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A GEOMETRIC APPROACH TO PULSE-WIDTH-MODULATED CONTROL DESIGN[1]

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Abstract

In this article a systematic method, of geometric nature, is proposed for the design of Pulse-Width-Modulated (PWM) control strategies in nonlinear dynamic systems. Necessary and sufficient conditions are presented for the existence of a local integral manifold of an ideal average system associated with the PWM controlled plant. This invariant manifold qualifies as a local sliding surface where ideal sliding trajectories coincide with the average PWM controlled response, provided the equivalent control and the duty ratio coincide as smooth feedback functions. Applications of the proposed method are presented for the feedback control design of Switchmode DC-to-DC Power Converters.

I. INTRODUCTION

A general equivalence is established among the sliding modes, resulting from a Variable Structure Control (VSC) strategy (Utkin,1978), and the response resulting from a Pulse-Width-Modulated (PWM) control scheme in nonlinear analytic sytems. This ideal equivalence constitutes the basis for a geometric framework in which PWM design problems can be systematically treated via the specification of a sliding surface on which the ideal sliding dynamics coincides with a well defined average PWM controlled response of the system.

Under the assumption of an infinite frequency duty cycle an ideal average PWM controlled response is precisely defined. This ideal response is shown to play the same role in PWM control strategies as the ideal sliding dynamics does in VSC schemes. Necessary and sufficiency conditions are presented for the existence of a local integral manifold for the average PWM controlled response. This manifold is shown to qualify as a local sliding surface for a VSC strategy which produces exactly the same average response of the PWM controlled system. This equivalence identifies the **duty ratio** of the PWM control option with the **equivalent control** (Utkin, 1978) of the VSC scheme.

The average behavior of the PWM controlled system is obtained from the sytem model just by replacing the discrete control input (switch position function) by a smooth feedback function of the state known as the **duty ratio** . The ideal sliding dynamics, obtained from the VSC scheme, is similarly obtained by replacing the switch position function by a smooth feedback input known as the **equivalent control**. The equivalent control and the duty ratio coincide when the integral manifold of the average PWM controlled system is taken as a sliding surface for the VSC option. Conversely, the ideal sliding dynamics of the VSC scheme adopts as an integral manifold that of the average PWM controlled system when the equivalent control is made to coincide with the corresponding duty ratio.

The above facts form the basis for a geometric method for the design of PWM feedback control strategies thrpugh VSC systems. The far simpler switching logic and drastically reduced feedback hardware demands of the equivalent VSC scheme make the approach especially attractive.

As an applications area of the proposed theory, a DC-to-DC power converter circuit is analyzed under the assumptions of a constant duty ratio (See Severns and Bloom, 1985, Middlebrook and Čuk, 1981, Venkataramanan *et al*, 1985). The computation of a local invariant set is considerably simplified by exploiting a time scale separation property of the linear average PWM controlled system (justified in terms of overdamped, nonoscillatory, response). This allows the use of a linear slow manifold as a sliding surface. This variety does not globally qualify as an integral manifold of the average system and therefore only local stable sliding regimes exist on such a switching surface.

Section II is devoted to present a general equivalence among PWM and VSC schemes on which the design method is based. Section III contains applications of the proposed geometric method for PWM feedback control design in a DC-to-DC switchmode power converter of the Boost type. Section IV summarizes the conclusions of the article.

II BACKGROUND AND GENERAL RESULTS

Consider the nonlinear analytic system defined in R^n :

$$\dot{x} = f(x) + ug(x) \qquad (2.1)$$

with f,g local smooth vector fields defined on an open neighborhood X of R^n. The scalar control function u represents a switch position function assumed to take values on the discrete set $U = \{0,1\}$.

A common feature of the VSC and PWM control schemes is the discontinuous character of the right hand side of (2.1) . Both schemes coincide in their essential features when their corresponding average behaviors are computed under the assumption of, respectively, infinitely fast switchings and infinite frequency duty cycles. The rest of the section is devoted to demonstrate this fact.

