\bar{z})v_{EQ}^{i-1}(\bar{z})]$$

with
$$\zeta_i(\bar{z},s) = w_{\Phi_i}(\bar{z},s) \times \prod_{j=i+1}^{m} w_{\Psi_{ij}}(\bar{z},s) \times \{v^{i+1}\} \times \prod_{j=1}^{i-1} w_{\Psi_{ij}}(\bar{z}) \times \Pi_{j=1}^{i-1} w_{jEQ}(\bar{z}) . \tag{3.12}$$

This procedure allows for the computation of the required gains and relay terms for the robust control law. In (3.11) the relay terms handle the interactions among the subsystems and the uncertainties as disturbance terms while the feedback portion of the controller with Variable Structure gains is primarily concerned with cautious regulation of the linear structure of the model. Caution is exercised by the enhancement of the gains through "worst case" parametric uncertainty effects on the individual sliding conditions. If the interactions are significantly strong, one may show that the relay terms could become unbounded or unable to guarantee appropriate sign requirements and, as a consequence, sliding motions may not exist.

3.2 DESIGN VIA THE SECOND METHOD OF LYAPUNOV

The next two design schemes have appeared in [20] for the general case and in [6] for the robot control problem. Our intent here is then merely to survey these ideas and our discussion in the next two sections will thus be less detailed than in the previous section. The interested reader is referred to the references for the details of these results.

Definition 3.1 : Given a solution

$$z(\cdot) : [t_0, \infty) \rightarrow R^{2n}, \; z(t_0) = z_0$$

of (2.17), we say the $z(\cdot)$ is *uniformly ultimately bounded* (u.u.b.) with respect to a set S if there is a nonnegative constant $T(z_0, S) < \infty$ such that $z(t) \in S$ for all $t \geqslant t_0 + T$.

Since the Brunovsky form (2.6) is unstable we first choose K so that $A + BK$ is stable and set $v = Kz + \Delta v$. Then (2.17) becomes

$$\dot{z} = \bar{A}z + B\{\Delta v + \bar{\eta}(z, \Delta v)\} \tag{3.13}$$

where $\bar{A} = A + BK$, and

$$\begin{aligned} \bar{\eta}(z, \Delta v) &= \Phi(z) + \Psi(z)(Kz + \Delta v) \\ &= \bar{\Phi}(z) + \Psi(z)\Delta v \end{aligned} \tag{3.14}$$

We shall henceforth drop the overbars in (3.14) for convenience and assume that A is now a Hurwitz matrix.

Suppose that we can satisfy simultaneously the inequalities

$$||\eta|| \leqslant \rho(z,t) \tag{3.15}$$

$$||\Delta v|| \leqslant \rho(z,t) \tag{3.16}$$

for a known function $\rho(z,t)$. ρ can be determined as follows. Suppose first that Δv satisfies (3.16). Then from (3.14) we have

$$\begin{aligned} ||\eta|| &\leqslant ||\Phi|| + ||\Psi||\rho(z,t) \\ &\leqslant a + b\,||z|| + \alpha\rho(z,t) := \rho(z,t) \end{aligned} \tag{3.17}$$

Thus, ρ is defined implicitly via (3.17). This definition of ρ is well defined since $\alpha < 1$ and we have

$$\rho(z,t) = (1-\alpha)^{-1}\{a + b\,||z||\}. \tag{3.18}$$

Since A is Hurwitz, choose a symmetric, positive definite matrix Q and let P be the unique positive definite solution to the Liapunov equation

$$A^T P + PA + Q = 0. \tag{3.19}$$

We can now prove the following

Theorem 3.2: The system (3.13) is u.u.b. with respect to the set S (defined below) if the control Δv is chosen as

$$\Delta v = \begin{cases} -\rho(z,t)\dfrac{B^T Pz}{\|B^T Pz\|} \ ; \text{if } \|B^T Pz\| > \epsilon \\ \dfrac{-\rho(z,t)}{\epsilon} B^T Pz \ ; \text{if } \|B^T Pz\| < \epsilon \end{cases} \tag{3.20}$$

for a given $\epsilon > 0$ and ρ, P, B, z as above.

