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Dynamical Discontinuous Feedback Control of Nonlinear Systems

Hebertt Sira-Ramirez
Departamento Sistemas de Control
Escuela de Ingenieria de Sistemas. Universidad de Los Andes.
Merida, VENEZUELA.

Pablo Lischinsky-Arenas
Departamento de Computacion
Escuela de Ingenieria de Sistemas. Universidad de Los Andes.
Merida, VENEZUELA.

Abstract. In this article, a technique is presented for the analysis of discontinuous dynamical feedback regulation in nonlinear systems. A PWM feedback interconnection scheme, with a general duty ratio function, is shown to be easily analyzable in terms of an average model which captures the essential features of the discontinuously feedback controlled system.

1. INTRODUCTION

Discontinuous feedback control of dynamical systems has been traditionally addressed under the assumption of static (or memoryless) feedback (See Utkin [1], Tsypkin [2]). Variable Structure Systems and other representatives of discontinuous control schemes, such as : Pulse-Width-Modulation and Pulse Frequency Modulation schemes have been restricted to classes of systems in which either output, output error, or, state feedback signals are directly pulsed, usually through a unity feedback loop, into the controlled system. The more realistic and general situation, within a discontinous feedback scheme, calls, however, for dynamical feedback, or interconnection, of the plant and the feedback subsystems constituted by state estimators, controllers, sensors and actuators whose dynamics can not be entirely neglected.

This article addresses, in full generality, the problem of analyzing dynamical discontinously fedback nonlinear controlled plants. The discontinuous feedback scheme is assumed to be constituted by a dynamical feedback plant, of nonlinear nature, and a controlled switch obeying a PWM type of switching strategy with sufficiently high sampling rate. It is found that the actual closed loop controlled responses of the system exhibit sliding regimes on certain average manifolds. These manifolds are inmmersed in the regions of the composite state space where the duty ratio function is not acting under saturation conditions. In fact, such sliding motions locally take place on integral manifolds of a suitable average system described in the (augmented) state space of the closed loop system. The average system is simply obtained by an infinite sampling frequency assumption on the PWM process. This article constitutes an extension, to nonstatic discontinuous feedback, of the work in Sira-Ramirez [3]-[5].

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Section 2 presents general results about PWM interconnection of dynamical systems in a feedback arrangement. Section 3 is devoted to an illustrative example of a discontinuous dynamical feedback scheme of a nonlinear system which includes a nonlinear feedback observer. Section 4 contains some conclusions and suggestions for further work in the area. The necessary background on PWM control is presented in the Appendix.

2. DEFINITIONS AND MAIN RESULTS

Consider the switched controlled interconnected system shown in figure 1. Such system is described by:

$$\begin{aligned} dx/dt &= f(x) + g(x)e_1 \\ y_1 &= h(x) \\ e_2 &= y_1 + u_2 \\ dz/dt &= \phi(z) + \gamma(z)e_2 \\ y_2 &= \eta(z) \\ e_1 &= u_1 - v\, y_2 \\ v &= PWM\,[s(t_k)] \\ s(t) &= s[e_1(t), e_2(t)] \end{aligned} \tag{2.1}$$

where f, g, ϕ and γ are smooth vector fields, x and z are smooth coordinate functions of R^n and R^p respectively, the functions h, η and s are smooth scalar functions of their arguments and the scalar signals u_1 and u_2 are assumed to be either external reference control inputs or external disturbances. The PWM operator is defined as :

$$PWM\,[s(t_k)] = \begin{cases} 1 & \text{for } t_k < t \le t_k + \tau[s(t_k)]T \\ 0 & \text{for } t_k + \tau[s(t_k)]T < t \le t_k + T \end{cases} \tag{2.2}$$

where T is a fixed (i.e., constant) sampling interval length, known as the duty cycle. and $\tau[s(t_k)]$ is a piecewise smooth function, known as the duty ratio function which takes values in the closed interval [0,1]. The duty ratio function represents the fractional length of the sampling interval in which the feedback interconnection is enabled, before it is switched off for the rest of the sampling interval. The notation $s(t_k)$ actually stands for $s[e_1(t_k), e_2(t_k)]$, for each t_k. If during a certain open interval of time the duty ratio function exhibits either the value 0 or 1, the PWM controller is said to be saturated or it is said to be acting under saturation conditions.