2.1 Generalities about Sliding Modes under VSC

The sliding mode control of (2.1) by means of a VSC scheme entitles the use of a feedback control law of the form :

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$$u = \begin{cases} 1 & \text{for} \quad s(x) > 0 \\ 0 & \text{for} \quad s(x) < 0 \end{cases} \qquad (2.2)$$

where the smooth function $s(x)$ determines a "switching surface" or a " sliding manifold " in R^n defined as :

$$S = \{ x \in R^n : s(x) = 0 \} \qquad (2.3)$$

It is assumed that the gradient of S , denoted by ds, is nowhere zero in X, i.e., S is constant rank. Also, it is assumed the S is a **locally regular manifold**, hence **locally integrable** (Boothby, 1975). S is oriented in such a way that ds points from the region where $s(x) < 0$ towards that where $s(x) > 0$

We shall henceforth qualify a result or assumption as <u>local</u> whenever x is restricted to X , an open neighborhood of R^n which has non-empty intersection with S.

Necessary and sufficient conditions for the local existence of a sliding mode on S are satisfied whenever the switching logic (2.2) and the controlled motion (2.1) are such that :

$$\lim_{s \to +0} L_{f+g}\, s < 0 \quad \text{and} \quad \lim_{s \to -0} L_f\, s > 0 \qquad (2.4)$$

where $L_h s$ denotes the **directional derivative** of the sliding surface coordinate function s in the direction of the vector field h , also denoted by $\langle ds, h \rangle$.

<u>Lemma 1</u> A necessary condition for the existence of a local sliding motion S , for the adopted switching logic (2.2), is that :

$$L_g s = \langle ds , g \rangle < 0 \qquad (2.5)$$

locally on S .

<u>Proof:</u> This is immediate upon substracting, on S, the inequalities representing the existence conditions (2.4), and the linearity of the directional derivative operator with respect to the sum of vector fields □.

The condition $L_g s < 0$ represents a **transversality condition** of the vector field g with respect to the surface S. Outside its region of validity, a sliding motion does not exist with the adopted switching logic.

According to (2.2) the state trajectories of the controlled motion of (2.1) are defined everywhere in X except at the surface of discontinuity S. Several definitions have been proposed to describe the solution of (2.1),(2.2) when a sliding regime exists locally on S. Filippov (1964) defines an average vector field, tangent to the sliding surface, describing the ideal sliding dynamics which generates the corresponding ideal trajectories. This average vector field is obtained by a **geometric average** or **convex combination** of the vector fields defined on each "side" of S. Utkin (1978) , on the other hand, defines the average controlled trajectories in terms of the response of the system to a smooth control function known as the **equivalent control** which renders the sliding surface as a **local integral manifold**. The equivalent control, denoted by $u_{EQ}(x)$ is obtained from the **invariance conditions** :

$$s = 0 \; , \; \dot{s} = L_{f + u_{EQ}(x) g}\; s = 0 \qquad (2.6)$$

The ideal sliding dynamics is then governed by :

$$\dot{x} = f(x) + u_{EQ}(x)\, g(x) \qquad (2.7)$$

Filippov's and Utkin's definitions of an ideal sliding mode generally lead to different results in more general settings. However, for the case at hand, they are totally equivalent.

We denote by $\mathcal{K}er\, ds$ the constant dimensional and involutive **tangent distribution** associated with S and defined as the $n-1$ dimensional subspace of the **tangent space** to X, $T_x X$, at each point of S which is orthogonal to ds. $\mathcal{K}er\, ds := \{h : \langle ds, h \rangle = 0\}$. From the definition of the ideal sliding dynamics (2.7) it follows immediately that :

$$f + u_{EQ}(x) g \in \mathcal{K}er\, ds \;\; \text{i.e.,} \;\; \langle ds, f + u_{EQ}(x)\, g \rangle = 0 \qquad (2.8)$$

In other words, the smooth vector field represented by $f + u_{EQ}(x)g$ will be locally tangent to the sliding surface at each point of existence of the sliding regime. As a consequence of this, the ideal sliding dynamics has as a local integral manifold (invariant set) the surface S.

<u>Lemma 2</u> The equivalent control, if it locally exists, is unique. i.e., the sliding manifold S uniquely determines the equivalent control for the system (2.1)

<u>Proof:</u> Let $u_{EQ1}(x)$ be also an equivalent control associated with a sliding mode locally created on S. We then have from the definition of equivalent control : $\langle ds, f + u_{EQ}(x)\, g \rangle = \langle ds, f + u_{EQ1}(x) g \rangle = 0$ and hence $(u_{EQ} - u_{EQ1}) \langle ds, g \rangle = 0$. Since by assumption the transversality condition is satisfied $\langle ds, g \rangle$ is non-zero. It follows that $u_{EQ} = u_{EQ1}$ locally on S. □

