A DIFFERENTIAL GEOMETRIC APPROACH FOR THE DESIGN OF VARIABLE STRUCTURE SYSTEMS*

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Abstract

This article presents a differential geometric approach for the design of feedback Variable Structure Controllers (VSC) acting on smooth systems whose trajectory evolves on a smooth manifold in R^n. Complete characterization of the ideal sliding dynamics and local reachability conditions are obtained in terms of geometric conditions which are intuitively appealing.

1. INTRODUCTION

Applications of differential geometry concepts to the solution of nonlinear control problems has been the subject of intensive research in the past few years. The theoretical and practical implications involved in the amount of available research is the topic of excellent surveys [1],[2], and books [3], where the reader is referred to for more detailed information.

The theory of Variable Structure Systems (VSS) and their associated "sliding mode" behavior [4] has also undergone extensive and detailed studies in the last twenty-five years, especially in the Soviet Union. Survey articles [5],[6], and books [7],[8] contain lucid expositions on the state of the art and its potentials for the future.

A surface, or manifold in the state space represents static relationships among the different state variables which describe the system. If these relationships are enforced on the dynamic description of the system, the resulting reduced order dynamics may sometimes contain highly desirable features. The idea is then to specify a feedback control action which guarantees accessibility of the prescribed surface and then proceeds to maintain the systems motions constrained to this surface. The task is usually accomplished by opportune, drastic changes on the structure of the feedback controller which induce motions invariably directed towards the surface.

In this article we explore, within the differential geometric framework, the problem of designing VSC leading to sliding regimes in nonlinear, stationary smooth plants with multiple inputs.

The approach allows for a well-defined characterization of the "equivalent control" and the ideal sliding dynamics representing the invariant motions of the system on the sliding submanifold. A geometric interpretation of the submanifold invariant condition calls for the annihilation of drift vector field components along directions not spanned by the sliding distribution (tangent space to the sliding sub-manifold). The sliding submanifold specification can then be viewed as an optimal control problem, a feedback stabilization problem, or simply as an algebraic nonlinear root-solving problem for the reduced ideal sliding system. For the single input case, general reachability conditions of the sliding submanifold can be proposed in a local sense. The relation of these conditions to the geometry of the surface and to the equivalent control are both transparent and appealing.

In Section 2 we formulate and solve the sliding motion creation process for a large class of nonlinear stationary smooth systems. In this section the interpretation of the equivalent control and the equations for the ideal sliding regime are obtained through a regularization procedure of the system equations. We obtain general existence conditions for the sliding submanifolds which renders desirable dynamic behavior of the reduced ideal sliding motions.

Section 3 is devoted to some illustrative examples. Section 4 contains the conclusions and suggestions for further research. Background material on differential geometry is used in the style of [3].

2. PROBLEM FORMULATION AND MAIN RESULTS

2.1. Notation, Definitions, and Main Assumptions

Consider the nonlinear dynamic system:

$$\Sigma : \frac{d}{dt} x = f(x) + G(x)\, u \quad ; \; x(t_0) = x_0$$
$$= f(x) + \sum_{i=1}^{m} g_i(x) u_i(x) \tag{2.1}$$

where x is a local coordinate system on a smooth n-dimensional manifold M which we usually take as R^n. The vectors f and g_i ($i \in \mathbf{m} =: \{1,2,\ldots,m\}$) are local representations of smooth vector fields in M.

We center our analysis on an open neighborhood N of x_0 in M on which the set of vector fields $\{g_i(x)\}$ is linearly independent, i.e., everywhere in N, they span a full rank smooth m-dimensional distribution of T_xM. This distribution is denoted by $\Delta_G(x)$. In our assertions, a property is <u>local</u> whenever it is valid only on N.

The control functions $u_i : M \to R$ are real valued discontinuous functions of the form

$$u_i(x) = \begin{cases} -u_i^+(x) & \text{for} \quad s_i(x) > 0 \\ -u_i^-(x) & \text{for} \quad s_i(x) < 0 \end{cases} \tag{2.2}$$

where $s_i : M \to R (i \in \mathbf{m})$ is a smooth function for which the set $\mathbb{S}_i = \{x \in M : s_i(x) = 0\}$ defines a smooth submanifold (hypersurface) of M. $\mathbb{S}_i$ is called the <u>i-th sliding submanifold</u>. $T_x \mathbb{S}_i$ is a smooth distribution of the tangent space and coincides with $\text{Ker}\, ds_i$. We denote this distribution as $\Delta\mathbb{S}_i(x)$. The <u>sliding submanifold</u> $\mathbb{S}$ is defined as $\mathbb{S} = \bigcap_{i=1}^{m} \mathbb{S}_i$ and it is assumed to be a smooth m-dimensional submanifold. $T_x\mathbb{S}$ is an n-m dimensional smooth distribution which we call the <u>sliding distribution</u> and express it by $\Delta_{\mathbb{S}}(x) = \bigcap_{i=1}^{m} \Delta_{\mathbb{S}_i} = \bigcap_{i=1}^{m} \text{Ker}\, ds_i$.

