

DISTRIBUTED SLIDING REGIMES IN SYSTEMS DESCRIBED BY NONLINEAR FIRST ORDER PARTIAL DIFFERENTIAL EQUATIONS

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Abstract

A complete characterization of Distributed Sliding Regimes is presented for dynamical systems described by discontinuous controlled nonlinear first order partial differential equations.

Key Words: Sliding Regimes, Distributed Parameter Systems, Varying Structures.

1. INTRODUCTION

In this article a geometric approach is presented for the characterization of sliding regimes (Utkin 1978), and their fundamental properties, in discontinuously controlled dynamical systems described by nonlinear first order partial differential equations (NFOPDE). This article constitutes a generalization of the results presented in Sira-Ramirez (1989a) for the case of distributed sliding mode control in first order linear and quasi-linear partial differential equations. The proposed approach for the treated class of distributed parameter systems constitutes a fundamental departure from the methods, based on Banach spaces, used by Broger *et al* (1980), Orlov and Utkin (1982), and from the approximate finite-dimensional method (Orlov and Utkin, 1987). It also differs from the Lyapunov-based approach (Utkin, 1990) used for a particular one-dimensional second order distributed heat process of the nonlinear parabolic type.

A distributed sliding manifold may be portrayed as the smooth solution manifold of a desirable closed loop dynamical system represented by a NFOPDE. The prolongation to the space of 1-jets of functions of the sliding surface is shown to play a fundamental role in both the characterization of the invariance properties of the ideal sliding dynamics and the determination of the sliding mode existence conditions. A controlled NFOPDE is geometrically characterized as a smooth manifold, parametrized by the feedback control function, in the space of 1-jets of functions. The ideal sliding dynamics is obtained by imposing a local invariance condition on Filippov's average (Filippov, 1988) of the variable structure controlled characteristic direction fields defined in such a jet space, with respect to the prolongation of the sliding manifold. Local existence of a sliding regime on the given sliding surface is then characterized in terms of the appropriate transversality of the corresponding characteristic direction fields associated to the extreme controlled systems with respect to the tangent distribution to the sliding manifold representing the solution of the ideally desirable dynamics. The results not only characterize sliding regimes in general nonlinear first order distributed parameter systems but they can also rederive known results for the linear and quasilinear cases (Sira-Ramirez, 1989a) while

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generalizing those related to the explicit and implicit finite dimensional nonlinear cases (Sira-Ramirez, 1988, 1989b). The approach here adopted can also be extended to systems described by a finite set of NFOPDE controlled by multiple inputs.

Section 2 of this article contains some background material on the geometrical aspects of nonlinear partial differential equations. Section 3 contains the main results of the article. Section 4 is devoted to some conclusions and suggestions for further research. Further background material related to this article can be found in the books by Olver (1986), Arnold (1984) and Arnold (1988).

2. MATHEMATICAL BACKGROUND

In this section we present some basic results about integration of NFOPDE's by means of characteristics. The reader is referred to the books by Olver (1986) and Arnold (1984, 1988) for more thorough and enlightening details.

2.1 Elements of the Geometric Theory of Nonlinear First Order Partial Differential Equations

Consider a NFOPDE:

$$\partial v / \partial t + \mathcal{F}(v, x, t, p) = 0 \quad (2.1)$$

where x represents the vector of local spatial coordinate functions x_i ($i = 1, \dots, n$) defining points on an open set in R^n , t denotes time. The function v is the unknown scalar function and p is an n dimensional vector with components p_i , representing the partial derivatives, $\partial v / \partial x_i$ ($i = 1, \dots, n$). We also denote by q the partial derivative $\partial v / \partial t$ and by π the vector of components (p, q) . \mathcal{F} is a smooth function of all its arguments.