<u>Theorem 3</u> A necessary and sufficient condition for the existence of a sliding mode on S is that the equivalent control satisfies

$$0 < u_{EQ}(x) < 1 \qquad (2.9)$$

<u>Proof</u> Suppose a sliding motion exists on S . Then, from (2.8) the equivalent control is given by :

$$u_{EQ}(x) = -(L_g s)^{-1} L_f s = - [(\partial s/\partial x) g]^{-1} [(\partial s/\partial x) f] \qquad (2.10)$$

From (2.4) and the transversality condition, the above quantity is locally positive, i.e., $u_{EQ}(x) > 0$. On the other hand, using again (2.4) one obtains :

$$(L_g s)^{-1} L_{f+g}\, s = (L_g s)^{-1} (L_f s) + 1 = -u_{EQ}(x) + 1 > 0 \qquad (2.11)$$

which implies $u_{EQ} < 1$, locally.

To prove the reverse implication of the theorem, let (2.9) hold true for a smooth control function $u_{EQ}(x)$ which turns S into a local invariant manifold and assume that a sliding motion does not exist locally on S. Then, the inequality : $0 < 1 - u_{EQ}(x) < 1$ also holds true. By assumption, the smooth vector field generated by u_{EQ} will locally belong to the integrable distribution associated with S. In this region the following relation will be satisfied :

$$\langle ds, f + u_{EQ}(x)\, g \rangle = u_{EQ}(x) \langle ds, f + g \rangle + (1 - u_{EQ}) \langle ds, f \rangle = 0$$

It follows, necessarily, that the quantities $\langle ds, f + g \rangle$ and $\langle ds, f \rangle$ are opposite in signs on the surface S. The linearity of the inner product implies that $\langle ds, g \rangle$ can not be zero and thus the transversality condition is locally satisified since its sign can be arbitrarily established. We then have :

$$\langle ds, f+g \rangle |_{s=0} = \lim_{s \to +0} \langle ds, f+g \rangle = \lim_{s \to +0} L_{f+g}\, s < 0$$

$$\langle ds, f \rangle |_{s=0} = \lim_{s \to -0} \langle ds, f \rangle = \lim_{s \to -0} L_f\, s > 0$$

(2.12)

which means that the control law (2.2) locally creates a sliding mode on S. This is a contradiction and hence the Theorem is proved. □

<u>Remark</u> The pair of inequalities (2.9) determine, on S, the region, or regions, of local existence of a sliding mode. The intersection of the open regions defined by each inequality with both the sliding surface candidate S and the region of validity of the transversality condition determine the portion on S where a sliding mode exists. The VSC law (2.2) acting on the system (2.1) achieves such a sliding motion provided the initial state is close enough to S.

2.2 Generalities about Pulse-Width-Modulated Control

In a PWM control option, the scalar control u is switched once within a **duty cycle** of fixed small duration Δ. The instants of time at which the switchings occur are determined by the sample value of the state vector at the begining of each duty cycle. The fraction of the duty cycle on which the control holds the fixed value, say 1, is known as the **duty ratio** and it is denoted by $D(x(t))$. The duty ratio is usually specified as a smooth function of the state vector x. The duty ratio evidently satisfies $0 < D(x) < 1$. On a typical duty cycle interval, the control input u is defined as (See Figure. 1) :

$$u = \begin{cases} 1 & \text{for} \quad t \leq \tau \leq t + D(x(t))\Delta \\ 0 & \text{for} \quad t + D(x(t))\Delta \leq \tau \leq t + \Delta \end{cases} \tag{2.13}$$

It follows then that, generally :

$$x(t+\Delta) = x(t) + \int_t^{t+D(x(t))\Delta} \{ f(x(\tau)) + g(x(\tau))\} d\tau + \int_{t+D(x(t))\Delta}^{t} f(x(\tau))\, d\tau$$

$$= x(t) + \int_t^{t+\Delta} f(x(\tau))d\tau + \int_t^{t+D(x(t))\Delta} g(x(\tau))\, d\tau$$

The ideal average behavior of the PWM controlled system response is obtained by allowing the duty cycle frequency tend to infinity with the duty cycle length Δ approaching zero. In the limit, the above relation yields :

$$\lim_{\Delta \to 0} [x(t+\Delta)-x(t)]/\Delta =$$

$$\lim_{\Delta \to 0}(1/\Delta)[\int_t^{t+\Delta} f(x(\tau))d\tau + \int_t^{t+D(x(t))\Delta} g(x(\tau))\, d\tau]$$

i.e.,

$$\dot{x} = f(x) + D(x)\, g(x) \tag{2.14}$$

As the duty cycle frequency tends to infinity, the ideal average dynamics of the PWM controlled system is represented by the smooth response of the system (2.1) to the smooth control function constituted by the duty ratio $D(x)$. The duty ratio replaces the discrete control function u in the same manner as the equivalent control u_{EQ}, of the VSC scheme, replaces u in (2.1) to obtain (2.7).