The analysis of (2.1),(2.2) is extremely difficult if one uses the discrete-time approximation scheme by which PWM systems have been traditional analyzed. This is so, even in the case of a linear dynamical plant interconnected to a static feedback system (See, for instance, Csaki [6, pp. 591]). Rather than using this route, we resort to a recent averaging technique, proposed in [3]-[5], used for studying nonlinear discontinuously controlled systems under static (memoryless) feedback. The essential features of this technique which are applicable to system (2.1) are summarized in the Appendix of this article.

Definition 2.1 We define the average system of (2.1) as the following dynamical interconnected system:

$$
\begin{aligned}
&dx/dt = f(x) + g(x)e_1\\
&y_1 = h(x)\\
&e_2 = y_1 + u_2\\
&dz/dt = \phi(z) + \gamma(z)e_2\\
&y_2 = \eta(z)\\
&e_1 = u_1 - w\, y_2\\
&w = \tau[s(t)]\\
&s(t) = s[e_1(t), e_2(t)]
\end{aligned}
\qquad (2.3)
$$

The average system (2.3) exhibits exactly the same structure as the original controlled system except for the fact that the feedback enabling switch, represented by the function v, is substituted by the duty ratio function $\tau[s(e_1,e_2)]$. It will be shown in the appendix that such a substitution process is justified by letting the sampling frequency, of the pulse modulator, reach an arbitrarily large rate. In other words, the average model (2.3) can be obtained from the original system (2.1) by allowing an infinite sampling frequency asumption on the PWM block. The advantage of the average model lies, precisely, in the smooth character of the controlled response which, incidentally, entirely coincides with that of the real system in the saturation regions of the PWM operator and is, moreover, arbitrarily close to the response of the real system in the nonsaturation regions. The nature of the approximation, on such nonsaturation regions, is characterized by the existence of a sliding regime about the average responses, or, more preceisely, by a sliding regime occuring about integral manifolds of the average model. (See Sira-Ramirez [3]-[5]).

The following theorem constitutes an extension of the main result presented in the Appendix .

Theorem 2.1 For identical initial conditions, the responses of system 2.1 entirely coincide with those of the average system 2.3 in the regions of the state space where the duty ratio function acts under saturation conditions. On the regions of nonsaturation (i.e., where the duty ratio function takes values in the open interval (0,1)), the responses of the actual PWM contolled system exhibit a sliding motion about integral manifolds, of the average system 2.3, containing the initial condition prescribed for (2.1).

Proof The first part of the proof is obvious. Consider system 2.1 in the augmented state space of coordinate functions (x,z).

$$
\frac{d}{dt}\begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} f(x) \\ \phi(z) + \gamma(z)h(x) \end{bmatrix} + \begin{bmatrix} g(x) & 0 \\ 0 & \gamma(z) \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} -g(x)\eta(z) \\ 0 \end{bmatrix} v
$$

$$
\begin{matrix} y_1 \\ y_2 \end{matrix} = \begin{matrix} h(x) \\ \eta(z) \end{matrix} \qquad (2.4)
$$

which we shall express as :

$$
\begin{aligned}
&d/dt\; x_e = f_e(x_e) + G_{1e}(x_e)u_e + g_{2e}(x_e)\; v\\
&y_e = h_e(x_e)
\end{aligned}
\qquad (2.5)
$$

with x_e = col (x,z) and the vectors fields; f_e, g_{2e} , the columns of G_{1e}, and the function h_e are trivially defined from the expression (2.4) of the closed loop system. The system (2.5) is of the same form (A.2)-(A.3), with discontinuous input function, v, governed by the PWM operator (2.2). The result of theorem A.1 immediately applies and the result follows. ◊

Remark Notice that in the event of a prescribed constant duty ratio function, $0 < \tau < 1$, the sliding motion described by theorem 2.1 occurs globally in the augmented state space of the closed loop system, about integral manifolds of the average model. This result should be clear since, in such a case, the saturation condition is never reached for the feedback switching device.

3. AN ILLUSTRATIVE EXAMPLE

Example Dynamic Discontinuous Feedback Control of a Flexible Joint Manipulator (Albert and Spong [7]).