Proof: Note that the control Δv in (3.20) does indeed satisfy (3.16) and hence ρ satisfies (3.15). The basic idea of the proof is then to show that the function $V(z) = z^T P z$, which is a Liapunov function for the linear system (A, B), remains a Liapunov function for the nonlinear system (3.13) provided that η satisfies (3.15). The details of the proof can be found in [6],[9], where it is shown that the uniform ultimate boundedness set S is the ellipsoid

$$S(\bar{k}) = \left\{ z \in R^{2n} \mid z^T Pz < \bar{k} \right\} \tag{3.21}$$

with $\bar{k} = \lambda_{\max}(P)\omega^2$ and ω defined as

$$\omega = \frac{\epsilon b}{4\lambda_{\min}(Q)} + \left[\left(\frac{\epsilon b}{4\lambda_{\min}(Q)} \right)^2 + \frac{\epsilon a}{2\lambda_{\min}(Q)} \right]^{\frac{1}{2}} \tag{3.22}$$

where $\lambda_{\min}(\cdot)$, $\lambda_{\max}(\cdot)$ denote, respectively, the minimun and maximum eigenvalue of a matrix.

Remark 3.2: The control law (3.20) is continuous for each $\epsilon > 0$. Looking at (3.22) we see that for any values of a and b the uniform ultimate boundedness set S can be made arbitrarily small by decreasing ϵ. For $\epsilon = 0$ the system is asymptotically stable. In this case the control law (3.20) is a discontinuous or "chattering" control law [6]. However, for nonzero ϵ, the uniform ultimate boundedness result of Theorem 3.2 remains valid even if there is measurement uncertainty (noise, etc.) present in the system. (see [6])

It is also possible to show using these techniques that a strictly linear high gain control law can result in uniform ultimate boundedness of z in (3.13). Specifically, we can show

Theorem 3.4 There exists $\gamma > 0$ sufficiently large so that the state $z(t)$ governed by (3.13) is u.u.b. with Δv chosen as

$$\Delta v(t) = -\gamma B^T Pz \tag{3.23}$$

Remark 3.4: Theorem 3.4 follows from the work of Thorp and Barmish [7] in the case that there is no measurement uncertainty present in the system and follows from the results in [8] in case there is measurement uncertainty. We refer the reader to references [7],[8] for the calculation of γ and the ultimate boundedness set S in these cases.

3.3. LINEAR DYNAMIC COMPENSATOR DESIGN

Our third design methodology is a linear dynamic compensation scheme based on the method of stable factorization. This scheme is detailed in [18],[20].

In what follows R_+ will denote the set of nonnegative real numbers, and R^n will denote the usual n-dimensional vector space over R endowed with the Euclidean or L_2 norm

$$\|x\| = \left[\sum_{j=1}^{n} |x_j|^2 \right]^{1/2}. \tag{3.24}$$

If A is an $n \times n$ matrix over R, $\|A\|$ will be the corresponding induced norm

$$\|A\| = [\lambda_{\max}(A^T A)]^{1/2} \tag{3.25}$$

Let $L_\infty^n(R_+)$ denote the set of Lebesgue measurable functions $F: R_+ \to R^n$ such that the L_∞^n-norm $\|f\|_\infty < \infty$} defined by

$$\|f\|_\infty = \operatorname*{ess\,sup}_{t \in [0,\infty)} \|f(t)\| \tag{3.26}$$

is finite.

The so-called *extended* L_∞^n-space is defined as

$$L_{\infty e}^n = \{ f : R_+ \to R^n \mid f_T \in L_\infty^n \text{ for all } T \}$$

where

$$f_T = \begin{cases} t, & 0 \leq t \leq T \\ 0, & T \leq t \end{cases} \tag{3.27}$$

For convenience, we use the notation $\|f\|_{T\infty}$ to denote the quantity $\|f_T\|_\infty$.

Set $G(s) = (sI - A)^{-1}B$. Note that $G(s)$ represents a set of m uncoupled chains of integrators of length κ_j, $j = 1, \cdots m$ since A, B is in Brunovsky form. $g(t)$ will denote the corresponding impulse response function of $G(s)$.

