DISTRIBUTED SLIDING MODE CONTROL IN SYSTEMS DESCRIBED BY LINEAR AND QUASILINEAR PARTIAL DIFFERENTIAL EQUATIONS[1]

Hebertt Sira-Ramírez,

Departamento Sistemas de Control. Escuela Ingeniería de Sistemas
Universidad de Los Andes. Mérida-VENEZUELA

Abstract The theory of Variable Structure Systems, and their associated Sliding Regimes, is extended to Distributed Dynamical Systems described by First Order Controlled Linear and Quasilinear Partial Diferential Equations.

1. INTRODUCTION

The theory of Variable Structure Systems (VSS) and their associated Sliding Regimes constitute an interesting theoretical field of the Control Systems discipline with a vast number of applications. A detailed account of the basic elements of the theory, as applied to dynamical systems described by Ordinary Differential Equations (ODE), is contained in the work of Utkin[1-3].

There are only few instances where this versatile control method has been studied in the context of systems described by Partial Differential Equations (PDE). In Orlov and Utkin[4], use was made of VSS for the Sliding Mode control of a distributed thermal process described by a second-order PDE of the parabolic type. The scheme resorted to a finite dimensional approximation technique of the distributed process and the sliding mode creation problem was defined on the associated finite dimensional controlled system approximation characterized by a set of ODE's. In Orlov and Utkin[5], the theory of sliding mode control was extended to indefinite dimensional systems described by differential equations defined in Banach spaces. Applications were given, in that article, for a multi-dimensional heat process.

In this article, using simple notions from elementary Differential Geometry, the theory of VSS, and their associated Sliding Regimes, is extended to dynamical systems described by First Order Linear and Quasilinear Partial Diferential Equations (LPDE and QPDE respectively). The key idea is to exploit the facts that 1) The solutions of a controlled QPDE are represented by surfaces constituted by integral manifolds of the controlled characteristic direction field and 2) that the variable structure controlled characteristic direction field determines, in turn, phase flows, known as characteristics, which must also slide on the switching manifold.

This paper is organized as follows: Section 2 presents the definitions and main results. The conclusions and suggestions for further work are given at the end of the article. Background material on the geometric aspects of LPDE's and QPDE's are directly taken from Arnold[6] and Tikhonov *et al* [7]. For general background on the subject of PDE's, the reader is referred to the extensive treatise by Courant and Hilbert[8].

[1] This work was supported by the Consejo de Desarrollo Científico, Humanístico y Tecnológico of the Universidad de Los Andes, under Research Grant # I-280-87.

2. MAIN RESULTS

Consider a dynamical system described by a first order feedback-controlled quasilinear PDE with scalar-valued output function y :

$$\partial v/\partial t + \Sigma_{i=1}^{n} [\partial v/\partial x_i] X_i(v,x,t,u) = b(v,x,t,u)$$
$$y = h(v,x,t) \quad (2.1)$$

where $u = u(v,x,t)$ is a, possibly discontinuous, distributed feedback control law taking values in R. v is the distributed scalar "state" and the X_i's are the smooth components of a time-varying control-parametrized vector field X defined on an open set of R^n and assumed to be nonzero everywhere. b and h are smooth functions of their arguments. All our considerations and results are of local character. Condition $y = 0$ is assumed to locally define an isolated smooth manifold solution $v = \varphi(x,t)$, i.e., $y(\varphi(x,t),x,t) \equiv 0$. The graph of v is a time varying surface: $S = \{ (v,x,t) \in R^{n+2}: v = \varphi(x,t) \}$ addressed to as the Sliding Manifold or Sliding Surface.

Available to the controller is a Distributed Variable Structure Feedback Switching law :

$$u = \begin{cases} u^+(v,x,t) & \text{for} \quad y > 0 \\ u^-(v,x,t) & \text{for} \quad y < 0 \end{cases} \quad (2.2)$$

with $u^+(v,x,t) > u^-(v,x,t)$ locally.

Definition 1 A Distributed Sliding Regime is said to locally exist on an open set of the manifold S iff the total derivative of the output function of system (2.1)-(2.2) satisfies :

$$\lim_{y \to +0} dy/dt < 0 \quad \text{and} \quad \lim_{y \to -0} dy/dt > 0 \quad (2.3)$$

To simplify notation we introduce the vector $z = \text{col}\,(v,x,t)$ and the control-parametrized vector field $\xi = \text{col}\,(b(z,u),X(z,u),1)$ referred to as the characteristic direction field of (2.1). The Lie derivative of the sacalar output function h(z) along the vector field ξ, for a given control input $u = u(z)$, is denoted by $L_{\xi(z,u(z))}h$.

Theorem 1 A Distributed Sliding Regime locally exists for system (2.1) on an open set of the sliding manifold S, if and only if the phase flows corresponding to the controlled characteristic direction field of (2.1) exhibit a local sliding regime under the influence of the switching law (2.2).