We refer to (2.14) as the <u>average PWM controlled system.</u>

<u>Definition 4</u> An n-1 dimensional manifold Σ is said to be a **local integral manifold**, on an open neighborhood X of R^n, of the average PWM controlled system (2.14) if for some smooth $0 < D(x) < 1$, there exists a smooth function $\sigma : R^n \to R$ defining a constant dimensional, regular, n-1 dimensional manifold $\Sigma = \{ x \in R^n : \sigma(x) = 0 \}$, with gradient $d\sigma$ nowhere zero in X, and such that locally on Σ :

$$\langle d\sigma , f(x) + D(x)\, g(x) \rangle = 0 \tag{2.15}$$

From the above definition it follows that the duty ratio satisfies :

$$D(x) = - (\langle ds,g\rangle)^{-1}(\langle ds,f\rangle) = - (L_g s)^{-1}(L_f s) \tag{2.16}$$

Notice that $\langle d\sigma , g \rangle = 0$ makes $D(x)$ unbounded unless $\langle ds , f \rangle$ is itself zero, in which case $f + ug$ trivially belongs to the involutive tangent distribution *Ker* $d\sigma$ for all u and thus the trajectories are locally constrained to Σ irrespectively of the control function u. We therefore assume that, locally on Σ, the quantity $\langle d\sigma , g \rangle \neq 0$. From this assumption, it follows that, without loss of generality, we may consider $\langle d\sigma , g \rangle$ as a negative quantity in the region of interest. To see this, suppose $\langle d\sigma , g \rangle > 0$ and consider the following alternative definition of the integral manifold $\Sigma = \{ x: \sigma_1(x) = - \sigma(x) = 0 \}$. On the region of interest we now have : $\langle d\sigma , g \rangle = \langle -d\sigma_1, g \rangle = - \langle d\sigma_1, g \rangle > 0$ i.e., $\langle d\sigma_1 , g \rangle < 0$. From the above assumption and the definition of the duty ratio, it follows that in order to have $0 < D(x) < 1$, necessarily, $\langle d\sigma , f \rangle > 0$, locally on Σ.

<u>Lemma 5</u> If Σ is a locally integral manifold for (2.14), and $\langle d\sigma,g \rangle \neq 0$ locally on S, then Σ is also a local integral manifold for $\dot{x} = f(x) + D_1(x)\, g(x)$ if and only if $D(x) = D_1(x)$ in the region of interest.

<u>Proof</u> Sufficiency is obvious. To prove necessity suppose $D_1(x) \neq D(x)$ locally on Σ, but assume that Σ is a local integral manifold for both controlled systems. It follows from Definition 4 that on Σ := $\langle d\sigma, f + D(x)g \rangle = \langle d\sigma, f + D_1(x)g \rangle = 0$. From this equality it is easy to see that $\langle d\sigma,(D(x)-D_1(x))g \rangle = (D(x)-D_1(x)) \langle d\sigma, g \rangle = 0$. Since by hypothesis $\langle d\sigma, g\rangle \neq 0$, then, necessarily, $D(x) = D_1(x)$ locally on Σ. This is a contradiction and the lemma is established. □

<u>Theorem 6</u> A necessary and sufficient condition for Σ to be a local integral manifold of (2.14) is that locally on Σ :

$$\langle d\sigma , f + g \rangle < 0 \quad \text{and} \quad \langle d\sigma , f \rangle > 0 \tag{2.17}$$

<u>Proof:</u> Let Σ be a local integral manifold for (2.14), then $f+D(x)g$ satisfies (2.15) and from the definition of duty ratio (2.16) :

$$0 < -(\langle d\sigma,g\rangle)^{-1}(\langle d\sigma,f\rangle) = D(x) = -(\langle ds,g\rangle)^{-1}(\langle ds,f\rangle) < 1 \tag{2.18}$$

Using the hypothesis that $\langle d\sigma, g \rangle < 0$, it follows from the right hand side of (2.18) that $- \langle d\sigma, f \rangle > \langle d\sigma,g \rangle$ and therefore $\langle d\sigma, f + g \rangle < 0$. On the other hand, using the first inequality of (2.17), it follows that $-\langle d\sigma,f\rangle < 0$ i.e., $\langle d\sigma, f \rangle > 0$.