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The functions s_i, $i \in \mathbf{m}$ may also be regarded as local coordinate functions. When this is the case we refer to them as the **surface coordinates**. The m-dimensional vector $s = \mathrm{col}(s_1(x), s_2(x), \ldots, s_m(x))$ is used to denote all surface coordinates at once.

Remark 2.1. According to the relative value of the state coordinate functions with respect to the value of the functions s_i, a unique structure will be valid for Σ. However, notice that the prescribed action for the VSC (2.2) leaves undefined the nature and value of the velocity vector field of Σ on the surfaces $\mathbb{S}_i$, $i \in \mathbf{m}$, and hence on $\mathbb{S}$. The specification of a smooth feedback control function which makes $\mathbb{S}$ into an invariant submanifold is known as the "equivalent control problem" [4]. The equivalent control function describes the motion of the system on $\mathbb{S}$ in an average sense. The actual motion of Σ in $\mathbb{S}$ under the persistent action of the VSC is the sliding regime. The ideal invariant motion resulting from the equivalent control is the ideal sliding dynamics or the equivalent dynamics [9].

2.2. Problem Formulation

We are required to specify a VSC of the form (2.2) so that the integral curves of Σ, locally, approach the smooth submanifold $\mathbb{S}$ and stay constrained to $\mathbb{S}$ thanks to the drastic active switching of the feedback controller. The average motion in $\mathbb{S}$ is deemed to be desirable in an appropriately formulated sense (stability, asymptotic stability, sustained oscillatory response, etc.).

2.3. Adaptation of Coordinates and Regularization

Definition 2.3. We define Σ to be in regular form [10] whenever a local coordinate system can be found in which the system is expressed as

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2) \\ \dot{x}_2 &= f_2(x_1, x_2) + G_2(x_1, x_2) \end{aligned} \tag{2.3}$$

with x_1 and x_2, n-m and m-dimensional vectors, respectively, of local coordinate functions.

Theorem 2.4. A local coordinate system exists in which (2.1) is in regular form if and only if, locally, $\Delta_G(x)$ is constant dimensional and involutive.

Proof: It is easy to see that Σ can be put in regular form if and only if $\Delta_G(x)$ is completely integrable [3]. By Frobenius theorem $\Delta_G(x)$ is completely integrable if and only if it is involutive and constant dimensional. □

Notice that from the constant dimensionality assumption on $\Delta_G(x)$ it follows that $\det G_2(x_1, x_2) \neq 0$ in N.

Proposition 2.5. $\det[\frac{\partial s}{\partial x} G(x)] \neq 0$ if and only if $\Delta_{\mathbf{S}}^{\perp}(x) \cap \Delta_G^{\perp}(x) = 0$.

Proof: Let $\det[\frac{\partial s}{\partial x} G(x)] = 0$ then either $G(x)$ or $\frac{\partial s}{\partial x}$ are not full rank m. Since $G(x)$ is assumed to be full rank, then the span of the unit vectors ds_i in the co-distribution that annihilates $\Delta_{\mathbf{S}}(x)$ is rank $k < n-m$. It then follows that $\Delta_{\mathbf{S}}(x)$ is also rank k and the span of $\Delta_{\mathbf{S}}(x) \oplus \Delta_G(x)$ is rank $m+k < n$. Therefore, $\Delta_{\mathbf{S}}^{\perp}(x) \cap \Delta_G^{\perp}(x) \neq 0$. On the other hand, the null intersection of the annihilating distributions is equivalent to $\Delta_{\mathbf{S}}(x) \oplus \Delta_G(x) = T_xM$, i.e., $\Delta_{\mathbf{S}}(x)$ and $\Delta_G(x)$ are full rank. The annihilating co-distribution of $\Delta_{\mathbf{S}}(x)$ is also full rank and then $\frac{\partial s}{\partial x}$ and G are both full rank. □

Corollary 2.6. $\det[\frac{\partial s}{\partial x_2} G_2(x_1, x_2)] \neq 0$.

Proof. In local coordinates x_1, x_2: $\frac{\partial s}{\partial x} G = \frac{\partial s}{\partial x_1} 0 + \frac{\partial s}{\partial x_2} G_2(x_1, x_2)$. □

Lemma 2.7. Let $\mathbb{S}$ be a smooth m-dimensional submanifold of M satisfying the assumptions of Section 2.1 then, locally, $\mathbb{S}$ can be always expressed as

$$\mathbb{S} = \{x \in M : \; s = x_2 + m(x_1) = 0\}\,. \tag{2.4}$$

Proof. By virtue of our assumptions on the smoothness of $\mathbb{S}$ and the full dimensionality of $T_x\mathbb{S} = \Delta_{\mathbf{S}}$, which is equivalent to the nonsingularity of the Jacobian matrix $\frac{\partial s}{\partial x_2}$, then, the implicit function theorem [3] guarantees, locally, the existence of the function $m(x_1)$. □

Without loss of generality we shall assume that $\mathbb{S}$ is of the form (2.4).