All our considerations and results are of local character on a given open set (manifold) N of R^{n+2} described by the vector of local coordinate functions (v, x, t) , denoted by η . The projection of such an open set N onto R^{n+1} , along the direction of v , is labeled as M , and it is equipped with local coordinates (x, t) which we simply denote by χ . Also, we denote by z the vector of local coordinates, $(v, x, t, p, q) = (\eta, \pi)$ in R^{2n+3} which we identify as the manifold of 1-jets of functions defined on M , labeled here as $J^1(M, R)$. By TN and $TJ^1(M, R)$ we denote, respectively, the tangent bundles of N and of $J^1(M, R)$.

Equation (2.1) can be interpreted as the expression of a $2n+2$ dimensional hypersurface in $J^1(M, R)$. We denote such an hypersurface by E and define it as:

$$E = \phi^{-1}(0) := \{z \in J^1(M, R) : \phi(z) = q + \mathcal{F}(v, x, t, p) = 0\} \quad (2.2)$$

The space $J^1(M, R)$ is equipped with the standard contact structure induced by the nowhere zero canonical 1-form $\alpha := dv - p dx - q dt = dv - \pi d\chi$. The contact structure in

the space of 1-jets, $J^1(M, R)$, is represented by the field of $2n+2$ dimensional planes -called the *contact planes* (or *contact distribution*) - annihilating the 1-form α . A contact plane at the point z in $J^1(M, R)$ is denoted by Π_z and it is a subspace of the tangent space of $J^1(M, R)$ at z , $T_z J^1(M, R)$. Since the exterior derivative of α , denoted by $\omega := d\alpha = dx \wedge dp + dt \wedge dq$, is a nondegenerate skew symmetric bilinear form on the field of even dimensional contact planes Π , the contact planes are, indeed, *symplectic vector spaces*. The distribution in $TJ^1(M, R)$ annihilating the 1-form $d\phi$ is constituted by the field of planes Θ tangent to the hypersurface E (*tangent distribution*). The plane tangent to E at z is denoted by Θ_z .

Definition 2.1 A surface E in $J^1(M, R)$ is *noncharacteristic* if its tangent planes Θ and the contact planes Π are transversal, i.e., if their direct sum at each z spans the tangent space $T_z J^1(M, R)$.

It is generally assumed that the manifold E is noncharacteristic at all points z of the space $J^1(M, R)$ under consideration. The intersections of the contact planes and tangent planes at points z in E are, again, a field of planes, of dimension $2n+1$, which are tangent to E . These are the *characteristic planes*. It is not difficult to see how the *symplectic structure* associated to the contact structure, naturally and uniquely, determines a *jet-field of directions* belonging to the characteristic planes. Such a field of directions, in turn, determines, by projection, the *characteristics* of the nonlinear equation (2.1) as described below.

Consider the 2-form ω , obtained by exterior differentiation of the contact 1-form α . Since the 1-form $d\phi$ is non identically zero at z , $d\phi$ acts on vectors belonging to Π_z as a nonzero linear form. Notice that Π_z is an even dimensional vector space, while ω is a nondegenerate skew-symmetric bilinear form on Π_z . Hence, Π_z is actually a *symplectic vector space*. Every symplectic vector space is known to be isomorphic to its dual space. Hence, one can identify 1-forms (covectors) with vectors of Π_z . We proceed to identify, by the above discussion, a nonzero vector ξ of Π_z with the nonzero 1-form $d\phi$, by imposing the equality: $d\phi(\cdot) = \omega(\xi, \cdot)$. The vector ξ , will be known as the *jet-characteristic vector* of E at the point z . The jet-characteristic vector ξ belongs to the characteristic distribution $\Pi \cap \Theta$ and it is skew-orthogonal to itself: $d\phi(\xi) = \omega(\xi, \xi) = 0$. This vector, however, is nonuniquely defined. The nonuniqueness of ξ arises from the fact that one may take any vector $\mu = \beta(z)\xi$ in the span of ξ , and still obtain the same previously described identification with the 1-form $d\phi$. Indeed, let $d\phi(\cdot) = \omega(\mu, \cdot)$ and then $d\phi(\mu) = \beta^2(z)\omega(\xi, \xi) = 0$, and μ is also characteristic. The above procedure does, however, uniquely determine, at each point z , the *jet-characteristic line* or, *jet-characteristic direction*, containing all vectors that are in the span of ξ . The $n+1$ -dimensional integral curves of the jet-characteristic directions, contained in E , are the *jet-characteristics* of the partial differential equation.