To prove sufficiency, suppose (2.17) holds true locally on Σ. Then, there exists positive smooth functions $a(x)$ and $b(x)$ such that on the region of interest. :

$$a(x)\langle d\sigma,f + g \rangle + b(x)\langle d\sigma,f \rangle = 0 \tag{2.19}$$

It then follows, rearranging the above expression, that $\langle d\sigma , f + a(x)[(a(x) + b(x)]^{-1} g \rangle = 0$ i.e., there exists a smooth control function $0 < D(x) = a(x)[a(x)+b(x)]^{-1} < 1$ such that, locally on Σ , $\langle d\sigma , f + D(x)\, g \rangle = 0$. In other words, in the region of interest, Σ is a local integral manifold of (2.14). □

In a similar form to the VSC case, equations (2.17) determine the regions of existence of a local integral manifold for the average PWM controlled system on Σ.

2.3 Ideal Equivalence of VS and PWM control.

The following theorem demonstrates that local integral manifolds of the average PWM controlled system, if they exist, qualify as a local sliding surfaces. When a sliding regime is created on such portions of the manifold, the corresponding equivalent control coincides with the duty ratio.

Theorem 7. Let Σ be a local integral manifold of (2.14). Then, a sliding mode exists on Σ in the same region where it qualifies as a local integral manifold. Moreover, the equivalent control corresponding to such a sliding motion coincides, locally with the duty cycle.

Proof: Since Σ is an integral manifold for the average PWM controlled system (2.14), then Theorem 6 applies and (2.17) holds true. It follows that locally on Σ :

$$\langle d\sigma, f+g \rangle = \lim_{\sigma \to +0} \langle d\sigma, f+g \rangle = \lim_{\sigma \to +0} L_{f+g}\sigma < 0$$

$$\langle d\sigma, f \rangle = \lim_{\sigma \to -0} \langle d\sigma, f \rangle = \lim_{\sigma \to -0} L_f \sigma > 0$$

i.e., the Variable Structure control law :

$$u = \begin{cases} 1 & \text{for } \sigma(x) > 0 \\ 0 & \text{for } \sigma(x) < 0 \end{cases} \quad (2.20)$$

used on system (2.1) creates a sliding mode locally on Σ.

To prove the second part of the theorem, notice that if a sliding mode exists on Σ then, necessarily, the transversality condition $\langle d\sigma, g \rangle = L_g \sigma < 0$ is satisfied. By definition, the corresponding equivalent control satisfies the invariance condition on Σ :

$$\langle d\sigma, f + u_{EQ}(x)\, g \rangle = 0 \quad (2.21)$$

but this implies that $u_{EQ}(x)$ also qualifies as a duty ratio. From the uniqueness of such a duty ratio, shown in Lemma 5, it follows that $u_{EQ}(x) = D(x)$ locally on Σ. □

The converse theorem completes the equivalence between VSC and PWM control schemes in their respective idealized features.

Theorem 8. If a local sliding motion exists on the n-1 dimensional, regular smooth manifold $S = \{x \in R^n : s(x) = 0\}$ then, locally, S qualifies as an integral manifold for the average PWM controlled system. Moreover, the duty ratio corresponding to this average system locally coincides with the equivalent control.

Proof: Suppose a sliding motion locally exists on S, then (2.4) holds true locally on S and hence the conditions for Theorem 6 are valid on S. Hence, S qualifies as a local integral manifold of the average PWM controlled system. Notice, furthermore, that from Theorem 3, $0 < u_{EQ}(x) < 1$ is satisfied in the region of interest. The equivalent control also turns S into a local integral manifold in the region where the inequalities (2.17) are valid. By virtue of the uniqueness of the duty ratio, the equivalent control necessarily coincides with the duty ratio as smooth functions of the state vector. □

Theorems 7 and 8 allow us to conclude :

The necessary and sufficient condition for the local existence of a sliding mode on an n-1 dimensional regular manifold S is that it also locally qualifies as an integral manifold for an average PWM controlled system in the region of existence of the sliding mode. In this region, the equivalent control and the duty ratio totally coincide.