2.4. Ideal Sliding Dynamics and the Equivalent Control

In order to be able to relate the effects of the control input functions on the reachability of $\mathbb{S}$, as well as to have an assessment of the "equivalent control" and its associated ideal sliding dynamics we replace the x_2 coordinate functions by the surface coordinate functions s and let $x_2 = s - m(x_1)$. We then obtain

$$\begin{aligned} \dot{x}_1 &= \bar{f}_1(x_1, s) \\ \dot{s} &= \bar{f}_2(x_1, s) + \bar{G}_2(x_1, s)\, u \end{aligned} \tag{2.5}$$

where

$$\begin{aligned} \bar{f}_1(x_1, s) &= f_1(x_1, s - m(x_1)) \\ \bar{f}_2(x_1, s) &= f_2(x_1, s - m(x_1)) + \frac{\partial s}{\partial x_1} f_1(x_1, s - m(x_1)) \\ &= f_2(x_1, s - m(x_1)) + \frac{\partial m}{\partial x_1} f_1(x_1, s - m(x_1)) \\ \bar{G}_2(x_1, s) &= G_2(x_1, s - m(x_1))\,. \end{aligned} \tag{2.6}$$

We let

$$\bar{f} = \begin{bmatrix} \bar{f}_1 \\ \bar{f}_2 \end{bmatrix} \quad \text{and} \quad \bar{G} = \begin{bmatrix} 0 \\ \bar{G}_2 \end{bmatrix}.$$

Geometrically, this new change of coordinates amounts to considering $\Delta_{\mathbf{S}}(x)$ and $\Delta_G(s)$ as projection subspaces of the tangent space T_xM. Thus, the velocity vector field defining $\dot{s}$ is the sum of the projections of the involved vector fields onto $\Delta_G(x)$ along $\Delta_{\mathbf{S}}(x)$. On the other hand, $f_1(x_1, s - m(x_1))$ is the projection on $\Delta_{\mathbf{S}}(x)$ of the component $f_1(x_1, x_2)$ of the drift field, along $\Delta_G(x)$ (Fig. 1 illustrates the geometry of the problem).

Thus, in the local coordinate frames $\bar{f}_1(x_1, s)$ is the component of $\bar{f} + \bar{G}u$ on $\Delta_{\mathbf{S}}$ while $\bar{f}_2(x_1, s) + \bar{G}_2(x_1, s)u$ is the component of $\bar{f} + \bar{G}u$ on Δ_G. This fact allows for a simple interpretation of the role of the equivalent control and the nature of the ideal sliding dynamics.

Proposition. $\mathbb{S}$ is a locally invariant manifold for the system (2.5) if and only if the following two conditions are satisfied

$$\begin{aligned} &(1)\;\; s = 0 \\ &(2)\;\; \bar{f} + \bar{G}u \in \Delta_{\mathbf{S}} \end{aligned} \tag{2.7}$$

Proof. If $\mathbb{S}$ is invariant it follows that the manifold condition $s = 0$ is satisfied and that in order to have the integral curves of the system confined to $\mathbb{S}$, the velocity vector field is an element of the tangent space of this submanifold. The conditions are also easily seen to be sufficient.

The ideal sliding motion corresponds then to having all the components of the defining velocity vector field annihilated on directions not corresponding to the sliding distribution $\Delta_{\mathbf{S}}$. The equivalent control role is then to nullify all the components of f_1 and f_2 along Δ_G, i.e.,

$$\begin{aligned} u_{EQ}(x_1) = -G_1^{-1}(x_1, -m(x_1))\,[f_2(x_1, -m(x_1)) \\ + \frac{\partial m}{\partial x_1} f_1(x_1, -m(x_1))]\,. \end{aligned} \tag{2.8}$$

It is seen that a necessary and sufficient condition for the existence of a unique equivalent control is that $\det G_2(x_1,-m(x_1)) \neq 0$. This condition was previously linked to the transversality of Δ_G and $\Delta_{\mathbb{S}}$. The particular form (2.4) adopted for $\mathbb{S}$ makes this condition appear as independent of the geometry of $\mathbb{S}$.

The existence and uniqueness of the equivalent control is a crucial factor in the determination of the necessary gains to achieve a sliding motion. The cases where either $G_2(x_1,-m(x_1)) = 0$, or $\det G_2(x_1,-m(x_1)) = 0$, are known as singular cases and an equivalent control does not exist or else is nonunique. In these cases the sliding condition can be lost by excursions of the velocity vector fields along directions not spanned by the control input map on T_xM. If, in the case: rank $G_2 = k < m$, the condition

$$f_2(x_1,-m(x_1)) + \frac{\partial m}{\partial x_1} f_1(x_1,-m(x_1)) \in \operatorname{Im} G_2(x_1,-m(x_1))$$

is satisfied, then a nonunique equivalent control exists which makes $\mathbb{S}$ invariant. Namely, $\tilde{u}_{EQ} = u_{EQ} + u$ with $u \in \mathbf{N}(G_2)$, the null space of G_2, is also an equivalent control.