Consider a single rigid link of mass M, whose length from the center of rotation is L, and inertia J_L . The link is to act as a manipulator with flexible joint coupled to an actuator motor characterized by inertia J_m about the axis of rotation, and vicsous friction parameter B_m. The flexibility of the joint is modeled by means of a torsional spring with constant k, and a torsional damping with parameter B_m. The mathematical model is given by the following set of nonlinear ordinary differential equations :

$$
\begin{aligned}
&dx_1/dt = x_2,\; dx_2/dt = -(MgL/J_L)\sin x_1 - (B_L/J_L)x_2 + (k/J_L)(x_1 - x_3)\\
&dx_3/dt = x_4,\; dx_4/dt = -(B_m/J_m)x_4 + (k/J_m)(x_1 - x_3) + (1/J_m)u\\
&y = x_1
\end{aligned}
\qquad (3.1)
$$

where x_1 and x_3 are the link and motor shaft angle respectively and u represents the applied input torque. In spite of the nonlinear nature of the model, if the output of the system is taken as the angular position x_1 of the link, an asymptotically stable observer of the Luenberger type can be prescribed for the system linear error dynamics (See [7]). Such an asymptotic observer is given by :

$$d\xi_1/dt = \xi_2 + g_1(y-\xi_1),$$
$$d\xi_2/dt = -(MgL/J_L)\sin y - (B_L/J_L)\xi_2 + (k/J_L)(\xi_1-\xi_3) + g_2(y-\xi_1)$$
$$d\xi_3/dt = \xi_4 + g_3(y-\xi_1),$$
$$d\xi_4/dt = -(B_m/J_m)\xi_4 + (k/J_m)(\xi_1-\xi_3) + (1/J_m)u + g_4(y-\xi_1) \qquad (3.2)$$

where the observer parameters g_i ; $i = 1,\ldots,4$, are chosen to place the poles of the linear time invariant error dynamics in suitable locations of the left half of the complex plane.

Resorting to exact feedback linearization results, a nonlinear, static, feedback controller has also been designed in [7], which globally stabilizes the nonlinear controlled system, given by :

$$u = (J_mJ_L/k)[-a_4(T_1(\xi)-r)-a_3T_2(\xi)-a_2T_3(\xi)-a_1T_4(\xi)] + \alpha(\xi) \qquad (3.3)$$

where $r =: \theta_L{}^d$, is a desired angular position for the link and :

$$T_1(\xi) = \xi_1 \ ; \ T_2(\xi) = \xi_2 \ ;$$
$$T_3(\xi) = -(MgL/J_L)\sin \xi_1 - B_L/J_L\xi_2 - (k/J_L)(\xi_1-\xi_3)$$
$$T_4(\xi) = (-MgL/J_L)\ \xi_2\cos(\xi_1) - (B_L/J_L)T_3(\xi) - (k/J_L)(\xi_2-\xi_4)$$

$$\text{with: } \alpha(\xi) = T_3(\xi)[(MgL/J_L)\cos\xi_1 + (B_L/J_L)^2-(k/J_L)] + (MgL/J_L)\xi_2{}^2\sin \xi_1 + (B_LMgL/J_L{}^2)\xi_2\cos \xi_1 + (B_Lk/J_L{}^2)(\xi_2-\xi_4)$$

The constants a_i ; $i = 1,\ldots,4$ are chosen to appropriately place the poles of the linearized controlled system in desired locations of the left half of the complex plane.

In order to study the effect of a discontinuous PWM feedback scheme on (3.1)-(3.3), we introduce a feedback enabling switch - of constant duty ratio τ - characterized by the function v. In this case, the average system is symply obtained from the actual switched-regulated system by simply replacing the switch device by a constant gain of value τ. In order to maintain the same quality of the continuous feedback design achieved in [7], and take such responses as the <u>average</u> responses, the controller parameters a_i ($i = 1,\ldots,4$) were scaled down by a factor equal to the duty ratio τ. In this manner, the overall gain of the feedback path is unaffected for the average PWM system with respect to the original design and the average response of our PWM scheme should be identical with those obtained in [7].