It is, generally speaking, very difficult to compute an integral manifold for a system of nonlinear differential equations. However, for the class of linear time-invariant systems exhibiting a two time scale separation property, known as singularly perturbed systems, affine varieties containing the slow manifold of the system can be explicitely computed with considerable simplicity (Kokotovic, Khalil and O'Reilly 1986). Generally speaking, hypersurfaces or affine varieties containing slow manifolds are not, themselves, global integral manifolds.

III SLIDING MODE CONTROL OF DC-TO-DC SWITCHMODE POWER CONVERTERS

3.1 Sliding Motions on the Slow Manifold of the Boost Converter

Consider the Boost converter of Figure 2, where the state variables are defined as: $x_1 = I\sqrt{L}$, $x_2 = V\sqrt{C}$, and parameters ; $b = E/\sqrt{L}$, $w_0 = 1/\sqrt{LC}$, $w_1 = 1/RC$. The control "input" is represented by a discrete valued variable $u \in \{0,1\}$ representing an ideal switch position.

$$\begin{aligned} \dot{x}_1 &= -w_0 x_2 + u\, w_0 x_2 + b \\ \dot{x}_2 &= w_0 x_1 - w_1 x_2 - u\, w_0 x_1 \end{aligned} \quad (3.1)$$

As it was demonstrated in Section II of this article, the average response of the system to a constant duty ratio μ in a PWM control scheme is obtained by replacing u by the constant value $0 < \mu < 1$. The resulting system is governed by a linear time-invariant system. The equilibrium point of the average system is given by the coordinates :

$$x_{1ss} = w_1 b[(1-\mu)w_0]^{-2} \; ; \; x_{2ss} = b[(1-\mu)w_0]^{-1} \quad (3.2)$$

The corresponding characteristic equation is given by

$$\lambda^2 + w_1 \lambda + (1-\mu)^2 w_0^2 = 0 \quad (3.3)$$

The roots $\lambda_1(\mu)$, $\lambda_2(\mu)$ of this equation have negative real parts and hence, the average trajectories are asymptotically stable towards the equilibrium point. The damping coefficient ξ of the system is $\xi = w_1/[2\, w_0(1-\mu)]$, while the natural frequency of the average system coincides with $(1-\mu)w_0$. The value of the damping coefficient ξ determines the nature of the average response. Thus, for values of $\xi < 1$ the response is oscillatory, while for values of $\xi \gg 1$ one of the modes in the response, say $\lambda_2(\mu)$ is overdamped, while the other, $\lambda_1(\mu)$ represents a fast transient. In this case both eigenvalues are real and the corresponding eigenlines translated to the equilibrium point qualify as local integral manifolds called the slow and fast manifolds respectively. The damping coefficient is also a measure of the ratio of the time constant of the output circuit $w_1 = 1/RC$ to the natural frequency w_0 of the LC input circuit.

The affine variety containing the slow manifold of the different trajectories of (3.1), obtained from the above fact, is given by any of the following expressions :

$$S_\mu = \{ x \in R^2 : \sigma = x_2 + [(1-\mu)w_0]^{-1}\lambda_2(\mu)x_1 - b[(1-\mu)w_0]^{-1}\,[1+w_1\lambda_2(\mu)[(1-\mu)w_0]^{-2}] = 0 \} \quad (3.4)$$

$$S_\mu = \{ x \in R^2 : \sigma = x_2 - [\lambda_2(\mu) + w_1]^{-1}(1-\mu)w_0 x_1 - [\lambda_2(\mu)+w_1]^{-1}\lambda_2(\mu) b\,[(1-\mu)w_0]^{-1} = 0 \} \quad (3.5)$$

It is easy to see that the surface coordinate function in (3.4) or (3.5) is a solution, thanks to (3.3), of the partial differential equation representing the manifold condition (2.15):

$$(\partial s/\partial x_1)[b - w_0(1-\mu)x_2] + (\partial s/\partial x_2)[(1-\mu)w_0 x_1 - w_1 x_2] = 0$$