The ideal sliding dynamics is then governed on $\Delta_{\mathbb{S}}$ by

$$\dot{x}_1 = f_1(x_1,-m(x_1)) \tag{2.9}$$

which may be viewed as

$$\dot{x}_1 = f_1(x_1,v)\,; \quad v = -m(x_1) \tag{2.10}$$

Since according to (2.4) the function m completely defines the submanifold $\mathbb{S}$, it is seen that the manifold specification problem for the sliding regime is equivalent to the specification of an appropriate smooth feedback control law of the form $v = -m(x_1)$ on the ideal sliding dynamic equations (2.10).

Several approaches have been proposed in the literature of VSC design in connection with this problem [7],[11]. Among these, the most important are: optimal control, parametric optimization and pole placement in the linear case. The most simple minded approach is to try to "solve" for the control function v (i.e., x_2) once the desirable dynamics has been established.

2.4. Existence of a Sliding Submanifold with Desirable Equivalent Dynamics

Assume $\dot{x}_1 = f_d(x_1)$ is a reduced order dynamics defined as a desirable ideal dynamic behavior for the given plant. The following theorem answers the question of existence of a submanifold where such ideal evolution may take place.

Theorem 2.9 . Let $f_d(x_1)$ be a given local smooth vector field of dimension n-m, defined on an open neighborhood N of R^n. Then a smooth submanifold exists where $f_1(x_1,x_2) = f_d(x_1)$, locally, if the Jacobian matrix $\dfrac{\partial(f_1(x_1,x_2) - f_d(x_1))}{\partial x_1}$ is nonsingular.

Proof. Let $g : M \to R^{n-m}$ be defined by $g(x_1,x_2) = f_1(x_1,x_2) - f_d(x_1)$, then it follows from the implicit function theorem that $g(x_1,x_2) = 0$ defines a submanifold $x_1 = \phi(x_2)$ if $\dfrac{\partial g}{\partial x_1}$ is nonsingular. The result follows. □

The above theorem provides only a sufficient condition for the existence of a submanifold with desirable ideal sliding motion. The following simple example stresses the fact that the conditions are not necessary.

Example 2.10. Consider the system $x^{(n)} = f(x,\dot{x},\ldots,x^{(n-1)}) + ug(x,\dot{x},\ldots,x^{(n-1)})$ where (n) denotes the n-th time derivative. Suppose the reduced order system $x^{(n-1)} = m(x,\dot{x},\ldots,x^{(n-2)})$ has a desirable behavior that one would like to emulate as close as possible. By defining $x_1 = x$, $x_2 = \dot{x}, \ldots, x_n = x^{(n-1)}$ the plant and the desired dynamics are both in regular form

$$\dot{x}_i = x_{i+1}\,; \qquad i \in \mathbf{n-1}$$

$$\vdots$$

$$\dot{x}_n = f(x_1,\ldots,x_n) + ug(x_1,\ldots,x_n) \qquad g(x_1,\ldots,x_n) \neq 0$$

thus

$$f_1(x_1,\ldots,x_{n-1},x_n) = \mathrm{col}(x_2,x_3,\ldots,x_n)$$

while $f_d(x_1,\ldots,x_{n-1}) = \mathrm{col}(x_2,x_3,\ldots,x_{n-1},\ m(x_1,x_2,\ldots,x_{n-1}))$. It is easy to see that the Jacobian matrix of $f_1 - f_d$ with respect to $\underline{x}_1 = (x_1,\ldots,x_{n-1})^T$ is singular. Yet, the manifold $x_n = m(x_1,x_2,\ldots,x_{n-1})$, obtained by direct comparison of f_1 and f_d, yields exactly the desired sliding dynamics with the equivalent control given by:

$$u_{EQ} = -[f(x_1,-m(x_1)) - \frac{\partial m}{\partial x_1} f_d(x_1)]/g(x_1,-m(x_1))\,.$$

2.5. Local Reachability Conditions

Here we shall address the problem of specifying the variable structure controller gains (2.2) in order to guarantee, at least locally, reachability of the sliding submanifold. The geometric nature of these conditions is better understood in the context of single input systems while providing us with a good feeling of what to expect in the more complex case of multiple inputs, and how to go about it.