We, hence, prescribe a discontinuous feedback controller of the form :

$$u = (J_mJ_L/k)\{ -(v/\tau)[a_4(T_1(\xi)-r) + a_3T_2(\xi) + a_2T_3(\xi) + a_1T_4(\xi)] \} + \alpha(\xi) \qquad (3.4)$$

with:

$$v = \begin{cases} 1 & \text{for} \quad t_k < t \le t_k + \tau T \\ 0 & \text{elsewhere} \end{cases}$$

The system parameter values found in [7] are also used here, with the same prescribed controller and observer gains : $M = 0.02$, $J_m = 0.0004$ Nms^2/rad, $J_L = 0.0004$ Nms^2/rad, $B_m = 0.015$ Nms/rad, $B_L = 0$, $k = 0.8$ Nm/rad, $g = 9.8065$ m/s^2, $L = 0.5$ m and $r = \theta_L{}^d = 1$ rad.

Simulations were run with a constant duty ratio of $\tau = 0.5$ and a sampling frequency for the PWM block of 250 samples per second. Figure 2 shows that a sliding motion actually appears about the average response of the controlled system in the augmented state space. The figure portrays the "jerk" variable, x_4, directly beeing affected by the discontinuous feedback signal. Such sliding mode is not observed in the rest of the state variables as the discontinuous feedback signal travels, in its way to the output, through the succesive integration steps representing the dynamical system. In fact, on the response of the link angular position of the system, the sliding motion, exhibited by the actual PWM feedback controlled system, is smoothed out to the point of practically coinciding with the response of the average system (See figure 3). This smoothing phenomena, aside from the integration effects, is due to the <u>regular canonical structure</u> [9] exhibited by the discontinuously controlled system. In such cases, sliding regimes are undergone by the variable in the last equation (fast sliding variable) while the rest of the variables behave as the smooth <u>ideal sliding dynamics</u> (slow sliding variables).

4. CONCLUSIONS

An averaging technique has been introduced for the accurate description of discontinous feedback interconnected nonlinear systems under a pulse width modulation scheme for the switching element. The averaging proces is based on an infinite switching frequency assumption on the feedback enabling device. The proposed average model was shown to entirely capture the main qualitative, and quantitative, features of the actual finite frequency switched controlled system. The existence of a sliding motion in the augmented state space of the closed-loop pulsed-controlled system, which closely follows the average trajectories, makes the approximation scheme amenable to arbitrary improvement under increased switching frequency specifications for the actual controlled system. This entirely obviates the need for cumbersome approximation schemes, based on traditional discrete-time considerations and the technical difficulties associated with the unbounded character of the PWM operator (See Skoog and Blankenship [10]). Such sliding regimes occur only on integral manifolds of the average system inmmersed in regions of the state space where the duty ratio function, associated to the controlled switch, exhibits a nonsaturation condition. In the other regions of the state space, the trajectories of the actual and the average sytstem just coincide. Several nontrivial illustrative examples were presented.

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APPENDIX

Let $s(t_k)$ denote $s(x(t_k))$, a given scalar function of a vector x. Consider, then, the nonlinear discontinuously controlled system, described by:

$$dx/dt = \begin{cases} f(x) + G_1(x)u + g_2(x) & \text{for } t_k < t \le t_k + \tau[s(t_k)]T \\ f(x) + G_1(x)u & \text{for } t_k + \tau[s(t_k)]\,T < t \le t_k + T \end{cases} \tag{A.1}$$

where the vector field $f(x)$, the columns of $G_1(x)$ and $g_2(x)$ are smooth vector fields defined on R^n. The t_k's represent regularly spaced instants of time where an ideal sampling process takes place. At each of these instants, the value of the <u>duty ratio function</u>, $\tau[s(x(t_k))] =: \tau[s(t_k)]$, is determined in correspondence with the value of the scalar function $s(x)$, at the sampled value of the state vector, $x(t_k)$. The sampling period T is assumed to be sufficiently small, as compared with the time constants associated with the dynamics of the system. Unless otherwise stated, it will be assumed that our considerations are restricted to a region of the state space where the duty ratio function, $\tau[s(x)]$, is not saturated i.e., $\tau[s(x)]$ takes values in the open interval $(0,1)$.