Proof Suppose a distributed sliding mode locally exists for (2.1)-(2.2), then the total time derivative of y, evaluated on points above and below the switching surface which lie in a small vicinity of the sliding surface, and computed along the direction of the controlled characteristic direction field, are given by :

r y > 0 :

/dt = [∂h/∂v]dv/dt +[∂h/∂x]dx/dt +[∂h/∂t] =

h/∂v]b(v,x,t,u^+)+[∂h/∂x]X(v,x,t,u^+) +[∂h/∂t] =

$(z,u^+(z))^h < 0.$

r y < 0 :

/dt =[∂h/∂v]b(v,x,t,u^-)+[∂h/∂x]X(v,x,t,u^-) +[∂h/∂t] =

$(z,u^-(z))^h > 0.$

other words, the system described by the set of dinary differential equations: $dz/dt = \xi(z,u(z))$ lso known as the characteristic equation of the derlying controlled PDE) exhibits a local sliding gime on the same open set of the zero manifold of output equation y = h(z), when u(z) is ruled by switching law (2.2). Sufficiency follows easily by suming a sliding mode exists for the controlled aracteristic system while hypothesizing that a stributed sliding mode does not exist. By reversing arguments presented above, a contradiction is sily established. ⅽ

eorem 2 Suppose a sliding regime locally exists for stem (2.1)-(2.2) on S. Then, there exists a unique ooth feedback control law $u = u^{EQ}(v,x,t)$ such that characteristic direction field of (2.1) controlled u^{EQ} locally adopts as an integral surface the graph the zero output function $v = \varphi(x,t)$. Moreover cally on S :

$$u^-(\varphi,x,t) < u^{EQ}(\varphi,x,t) < u^+(\varphi,x,t). \qquad (2.4)$$

oof The proof amounts to demonstrating the istence of a smooth equivalent control for the rresponding sliding regime of the characteristic stem $dz/dt = \xi(z,u)$ on $y = h(z) = 0$. Indeed, since local invariance of $y = 0$ necessarily requires /dt = 0, one has: $L_{\xi(z,u)}h = 0$. On the other hand, existence of a local sliding regime on S implies, cessarily, that locally on S: $\partial[L_\xi h]/\partial u \neq 0$ i.e. $(z,u)^h = 0$ has, according to the implicit function eorem, a unique solution $u = u^{EQ}(z)$. Since $\xi(z,u^{EQ})$ pointwise orthogonal to the gradient of h on $y = 0$, zero level set of h is locally a smooth integral nifold for the flows generated by ξ under the fluence of $u^{EQ}(z)$. By definition of φ, the first rt of the theorem is true. The second part of the eorem was established in full generality by ra-Ramirez[9] using the Mean Value theorem and the oof will not be presented here. ⅽ

mark In the above theorem, if the extreme values of feedback control law, u^+, u^- are not fixed at e outset, the existence of an equivalent control tisgying (2.4) also suffices to establish the istence of a local sliding regime (See ra-Ramirez[10]) i.e., in such a case, the existence of intermediate equivalent control $u^{EQ}(z)$ locally tween $u^+(z)$ and $u^-(z)$ constitutes a necessary and fficient condition for the existence of a sliding gime.

The results are easily particularized for the case systems described by controlled homogeneous and n- homogeneous LPDE. The details of these particular cases are left for the reader and will not be presented here. It will only be remarked, however, that eventhough the definition of caracteristics, for LPDE's, is not usually made in the space of zero-jets of the controlled PDE (i.e., on the space of local coordinates (v,x,t)), but on the underlying manifold of local coordinates (x,t), they can be lifted in a trivial manner (See ref.7, Chapter 3). The general results for the quasilinear case remain obviously valid for such particular classes of systems.

3. CONCLUSIONS AND SUGGESTIONS FOR FURTHER WORK

The theory of Variable Structure Systems undergoing sliding motions can be easily extended to controlled systems described by first order quasi-linear PDE. The key property of such class of dynamical systems is the possibility of relating properties of their solution to those of a characteristic controlled system described by a set of ordinary differential equations (known as the characteristic equation). This property is used in this article to establish conditions for the local existence of a distributed sliding regime on a given switching surface. A distributed sliding mode locally exists whenever the controlled characteristic system exhibits such a controlled motion on the sliding surface. This surface must also qualify as a local integral manifold of an "equivalent direction field". The equivalent direction field is the average controlled field prescribed by the equivalent control method on the characteristic system. The case of systems described by implicit partial differential equations can also be treated from an entirely geometrical viewpoint using contact structures and the theory of jet bundles. Such direction may lead to a generalization of the results presented here. The geometric theory of second order PDE's (ref. 8) could also be taken as a starting point for the adequate treatment of distributed sliding regimes in controlled systems described by such distributed dynamical models.

4. REFERENCES

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