With slight abuse of notation we will use the convention that if $M(s)$ is a transfer function matrix and $x = x(t)$ is a (Laplace transformable) signal then by Mx we shall mean $(m*x)(t)$, where $m(t)$ is the impulse response of $M(s)$ and $*$ denotes the convolution operator. With this notation then the system (2.15) may be described by the block diagram of Figure 1 where the lower loop has been closed by a linear dynamic compensator.

$$\begin{aligned} z &= G\epsilon, \ \epsilon = \eta + v, \ v = Cz, \\ \eta &= \Phi(z) + \Psi(z)v. \end{aligned} \tag{3.28}$$

The first three equations represent the linear part of the system, while the last equation represents the nonlinear coupling between η and the other signals. The first three equations can be solved to give

$$\begin{aligned} z &= G(I - CG)^{-1}\eta \\ v &= CG(I - CG)^{-1}\eta. \end{aligned} \tag{3.29}$$

Let $P_1 = G(I - CG)^{-1}$, $P_2 = CG(I - CG)^{-1}$. Define the norm of a transfer matrix as follows[10]:

$$\|P_i\|_{\bar{A}} = \sup_{x \in L^{n\infty} - \{0\}} \frac{\|P_i x\|_\infty}{\|x\|_\infty}. \tag{3.30}$$

Finally, let β_i denote $\|P_i\|_\infty$. Then [10,15]

$$||P_i v||_{T\infty} \leqslant \beta_i ||v||_{T\infty} \tag{3.31}$$

for all i, v, T.

From (3.29)

$$||z||_{T\infty} \leqslant \beta_1 ||\eta||_{T\infty} \tag{3.32}$$

$$||v||_{T\infty} \leqslant \beta_2 ||\eta||_{T\infty} . \tag{3.33}$$

Combining the above inequalities with (3.28), (2.24), (2.25) yields

$$\begin{aligned} ||\eta||_{T\infty} &\leqslant ||\Phi(z)||_{T\infty} + ||\Psi(z)v||_{T\infty} \\ &\leqslant a + b||z||_{T\infty} + \alpha ||v||_{T\infty} \\ &\leqslant (b\beta_1 + \alpha\beta_2)||\eta||_{T\infty} + a \end{aligned} \tag{3.34}$$

Thus from (3.34) we see that if

$$b\beta_1 + \alpha\beta_2 < 1 \tag{3.35}$$

then setting

$$\Delta = 1 - b\beta_1 - \alpha\beta_2 \tag{3.36}$$

and letting $T \to \infty$ we have

$$||\eta||_\infty \leqslant \frac{a}{\Delta} \tag{3.37}$$

$$||v||_\infty \leqslant \frac{\beta_2 a}{\Delta} \tag{3.38}$$

The above result is actually a special case of a multi-loop version of the *small gain theorem*[10] first proved in [12] and shows that the control signal $v(t)$ and the "uncertainty" η are bounded in L_∞ provided the modeling assumptions and (3.35) are satisfied. Since the output $z(t)$ is given by

$$z = P_1 \eta \tag{3.39}$$

a simple calculation gives an explicit bound on z as a function of the uncertainty (represented by η)

$$||z||_\infty \leqslant \frac{\beta_1 a}{\Delta} . \tag{3.40}$$

Thus the condition $\Delta > 0$, i.e., (3.35) is a sufficient condition for tracking with internal stability of the system represented by (2.18).

It remains to show that a compensator $C(s)$ can be designed in such a way that the stability condition (3.35) is satisfied. It can be shown (see [18]) that it is possible to choose a compensator making β_1 arbitrarily close to zero while making β_2 arbitrarily close to one simultaneously. Thus under the assumption $\alpha < 1$ we see that the stability condition (3.35) can be satisfied. Note that is also follows from (3.40) that the L_∞ norm of the output z can be made arbitrarily close to zero. The basis for the design is the stable factorization approach developed during recent years by various researchers and given an exposition in [15]. The reader is referred to [18] for the details of the calculation.

4. EXAMPLE: RIGID ROBOT CONTROL

Consider, as an illustrative example the problem of trajectory tracking for an n-link rigid manipulator whose equations of motion are given by [1],

$$M(q)\ddot{q} + h(q,\dot{q}) = u \tag{4.1}$$

$M(q)$ is the $n \times n$ inertia matrix, $h(q,\dot{q})$ is the vector of Coriolis, centrifugal, and gravitational terms and $u(t)$ represents the input torque to each joint at time t .