Let m = 1, then g(x) is one-dimensional and therefore involutive, $\Delta_{\mathbb{S}}(x)$ is n-1 dimensional and Δ_G is a one dimensional distribution. The vector x_2 is reduced to the last scalar component function x_n of x. The ideal sliding dynamics conditions for this case reduce to $s = 0$; $L_{f+gu}s = 0$ where L is the Lie derivative (directional derivative) of s in the direction of the vector field f + gu. It follows that the equivalent control exists and it is unique whenever $g_n(x_1,-m(x_1)) \neq 0$. In this instance we have

$$u_{EQ} = -\frac{f_n(x_1,-m(x_1)) + \dfrac{\partial m}{\partial x_1} f_1(x_1,-m(x_1))}{g_n(x_1,-m(x_1))}\,. \tag{2.11}$$

In order to have surface reachability the velocity vector field has to be pointed towards the surface at points in the neighborhood of s = 0. In other words, the directional derivative of the scalar function representing the surface with respect to the velocity vector field f + gu must have different signs on both sides of the surface. In the limit these conditions must hold and we have

$$\lim_{s\to 0^+} L_{f+ug}s < 0 \quad \text{and} \quad \lim_{s\to 0^-} L_{f+ug}s > 0$$

i.e., if the trajectories approach $\mathbb{S}$ from negative values of s, the control function is to produce a positive rate of approach to the surface in order to hit it. If the trajectories approach $\mathbb{S}$ by positive values of S, the control function should make S decrease and therefore have a negative rate of approach, in order to achieve s = 0.

Notice that

$$L_{f+ug}s = \frac{\partial s}{\partial x_1}\bar{f}_1(x_1,s) + \bar{f}_n(x_1,s) + \bar{g}_n(x_1,s)u$$

$$= \frac{\partial m}{\partial x_1} f_1(x_1,s-m(x_1)) + f_n(x_1,s-m(x_1)) + g_n(x_1,s-m(x_1))u$$

$$\lim_{s\to 0^+} L_{f+ug}s < 0 \Rightarrow \frac{\partial m}{\partial x_1} f_1(x_1,-m(x_1)) + f_n(x_1,-m(x_1)) + g_n(x_1,-m(x_1))u^+ < 0$$

i.e.,

$$u^+(x_1) > -\frac{\dfrac{\partial m}{\partial x_1} f_1(x_1,-m(x_1)) + f_n(x_1,-m(x_1))}{g_n(x_1,-m(x_1))}$$

$$= u_{EQ}(x_1)\,. \tag{2.12}$$

Similarly

$$\lim_{s \to 0^-} L_{f + ug} s > 0 \Rightarrow$$

$$u^-(x_1) < -\frac{\frac{\partial m}{\partial x_1} f_1(x_1, -m(x_1)) + f_n(x_1, -m(x_1))}{g_n(x_1, -m(x_1))}$$

$$= u_{EQ}(x_1) \qquad (2.13)$$

The variable structure feedback gains are responsible for the transversality condition that makes the trajectory reach the surface. Their stipulation is highly dependent upon the value of the equivalent control.

Essentially due to the conflictive situation that may arise among the input command actions, due to dynamic interaction in the multiple input case, several approaches have been proposed for the specification of multivariable feedback gains which lead to sliding surface reachability. These methods group under two categories: diagonalization (through either input space coordinate transformation, or else through surface coordinate transformation) and the "hierarchy of controls" method [5],[7]. In the hierarchy of controls method the collective achievement of $\mathbb{S}$ is obtained as a consequence of sequentially ordered individually efforts by the inputs to drive the state trajectory towards their respective individual submanifold $\mathbb{S}_i$ (though arbitrary, the hierarchical order is assumed given). The reachability conditions are enforced, and the necessary gains computed, for each input, under the assumption of having all members in higher positions of the hierarchy already in sliding regime, while those in lower positions apply only one of a finite number of possible individual feedback structures.

The diagonalization method through surface coordinates transformation can be viewed as a rather restrictive decoupling problem in which a diffeomorphic transformation of the form $\sigma = \Omega(s)$ is sought in order to obtain each coordinate σ_i affected only by u_i

$$\begin{aligned} \dot\sigma &= \frac{\partial \Omega}{\partial s} \bar f_2(x_1, \Omega^{-1}(\sigma)) + \frac{\partial \Omega}{\partial s} \bar G_2(x_1, \Omega^{-1}(\sigma))\, u \\ &= \tilde f_2(x_1,\sigma) + \tilde G_2(x_1,\sigma)\, u = \tilde f_2(x_1,\sigma) + \sum_{i=1}^{m} \tilde g_{2i}(x_1,\sigma)\, u_i \end{aligned} \qquad (2.14)$$

If we let Δ_{σ_i} denote the distribution Δ_{S_i} in the new coordinates and $\tilde g_i = \begin{bmatrix} 0 \\ \tilde g_{2i} \end{bmatrix}$, then the condition for input-surface coordinates decoupling is, in geometric terms

$$\text{span}\, \tilde g_i(x_1,\sigma) \subset \bigcap_{j \neq i} \Delta_{\sigma_j}(x_1,\sigma) = \bigcap_{j \neq i} \ker d\sigma_j . \qquad (2.15)$$