Particularizing the existence conditions (2.17) of Theorem 6 for the dynamical system (3.1) with $u = \mu$, leads to the following region of existence of a local integral manifold for the average PWM controlled system :

$$x_2 > \lambda_2(\mu) b \, [(1-\mu)\omega_o \omega_1]^{-1} \qquad (3.6)$$

$$\omega_o x_2 < [(1-\mu)\omega_1 + \lambda_2(\mu)]^{-1} \{ (1-\mu)\omega_o{}^2 x_1 + \lambda_2(\mu) b \} \qquad (3.7)$$

The condition : $\langle ds, g \rangle < 0$ is represented by an open hemispace given by the inequality constraint :

$$x_2 > - [(1-\mu)\omega_o]^{-1} (\lambda_2(\mu) + \omega_1) x_1 \qquad (3.8)$$

As it was justified in the previous section we restict our attention to the region on S_μ where these constraints (3.6)-(3.8) are valid. This unbounded subset of R^2 is shown in Figure 3. The local integral manifold is constituted by the unbounded portion of S_μ to the right of the point P_1 shown in that figure. S_0 and S_1 represent the slow manifolds for $u = 0$ and $u = 1$ respectively.

The average PWM trajectories locally evolve on the integral manifold (3.4) or (3.5). Using these relationships, the expressions for the average dynamics result in :

$$\dot{x}_1 = \lambda_2(\mu) \, (x_1 - \omega_1 b [(1-\mu)\omega_o]^{-2})$$
$$\dot{x}_2 = \lambda_2(\mu) \, (x_2 - b \, [(1-\mu)\omega_o]^{-1}) \qquad (3.9)$$

which represent asymtotically stable motion towards the equilibrium point (3.2)

Example 1.

Figure 4 depicts simulated state trajectories in a Boost converter, for $u = 0$, $u = 1$ and a PWM control scheme with constant duty ratio $\mu = D = 0.5$. The component values of the circuit are $L = 4$ mH, $C = 0.1\mu F$, $R = 100\ \Omega$ and $E = 20$ Volts. Here $\omega_1 = 10^5$ and $\omega_o = 5 \times 10^4$. The damping coefficient $\xi = 2$.

3.2 A Variable Structure Control Approach

The preceeding developments identified a portion of S_μ where a local integral manifold exists for the average PWM controlled trajectories. Next, S_μ is taken as the switching surface S for a VSC scheme creating a local sliding regime leading to stable equilibrium.

$$S_\mu = \{ x \in R^2 : s = x_2 + [(1-\mu)\omega_o]^{-1} \lambda_2(\mu) x_1$$
$$- b[(1-\mu)\omega_o]^{-1} [1 + \omega_1 \lambda_2(\mu) [(1-\mu)\omega_o]^{-2}] = 0 \} \qquad (3.10)$$

Using the definition of s in (3.10), the integrable distribution *Ker* ds is given by :

$$Ker\ ds = span\{(1-\mu)\omega_o [\lambda_2(\mu) + \omega_1]^{-1} \partial/\partial x_1 + \partial/\partial x_2\} \qquad (3.11)$$

The invariance condition $f + u_{EQ}(x)\, g \in Ker\ ds$ on $s = 0$ translates into :

$$\omega_o [\lambda_2(\mu) + \omega_1]^{-1} (\omega_1 x_1 + \lambda_2(\mu) b \omega_1 [(1-\mu)\omega_o]^{-2} + b)(\mu - u_{EQ}(x)) = 0$$

i.e.,

$$u_{EQ} = \mu \qquad (3.12)$$

which means that, along the valid portion of S_μ , the average behavior of the PWM controlled system and the ideal sliding dynamics are totally equivalent. As expected, the necessary and sufficient condtions for the existence of a sliding mode on S_μ , obtained from Theorem 3 (or, equivalently, from conditions (2.4)) lead to the determination of the region of existence, totally coincident with that found for the local integral manifold of the average PWM controlled system. The variable structure control law guaranteeing the existence of a sliding motion on S_μ is obtained from conditions (2.4) as :

$$u = \begin{cases} 1 & \text{for } s > 0 \\ 0 & \text{for } s < 0 \end{cases} \qquad (3.13)$$

whenever :

$$[x_1 + \lambda_2(\mu) b \, [(1-\mu)\omega_o]^{-2} + b\omega_1{}^{-1}] > 0 \qquad (3.14)$$

which along $s = 0$ is equivalent to :

$$x_2 > \lambda_2(\mu) b [(1-\mu)\omega_o \omega_1]^{-1} \qquad (3.15)$$

The existence conditions (2.4) reproduce the regions (3.6)and (3.7) previously computed in the PWM case. A sliding motion thus exists on S_μ to the right of the point P_1 in Figure 3.