In terms of an ideal switching function v, taking values in the discrete set $\{0,1\}$, the above system can be equivalently represented as :

$$dx/dt = v[f(x) + G_1(x)u + g_2(x) + (1-v)[f(x) + g_1(x)u]$$

i.e.,

$$dx/dt = f(x) + G_1(x)u + v\, g_2(x) \tag{A.2}$$

with v obeying a switching policy of the form:

$$v = \begin{cases} 1 & \text{for } t_k < t \le t_k + \tau[s(t_k)]\ T \\ 0 & \text{for } t_k + \tau[s(t_k)]\ T < t \le t_k + T \end{cases} \tag{A.3}$$

The following lemma is a straightforward consequence of the Fundamental Theorem of Calculus.

<u>Lemma A.1</u> Let f be a smooth vector field and let $I_f(t) := \int_0^t f(x(s))ds$. Then for any smooth, strictly positive, function $\mu(x)$:

$$\lim_{T\to 0,\ t_k \to t} (I_f(t_k+\mu[x(t_k)]T) - I_f(t_k))/T = \mu[x(t)]f(x(t)) \tag{A.4}$$

The next theorem determines the nature of the infinte-frequency average dynamics of (A.2),(A.3) under nonsaturating conditions.

<u>Theorem A.1</u> Consider a region where the PWM controller is not saturated. Then, as the sampling frequency 1/T tends to infinity in system (A.2),(A.3), the discontinuous system (A.1) coincides with <u>Filippov's average model</u> .

$$dx/dt = \mu(x)\ [f(x)+G_1(x)u + g_2(x)] + [1-\mu(x)]\ [f(x) + G_1(x)u] = f(x) + G_1(x)u + \mu(x)\ g_2(x) = f_{av}(x,u) \tag{A.5}$$

with a corresponding convex combination function, $\mu(x)$, exactly represented by the duty ratio function $\tau(x)$. Moreover, in such a region, a sliding regime is exhibited by the actual PWM controlled system (A.2),(A.3) about an integral manifold M of (A.5).

Proof Let $f_1(x,u) = f(x) + G_1(x)u + g_2(x)$ and $f_2(x,u) = f(x) + G_1(x)u$, and, as before, let $s(t_k)$ denote $s(x(t_k))$. From (A.2), (A.3), the state x at time $t_k + T$ is exactly computed as:

$$x(t_k+T) = x(t_k) + \int_{t_k}^{t_k+\tau[s(t_k)]T} f_1(x(\sigma),u(\sigma))d\sigma + \int_{t_k+\tau[s(t_k)]T}^{t_k+T} f_2(x(\sigma),u(\sigma))d\sigma$$

$$= x(t_k) + \int_{t_k}^{t_k+\tau[s(t_k)]T} f_1(x(\sigma), u(\sigma))\, ds + \int_{t_k}^{t_k+T} f_2(x(\sigma),u(\sigma))\,d\sigma - \int_{t_k}^{t_k+\tau[s(t_k)]T} f_2(x(\sigma),u(\sigma))\,d\sigma$$

assuming that $\tau(s(x))$ is neither 0 or 1 in the region of interest, and using the result of lemma A.1, one has :

$$\lim_{T\to 0,\ t_k\to t} [x(t_k+T) - x(t_k)]/T$$

$$= \lim_{T\to 0,\ t_k\to t} \left[\int_{t_k}^{t_k+\tau[s(t_k)]T} f_1(x(\sigma),u(\sigma))\,d\sigma + \int_{t_k}^{t_k+T} f_2(x(\sigma),u(\sigma))\,d\sigma - \int_{t_k}^{t_k+\tau[s(t_k)]T} f_2(x(\sigma),u(\sigma))\,d\sigma\right](1/T)$$

$$= \tau(s(t))\, f_1(x(t)) + [1-\tau(s(t))]\, f_2(x(t))$$

or:

$$dx/dt = \tau[s(x)]\, f_1(x) + [1-\tau[s(x)]]\, f_2(x) =$$

$$= f(x) + G_1(x)u + \tau(x)\, g_2(x) = f_{av}(x,u) \qquad (A.6)$$

i.e., the infinite frequency model of (A.2)-(A.3) coincides with Filippov's Geometric Average model in which the convex combination function $\mu(x)$, defining the average vector field $f_{av}(x)$, is precisely taken as the duty ratio function $\tau[s(x)]$. It is clear that on an integral manifold of (A.6), described by, say, $S = \{ x \in R^n : m(x) = 0 \}$, the controlled vector field of (A.6) is pointwise orthogonal to the gradient of m(x) i.e :

$$(\partial m/\partial x)[\, f(x)+G_1(x)u + \tau(x)\, g_2(x)] = 0 \quad \text{on} \quad m = 0 \qquad (A.7)$$

the duty ratio function admitts, then, a geometrically based definition as :

$$\tau(x) = -(\partial m/\partial x)[f(x) + G_1(x)u\,]/(\partial m/\partial x)g_2(x) \qquad (A.8)$$