If this condition is satisfied with $\tilde g_{2i}(x_1,\sigma) \neq 0$ and $\text{span}\{\tilde g_i\} \oplus \Delta_{\sigma_i} = T_x M$ u_i may proceed to create a sliding motion on $\sigma = 0$. The problem for each submanifold reachability is reduced to a single input problem in which the interactions may be ignored. The individual achievements of $\mathbb{S}_i$ result in full sliding as it is easily seen from

$$\frac{d}{dt} \sigma^T \sigma = 2 \sum_{i=1}^{m} \sigma_i \dot\sigma_i = 2 \sum_{i=1}^{m} \sigma_i L_{\tilde f + \sum_{i=1}^{m} \tilde g_i u_i} \sigma_i$$

$$= \sum_{i=1}^{m} \sigma_i (L_{\tilde f + \tilde g_i u_i} \sigma_i + \sum_{j \neq i} u_j L_{\tilde g_j} \sigma_i) = \sum_{i=1}^{m} \sigma_i L_{\tilde f + \tilde g_i u_i} \sigma_i < 0 .$$

In the last equality we used the fact that $\tilde g_j \in \ker d\sigma_i$ $\forall i \neq j$. $\sigma = 0$ is therefore a "conditionally attractive manifold" [8]. As before, the feedback gains for the i-th input, which guarantee reachability of the particular $\mathbb{S}_i$, are computed taking as a reference level the i-th input equivalent control $u_{i_{EQ}}$ (see (2.12), (2.13)).

3. EXAMPLES

Example 3.1. Consider the system $\dot x_1 = -x_2 - x_1(x_1^2 + x_2^2 - u)$; $\dot x_1 = x_1 - x_2(x_1^2 + x_2^2 - u)$. For $u = 1$ this system has an asymptotically stable limit cycle on the circle $x_1^2 + x_2^2 = 1$. This example shows that the limit cycle may be reachable in finite time on any circle of radius r in the plane, by means of a VSC.

In this case $f = (-x_1 - x_1(x_1^2 + x_2^2)) \frac{\partial}{\partial x_1} + (x_1 - x_2(x_1^2 + x_2^2)) \frac{\partial}{\partial x_2}$ and $g = x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}$. $\Delta_G(x) = \text{span}\,\{x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}\}$ is one dimensional and involutive.

A transformation to regular form is accomplished by describing the system in polar coordinates: $\rho = \sqrt{x_1^2 + x_2^2}$; $\theta = tg^{-1}(x_2/x_1)$. Then $\frac{d}{dt}\theta = 1$ and $\frac{d}{dt}\rho = -\rho(\rho^2 - u)$ describe the system. The vector fields are now $\bar f = \frac{\partial}{\partial\theta} + \rho^3 \frac{\partial}{\partial\rho}$; $\bar g = -\rho \frac{\partial}{\partial\rho}$ while $\Delta_G = \text{span}\{-\rho \frac{\partial}{\partial\rho}\}$. Let $s = \rho - r$ be the surface coordinate equation with $r > 0$, constant. A sliding surface candidate, represented in this case by a circle of radius r, is obtained when $s = 0$. Then $\Delta_{\mathbb{S}} = \text{span}\{\frac{\partial}{\partial\theta}\}$ and $\Delta_{\mathbb{S}}^{\perp} = \text{span}\{\frac{\partial}{\partial\rho}\}$ while $\rho = s + r$. The equations for the sliding motion are simply

$$\frac{d}{dt}\theta = 1 = \frac{\rho - s}{r}$$

$$\frac{d}{dt} s = -(s + r)[(s + r)^2 - u] .$$

The ideal sliding conditions require that $s = 0$ and $\frac{\partial}{\partial\theta} - r[r^2 - u] \frac{\partial}{\partial\rho} \in \text{span}\{\frac{\partial}{\partial\theta}\}$ which is only possible if $r^2 - u = 0$, i.e., $u_{EQ} = r^2$ and the ideal sliding dynamics take place on $\rho = r$. It follows from the reachability conditions that the controller

$$u = \begin{cases} \alpha > r^2 & \text{for} \quad s < 0 \\ \beta < r^2 & \text{for} \quad s > 0 \end{cases}$$

achieve sliding motions on $s = 0$. In the original coordinates the control law is then $u = \alpha > r^2$ for $x_1^2 + x_2^2 - r^2 < 0$ and $u = \beta > r^2$ for $x_1^2 + x_2^2 - r^2 < 0$. Since the trajectories reach $\mathbb{S}$ traversally, the induced limit cycle is achieved in finite time (see Figs. 2a, 2b, and 2c).

Example 3.2. (Sliding modes on the Torus)

Consider the system of equations

$$\frac{dx_1}{dt} = x_2 ; \quad \frac{dx_2}{dt} = -\omega_1^2 x_1 , \quad \frac{dx_3}{dt} = u x_4 ; \quad \frac{dx_4}{dt} = -u\omega^2 x_3$$

in R^4. With u = constant, this system evolves on the direct product of two, 2-dimensional circles. The motion is thus representable as confined to a Torus in R^3. This surface in turn can be diffeomorphically represented in the plane R^2 by specifying two angular coordinates θ_1; longitude, and θ_2; latitude, modulo 2π. If additionally, a "pasting" of points in the square: $0 \leq \theta_1 \leq 2\pi$; $0 \leq \theta_2 \leq 2\pi$ is exercised by identifying the points $(\theta_1, 0)$ and $(\theta_2, 2\pi)$ as well as the pair of points $(0, \theta_2)$ and $(2\pi, \theta_2)$. The motion can be analyzed on this "mapping" of the torus by means of the differential equations