Let

$$q^d(t) = (q_1^d(t), \cdots, q_n^d(t))^T \tag{4.2}$$

represent a desired path in joint space that we wish the manipulator to track. We shall assume that $q^d(t)$ is continuously differentiable with q^d , $\dot{q}^d$, $\ddot{q}^d$ belonging to $L_\infty^n(R_+)$.

For the problem of tracking the desired trajectory (4.2) and its velocity we form the position and velocity error vectors

$$\dot{z}_1 = q - q^d \quad , \quad \dot{z}_2 = \dot{q} - \dot{q}^d \tag{4.3}$$

The error dynamics may then be written as a first order vector differential equation

$$\dot{z}_1 = z_2 \tag{4.4}$$

$$\dot{z}_2 = -M^{-1}h + M^{-1}u - \ddot{q}^d \tag{4.5}$$

so that in "error space" the problem of path tracking reduces to the problem of stabilizing the system (4.4)-(4.5).

We define a so-called "available model" of (4.1) as

$$\hat{M}(q)\ddot{q} + \hat{h}(q,\dot{q}) = u \tag{4.6}$$

where $\hat{M}$, $\hat{h}$ represent simplified or estimate values of M, h, respectively.

Given the plant (4.1) and the available model (4.6) the feedback linearizing control (2.15) is given by

$$u(t) = \hat{M}(q)(\ddot{q}^d + v) + \hat{h}(q,\dot{q}) \tag{4.7}$$

Equation (4.7) must be computed in real-time for a given sample rate (typically 60-100 Hz) for a six-link manipulator. Thus it is necessary both to consider carefully the micro-processor architecture chosen to implement (4.7) and also to simplify (4.7) as much as possible while still guaranteeing an acceptable response from the system.

Substituting the control law (4.7) into (4.5) we have

$$\dot{z}_1 = z_2 \tag{4.8}$$

$$\dot{z}_2 = -M^{-1}h + M^{-1}(\hat{M}(\ddot{q}^d + v) + \hat{h}) - \ddot{q}^d \tag{4.9}$$

Set

$$\Psi = M^{-1}\hat{M} - I \tag{4.10}$$

A straightforward calculation shows that

$$\dot{z}_2 = v + \eta \tag{4.11}$$

where η is given by

$$\begin{aligned} \eta &= \Psi(\ddot{q}^d + v) + M^{-1}(\hat{h} - h) \\ &= \Psi v + \Phi \end{aligned} \tag{4.12}$$

with $\Phi = \Psi\ddot{q}^d + M^{-1}(\hat{h} - h)$ Finally, the error equations (4.8),(4.9) may be written in matrix form

$$\dot{z} = Az + B(v + \eta) \tag{4.13}$$

where

$$A = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} \; ; \; B = \begin{bmatrix} 0 \\ I \end{bmatrix} \; ; \; z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}, \qquad (4.14)$$

and we have recovered the form of (2.18) to which our design schemes can now be applied. The modeling assumptions become in this context,

1) There exists a nonnegative constant $\alpha < 1$ such that for $q \in R^n$

$$|| M(q)^{-1}\hat{M}(q) - I || \leq \alpha \qquad (4.5)$$

and

2) There exist nonnegative constants a and b such that for q , $\dot{q} \in R^n$

$$|| \Psi \ddot{q}^d + M^{-1}(\hat{h} - h) || \leq a + b || z || \qquad (4.6)$$

Now we define the sets

$$\Omega_\Psi(z) = \{ R^{2\times 2} : || \Psi(z) || \leq \epsilon < 1 \}$$

$$\Omega_\phi(z) = \{ \phi(z) \in R^2 : || \phi(z) || \leq a + b || z || \}$$

with $a, b > 0$. We define the switching surfaces s_i as $\{ z : m_{i1} z_{i1} + z_{i2} = 0 \}$; $i=1,2$. In ideal sliding conditions, the resulting dynamics are simply

$$\dot{z}_{i1} = -m_{i1} z_{i1}$$

$$\dot{z}_{i2} = -m_{i2} z_{i2}$$

which represents an asymptotically stable motion of the state trajectory to the origin when m_{i1} , m_{i2} are positive.