The proposed approach, using the slow manifold characterized by a linear variety, exploits simple expressions for the sliding surface S_μ and only involves measurement of output voltage and input current while being realizable with operational amplifiers and resistors.

IV CONCLUSIONS

By establishing an ideal equivalence among Variable Structure and Pulse-Width-Modulated control schemes, a design procedure has been found which proposes the creation of a sliding regime on the integral manifold of an average PWM controlled system. The advantages of a VSC option, over a PWM control alternative, lies in the hardware simplicity needed for feedback synthesis and closed loop robustness. If on the contrary, a PWM control scheme is still prefered, the proposed design procedure allows for the computation of the necessary duty ratio as a truly feedback control law. The duty ratio is obtained as the equivalent control associated with the prescribed sliding manifold. This result unifies both approaches and renders an equivalence which was used to cast the design issues, associated with PWM control techniques, into a geometric framework, with the advantages of a more intuitive and fully systematic methodology.

In realistic applications, high frequency switchings or high frequency duty cycles are necessry to approximate the ideal behavior exploiting the equivalence among both approaches.

The method was used to obtain a Variable Structure Control law leading to a sliding mode on an affine variety containing the slow manifold of a DC-to-DC Switchmode Power Converter. Such possibilities were derived from desirable time scale separation properties among the natural frequency of LC input filter and the time constant associated with the RC output circuit. Being only a local integral manifold, the proposed affine variety does not globally satisfy the sliding mode existence conditions. However, the simplicity of the approach makes it attractive for hardware implementation since the switch position is determined only from sign information (i.e one bit data) about a scalar affine function of the converter's state.

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FIGURES

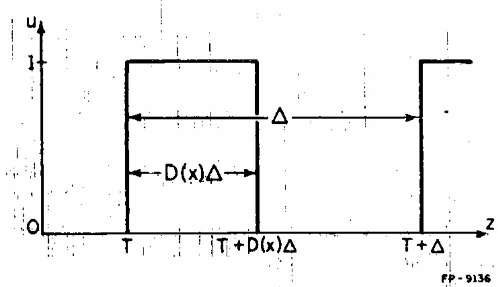

Figure 1.

Typical Duty Cycle in PWM Control

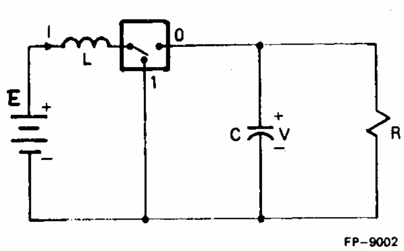

Figure 2.

Boost Converter

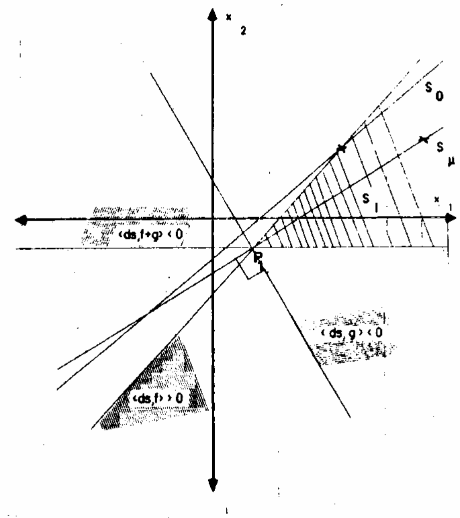

Figure 3.

Region of Existence of Integral (Sliding) Manifold

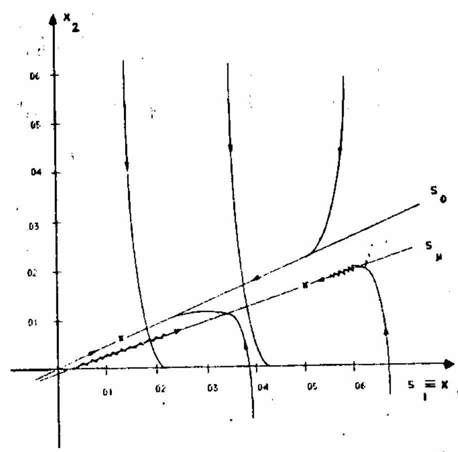

Figure 4.

Sliding Motions on Slow Integral Manifold