$$\dot\theta_1 = \omega_1 ; \quad \dot\theta_2 = u\omega_2$$

where θ_1 is the angle measured from x_2 towards x_1 in the plane x_1, x_2 and similarly θ_2 is measured from x_4 towards x_3 in the plane x_3, x_4. When $u = +1$ we have "inwards forward winding" and with $u = -1$ we obtain "outwards forward winding." Also $\frac{\omega_2}{\omega_1} > 1$ provides us with "fast winding" while $\frac{\omega_2}{\omega_1} < 1$ represents "slow winding." Let us assume $\frac{\omega_2}{\omega_1} > 1$ in our example.

In this case $f = \omega_1 \frac{\partial}{\partial\theta_1}$, $g = \omega_2 \frac{\partial}{\partial\theta_2}$; $\Delta g = \text{span}\{\frac{\partial}{\partial\theta_2}\}$. Let $s = \theta_2$. s=0 describes the zero latitude greater circle of the Torus. The tangent space to this submanifold is $\Delta_S = \text{span}\{\frac{\partial}{\partial\theta_1}\}$, then, $\dot{\theta}_1 = \omega$, $\dot{s} = u\omega_2$; the ideal sliding requires $\omega_1 \frac{\partial}{\partial\theta_1} + u\omega_2 \frac{\partial}{\partial\theta_2} \in \text{span}\{\frac{\partial}{\partial\theta_1}\}$, i.e., $u\omega_2 = 0$, $u_{EQ} = 0$. Thus the Variable Structure Controller $u = +1$ if $\theta_2 < 0$ and $u = -1$ if $\theta_2 > 0$ produces a sliding motion on $\theta_1 = 0$ as desired (see Fig. 3a).

A sliding motion can also be created around a closed or otherwise dense winding line of the torus. In this case $\theta_2 = K\theta_1$ for $0 < K < 1$ (deceleration of winding). Then $s = \theta_2 - K\theta_1$ and $\dot{\theta}_1 = \omega_1$, $\dot{s} = -K\omega_1 + u\omega_2$. $\Delta_S = \text{span}\{\frac{\partial}{\partial\theta_1} + K\frac{\partial}{\partial\theta_2}\}$ and $\Delta_G = \text{span}\{\frac{\partial}{\partial\theta_2}\}$. The ideal sliding conditions require $f + gu \in \Delta_S$, i.e.,

$$\omega_1 (\frac{\partial}{\partial\theta_1} + K\frac{\partial}{\partial\theta_2}) + (-K\omega_1 + u\omega_2)\frac{\partial}{\partial\theta_2} \in \text{span}\{\frac{\partial}{\partial\theta_1} + K\frac{\partial}{\partial\theta_2}\}$$

$\Rightarrow \omega_2 u = K\omega_1$; then $u_{EQ} = K\frac{\omega_1}{\omega_2} < 1$, thus obtaining a deceleration of the winding motion which is now ideally described by $\dot{\theta}_1 = \omega_1$, $\dot{\theta}_2 = K\omega_1$. Reachability is accomplished then by the VSC: $u = +1$ for $\theta_2 < K\theta_1$ and $u = -1$ for $\theta_2 > K\theta_1$ (see Fig. 3b). The following proposition is an easy consequence of well-known results about differential equations on the Torus (see Arnold [12]).

Proposition 3.1. If K is rational then the sliding motion occurs in a closed winding line (i.e., open subsets of the Torus are not reachable). Otherwise, the sliding submanifold is dense in the Torus and every point of the Torus is made reachable.

4. CONCLUSIONS AND SUGGESTIONS FOR FURTHER RESEARCH

In this article we have explored the possibility of using differential geometric concepts in the treatment of the design problem related to the induction of sliding regimes in systems regulated by Variable Structure Controllers. The approach is general enough to be directly applied to cases of nonlinear smooth systems whose state naturally evolves on differentiable manifolds embedded in R^n.

The interpretation, in geometric terms, of the key ingredients to be considered in the design problem: namely, specification of the sliding submanifold in terms of desirable invariant behavior, the notion of the equivalent control and finally, the local reachability conditions for the existence of a sliding regime result in a convenient, simple and sufficiently general methodology for the attack of this class of problems. The equivalent control is seen to play an essential role in the specification of the variable structure feedback gains by providing a reference level on which to assess the necessary feedback control action that achieves submanifold reachability. The approach also provides avenues in which the singular case can be conveniently treated and understood.

A number of issues deserve further attention in the future. Among them, exploration of the consequences of local controllability in the design problem. Applications to systems evolving on Lie groups may be of interest in certain aerospace applications. The possibilities of using the approach in the area of Power Systems has been only recently addressed in connection with the design of feedback regulators of different nature. The Variable Structure Control approach has been of significance in this area for the case of linearized plants; thus, consideration of the nonlinear problem from this viewpoint may prove fruitful. Finally, the area of asymptotic observers for nonlinear systems with variable structure gains is open for contributions. The geometric approach may hold the answers to many interesting practical problems in this field.