Let $\bar{z} = (z_{ii}, z_{21})^T$. In this case the set $\Omega_{EQ}(\bar{z})$ is equal to the set

$$\{ v(\bar{z}) \in R^2 : || v(\bar{z}) || \leq \frac{1}{1-\epsilon} [a + (b + || N ||) || \bar{z} ||] \}$$

with $|| N || = (\sum_{j=1}^{2} m_{j1}^4)^{\frac{1}{2}}$.

A controller of the form

$$v_i = -f_{i1} z_{i1} - d_{v_i} \; ; \; i=1,2$$

$$with f_{11} = \begin{cases} \alpha_{11} > 0 \text{ for } s_1 z_{11} > 0 \\ \beta_{11} < 0 \text{ for } s_1 z_{11} < 0 \end{cases}$$

$$f_{21} = \begin{cases} \alpha_{21} > 0 \text{ for } s_2 z_{21} > 0 \\ \beta_{21} < 0 \text{ for } s_2 z_{21} < 0 \end{cases}$$

$$d_{v_1} = d_1 sgn \; s_1$$

$$d_{v_2} = d_2 sgn \; s_2$$

achieves sliding motions on $s_1=0$, $s_2=0$. Under the hierarchy $s_1 \rightarrow s_2$ the required gains and relay terms are obtained as :

$$(1-\epsilon)\alpha_{i1} \geq -m_{i1}^2$$

$$(1+\epsilon) d_2 \geq \frac{1+\epsilon}{1-\epsilon} [a + (b + || N ||) \sqrt{|| \bar{z} ||^2 + || s ||^2}$$

$$:= h_2(|| \bar{z} ||, \epsilon)$$

$$-(1+\epsilon) d_2 \geq -h_2(|| \bar{z} ||, \epsilon)$$

(which is automatically satisfied)

$$(1-\epsilon) d_1 \geq \frac{1}{1-\epsilon} (a + (b + || N ||) \sqrt{|| \bar{z} ||^2 + || s ||^2}$$

$$+ \epsilon \sqrt{\sum \max\{\alpha_{2j}^2, \beta_{2j}^2\}} \; || \bar{z} || + d_2)$$

$$:= h_1(|| \bar{z} ||, \epsilon)$$

$$(1+\epsilon) d_1 \leq -h_1(|| z ||, \epsilon)$$

(which is automatically satisfied).

Given a set of initial positions

$$|| \bar{z} ||^2 = z_{11}^2 + z_{21}^2 \leq \delta^2$$

$$s_1^2 + s_2^2 \leq \mu^2$$

the required relay terms are established.

4. REFERENCES

[1] Bejczy, A.C., "Robot Arm Dynamics and Control", JPL Technica Memo., 33-369, Feb. 1974.

[2] Hunt, L.R., Su,R., and Meyer, G., "Design for Multi-input Non linear Systems", *Differential Geometric Control Theory Conf.*, Bir khauser, Boston, pp. 268-298, (1983).

[3] Paul, R.P., *Robot Manipulators: Mathematics, Programming and Control*, MIT Press, Cambridge, Mass., (1982).

[4] Slotine, J.J., and Sastry, S.S., "Tracking Control of Nonlinea Systems Using Sliding Surfaces with Application to Robot Mani pulators", *Int. J. Control*, Vol. 38, no. 2, pp 465-492, (1983).

[5] Gutman, S., "Uncertain Dynamical Systems--A Liapunov min max Approach", *IEEE Trans. Aut. Cont.* Vol. AC-24 No. 3 pp437-443, June 1979.

[6] Spong, W.W., Thorp, J.S., and Kleinwaks, J.M., "The Control o Robot Manipulators with Bounded Input Part II: Robustness and Disturbance Rejection", *Proc. 23rd IEEE Conference on Decision And Control*, Las Vegas, (Dec. 1984).

[7] Thorp, J.S., and Barmish, B.R., "On Guaranteed Stability of Unc ertain Linear Systems via Linear Control", *J. of Opt. Theory and Appl.*, Vol. 35, No. 4, pp. 559-579, Dec. 1981.

[8] Barmish, B.R., Petersen, I.R., and Feuer, A., "Linear Ultimat Boundedness Control of Uncertain Dynamical Systems" *Automatica*,V. 19, No. 5, pp 523-532, (1983).