One can explicitly compute the set of ordinary first order differential equations generating the jet-characteristics for the equation (2.1) as follows:

Consider a nonzero vector :

$$\zeta = V \partial/\partial v + X \partial/\partial x + T \partial/\partial t + P \partial/\partial p + Q \partial/\partial q$$

belonging to Θ_z , the tangent plane to E in $T_z J^1(M, R)$. For such a vector we have:

$$d\phi(\zeta) = \phi_v V + \phi_x X + \phi_t T + \phi_p P + Q = 0 \quad (2.3)$$

The vector ζ lies in the contact distribution Π_z if it belongs to the null space of the 1-form: α i.e., $\alpha(\zeta) = V - pX - qT = 0$. The vector ζ is then, necessarily, of the form $\zeta = (pX + qT) \partial/\partial v + X \partial/\partial x + T \partial/\partial t + P \partial/\partial p + Q \partial/\partial q$. The vector

ζ belongs to the characteristic plane $\Pi_z \cap \Theta_z$ if and only if:

$$[\phi_v p + \phi_x] X + [\phi_v q + \phi_t] T + \phi_p P + Q = 0 \quad (2.4)$$

The jet-characteristic vector ξ has components $(\dot{v} = \pi \dot{x}, \xi, 1, \dot{p}, \dot{q})$. These components are determined from the condition that the skew scalar product of ξ with all vectors of the form: $(pX + qT) \partial/\partial v + X \partial/\partial x + T \partial/\partial t + P \partial/\partial p + Q \partial/\partial q$, is identically zero. Evaluating the 2-form: $d\alpha = dX \wedge dp + dT \wedge dq$ on the pair of vectors: $(pX + qT) \partial/\partial v + X \partial/\partial x + T \partial/\partial t + P \partial/\partial p + Q \partial/\partial q$ and ξ , one readily obtains:

$$-pX - qT + xP + Q = 0 \quad (2.5)$$

Consequently, letting the coefficients of X, T, P in (2.4) and (2.5) be identical and using $\dot{v} = \pi \dot{x} = p\dot{x} + q$, one obtains an expression for the components of the jet-characteristic vector field ξ as :

$$\begin{aligned} \dot{v} &= p\dot{x} + q = p\mathcal{F}_p + q \\ \dot{x} &= \phi_p = \mathcal{F}_p, \quad \dot{t} = \phi_q = 1 \\ \dot{p} &= -\phi_v p - \phi_x = -\mathcal{F}_v p - \mathcal{F}_x, \quad \dot{q} = -\phi_v q - \phi_t = -\mathcal{F}_v q - \mathcal{F}_t \end{aligned} \quad (2.6)$$

or briefly, in terms of the local coordinates z of $J^1(M, R)$:

$$\dot{z} = \xi(z) \quad (2.7)$$

Definition 2.2 Let γ be an n -dimensional submanifold of M and let $\varphi: \gamma \rightarrow R$ be a smooth function. The *initial manifold* \mathcal{G} , defined in N , is constituted by the set $\{(v, x, t) \in N : v = \varphi(x, t), \text{ for } (x, t) \in \gamma\}$. The pair (φ, γ) is referred to as the *Cauchy data*. The *initial jet-manifold* Γ , constructed in $J^1(M, R)$ on the basis of the Cauchy data, is the set consisting of all 1-jets of functions on M satisfying the following requirements: 1) The base point x of the jet (η, x) lies on γ , 2) the value of the function v at the point x is equal to φ , 3) The value of the total differential of the function v at $x \in \gamma$ is such that its restriction to the tangent planes (tangent distribution) to γ is equal to the total differential of the initial condition φ evaluated on such planes, 4) the jet $\eta = (\eta, x)$ is a point of $J^1(M, R)$ belonging to E .