From known results about the relation between Filippov's average dynamics and sliding regimes [1], and the assumption that the duty ratio function is locally bounded in the open interval (0,1), it follows that a *sliding regime* exists locally on the manifold S for the VSS (A.2),(A.3).

The equivalent control $v^{EQ}(x)$, associated with such a sliding regime, is simply obtained from the *invariance conditions* [1],[8] of the ideal sliding mode taking place on the integral manifold $S = \{ x : m(x) = 0 \}$ of the average system : i.e. from the conditions: $dm/dt = 0$ on $m = 0$, written in local coordinates, one obtains:

$$dm/dt = \partial m/\partial x[\, v^{EQ}(x)\, f_1(x,u) + (1-v^{EQ}(x))\, f_2(x,u)\,]$$
$$= v^{EQ}(x)\,[\partial m/\partial x]\, f_1(x,u) + (1-v^{EQ}(x))\,[\partial m/\partial x]\, f_2(x,u) = 0$$

The corresponding equivalent control $v^{EQ}(x)$ is then obtained as :

$$v^{EQ}(x) = -[\partial m/\partial x]\, f_2(x,u)\, /[\,\partial m/\partial x](f_1(x,u)-f_2(x,u))$$

i.e.,

$$v^{EQ}(x) = [\partial m/\partial x](f(x) + G_1(x)u)/[\partial m/\partial x]g_2(x) \qquad (A.9)$$

It follows, from (A.8)-(A.9) and the uniqueness of the equivalent control [8], that :

$$v^{EQ}(x) = \tau(x) \qquad (A.10)$$

i.e., the *equivalent control* of the sliding motion associated with (A.2) and (A.3) is then, precisely, constituted by the *duty ratio* associated to the proposed PWM control scheme. The corresponding ideal sliding dynamics is then represented by :

$$dx/dt = v^{EQ}(x)\, f_1(x,u) + [1-v^{EQ}(x)\,]\, f_2(x,u)$$
$$= \tau(x)f_1(x,u) + [1-\tau(x)]\, f_2(x,u) = f(x)+ G_1(x)u + g_2(x)\,\tau(x)$$

which is just the Average PWM model (A.6).

It was shown in [8] that the region of existence of a sliding motion is determined by the region on S where $\tau[s(x)]$ satisfies the following conditions:

$$0 < \tau[s(x)] = v^{EQ}(x) < 1$$

By definition of the duty ratio, the above condition is evidently satisfied, along the integral manifold S, in all regions of the state space where the PWM controller is not saturated. ◊

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FIGURES

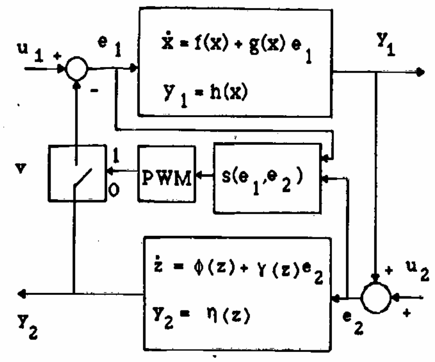

Figure 1. A PWM Discontinuous Feedback Interconnected System

Figure 2. Actual and Average PWM response of jerk variables

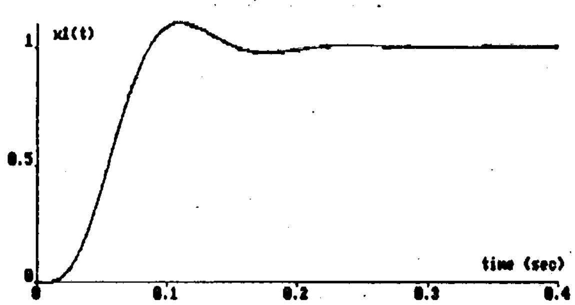

Figure 3. Actual and Average PWM response of link angular position