[9] Leitmann, G., "On the Efficacy of Nonlinear Control in Uncertain Linear Systems", *Journal of Dynamic Systems, Measurement, and Control*, June 1981, Vol. 102.

[10] Desoer, C.A., and Vidyasagar, M., *Feedback Systems: Input Output Properties Academic Press*, New York, 1975.

[11] Vidyasagar, M., *Nonlinear Systems Analysis*, Prentice-Hall Englewood Cliffs, N. J., 1978.

[12] Porter, D. W. and Michel, A. N., "Input-Output Stability of Time Varying Nonlinear Multiloop Feedback Systems" *IEEE Trans Auto. Control*, Vol.AC-19, pp. 422-427, 1974.

[13] Michel, A. N. and Miller, R. K., *Qualitative Analysis of Large Scale Dynamical Systems*, Academic Press, New York, N. Y. (1977).

[14] Siljak, D. D., *Large-Scale Dynamic Systems: Stability and Structure*, North-Holland, Amsterdam, Holland, 1978.

[15] Vidyasagar, M., *Control System Synthesis: A Factorization Approach*, M. I. T. Press, Cambridge, MA, 1985 (to be published)

[16] Nett, C. N., Jacobson, C. A. and Balas, M. J., "A Connection Between State-Space and Doubly Coprime Fractional Representa tions," *IEEE Trans. Auto. Control*, Vol. AC-29, number 9, pp 831-832, September 1984.

[17] Spong, M.W., and Vidysagar, M., "Robust Linear Compensator Design for Nonlinear Robotic Control", *Proc. 1985 IEEE International Conf. on Robotics and Automation*, St. Louis, March 1985.

[18] Spong, M.W., and Vidyasagar, M., "Robust Nonlinear Control of Robot Manipulators", *Proc. 14th IEEE CDC*, Fort Lauderdale, December, 1985.

[19] Marino, R., and Nicosia, S., "Linear Model Following Control and Feedback Equivalence to Linear Controllable Systems", *Int. J. of Control*, Vol. 39, No. 3, pp. 473-485, 1984.

[20] Spong, M.W., "Robust Stabilization for a Class of Nonlinear Systems", *Proc. 7th Int. Symp. on the Mathematical Theory of Networks and Systems (MTNS), Stockholm, June 1985.*

[21] Utkin , V.I., *Sliding Modes and Their Applications to Variable Structure Systems*, MIR Publishers, Moscow, 1978.

[22] Utkin , V.I., "Variable Structure Systems: Present and Future", *Automation and Remote Control*, Vol.44, pp 1105-1120, 1984.

[23] BAllestrino, A., DeMaria, G., and Zinober, A.,"Nonlinear Adaptive Model-Following Control", *Automatica*, Vol.20, No.5, pp.559-568, 1984.

[24] Young , K.K.D., "Controller Design for a Manipulator Using the Theory of Variable Structure Systems ", *IEEE Trans. on Systems, Man, and Cybernetics*, Vol. SMC-8, No. 2, February 1978.

[25] Utkin, V.I., and Young, K.K.D., "Methods for Constructing Discontinuity Planes in Multidimensional Variable Structure Systems", *Automation and Remote Control*, Vol. 26, pp 1466-1470, 1978.

[26] Ha, In-Joong, and Gilbert, E.G., "Robust Tracking in Nonlinear Systems and Its Application to Robotics", preprint (1985).

[27] Singh, S.N., "Adaptive Model Following Control of Nonlinear Robotic Systems", *IEEE Trans. Automatic Control*, vol. AC-30, No. 11, pp 1099-1100, Nov. 1985.

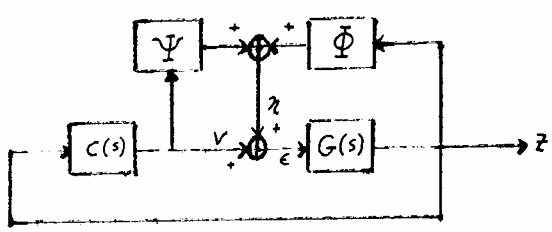

Figure 1 : Block Diagram for Linear Dynamic Compensation scheme.