REFERENCES

[1] Andreev, Y N., "Differential geometric methods in control theory, *Automation and Remote Control, vol. 43, no. 10, pt. 1*, pp. 1231-1265, March 1983.

[2] Isidori, A., *The geometric approach to nonlinear feedback control: A survey*, Lecture Notes in Control and Information Sciences, vol. 44, Springer-Verlag, New York, 1982.

[3] Isidori, A., *Nonlinear Control Systems: An Introduction*, Lecture Notes in Control and Information Sciences, vol. 72, Springer-Verlag, New York, 1985.

[4] Utkin, V. I., *Sliding Modes and Their Applications to Variable Structure Systems*, MIR Publishers, Moscow, 1978.

[5] Utkin, V. I., "Variable structure systems with sliding modes," *IEEE Trans. on Automatic Control, vol. AC-22, no. 2*, April 1977.

[6] Utkin, V. I., "Variable structure systems: Present and future," *Automation and Remote Control, vol. 44, no. 9, pt. 1*, pp. 1105-1119, Sept. 1983.

[7] Utkin, V. I., *Sliding Modes in Problems of Optimization and Control*, Nauka, Moscow, (in Russian), 1981.

[8] Itkis, U., *Control Systems of Variable Structure*, Wiley, New York, 1976.

[9] Marino, R., "High-gain feedback in nonlinear control systems," *Int. J. Control, vol. 42, no. 6*, pp. 1369-1386, Dec. 1985.

[10] Luk'yanov, A. G. and V. I. Utkin, "Methods of reducing equations for dynamic systems to a regular form," *Automation and Remote Control, vol. 4*, pp. 5-15, 1981.

[11] Utkin, V. and K. K. D. Young, "Methods for constructing planes of discontinuity in multidimensional systems with variable structure," *Avtomatika i Telemehanika, vol. 10*, pp. 72-77, 1978.

[12] Arnold, V. I., *Ordinary Differential Equations*, MIT Press, Cambridge, 4th Edition, 1985.

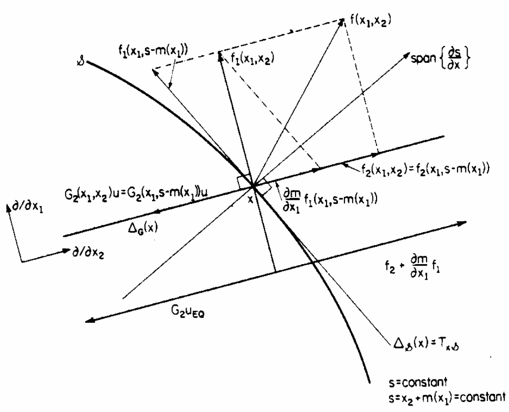

Fig. 1

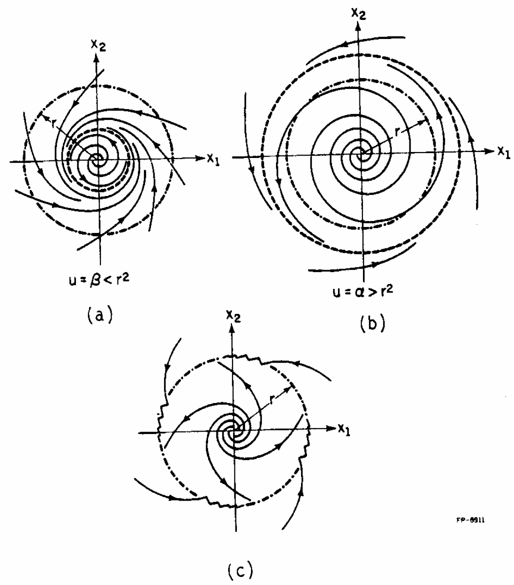

Fig. 2

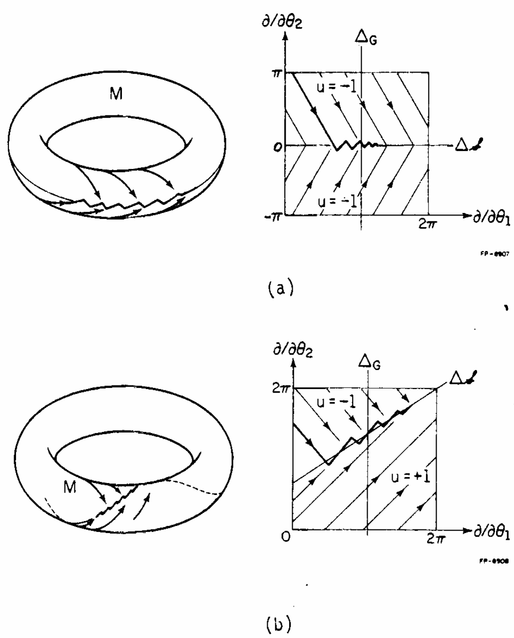

Fig. 3