In other words, an initial condition for the equation $\phi = 0$ is an assignment of a particular value φ to the unknown function v on the points of an n -dimensional hypersurface γ defined in the $n+1$ dimensional space of coordinates $x = (x, t)$.

Definition 2.3 A point of the initial jet-manifold Γ is said to be *noncharacteristic* for the system (2.1) if the projection of the *jet-characteristic direction* at this point onto M is transversal to γ . It can be shown (Arnold, 1988 pp.82) that an explicit condition, over the initial data (γ, φ) and the function ϕ , for which an initial point $z^0 = (\eta^0, x^0) = (v^0, x^0, t^0, p^0, q^0)$ is non characteristic, with respect to the equation $\phi = 0$, is that the $n+1$ -dimensional vector $\phi_{\pi}(z^0) = [\mathcal{F}_p(\eta^0, x^0), 1]$ is not tangent to γ .

For a given noncharacteristic point z^0 of the initial jet-manifold Γ in $J^1(M, R)$ there exists, defined in some open neighborhood U of $x^0 = (x^0, t^0)$, a locally unique solution v to the equation $\phi = 0$. This means that any two solutions of (2.1), which are made to satisfy the same noncharacteristic initial condition $v|_{U \cap \gamma} = \varphi|_{U \cap \gamma}$, $v(x^0, t^0) = v^0$, $dv(x^0, t^0) = (p^0, q^0) = \pi^0$, necessarily coincide over some open subset of U .

Definition 2.4 The 1-graph of a function $f: M \rightarrow R$ of $n+1$ variables is the submanifold constituted by 1-jets of f at all points of M , i.e., it is an $n+1$ dimensional surface in a $2n+3$ dimensional space.

The solution of equation (2.7) generates a one-parameter group of diffeomorphic transformations, on the subset E of the jet space $J^1(M, R)$, for which the field of contact planes Π remains invariant. From each noncharacteristic point of the initial jet-manifold Γ in $J^1(M, R)$ one locally obtains a jet-characteristic, defined as the integral curve of the vector field ξ . The solutions of equation (2.1) are constituted by those functions on M whose 1-graphs coincide with jet-characteristics on E . To find such a solution, one solves the set of $2n+3$ first order ordinary differential equations (2.7) for the jet-characteristics of (2.1) on E , and performs a number of algebraic operations (see Arnold, 1984, pp.370) to obtain the graph of the mapping $v = v(x, t)$, $p = \partial v(x, t)/\partial x$, $q = \partial v(x, t)/\partial t = \mathcal{F}(v(x, t), x, t, p(x, t))$ in $J^1(M, R)$. The function $v(x, t)$ is a solution of (2.1) with initial condition $v|_{t=0} = \phi$. We call a characteristic for (2.1) the functions defined on M which correspond to the jet-characteristics of (2.1) in E . In other words, a function f , defined on M with values in N , is a characteristic of (2.1) if its prolongation to $J^1(M, R)$ is a jet-characteristic of (2.1). The field of directions in TN whose associated flow coincides locally with the characteristics of (2.1), will be termed the characteristic direction of (2.1). We specify a characteristic vector field, $\kappa(\eta)$, defined in TN , as a vector field whose span is the characteristic direction of (2.1).

Notice that since $\xi(z)$ is a prolongation of $\kappa(\eta)$ to $J^1(M, R)$, the first $n+2$ components of $\xi(z)$ necessarily coincide with those of κ . According to the prolongation formula for vector fields (Olver, 1986, pp. 108-111), the last $n+1$ components of $\xi(z)$ are therefore discarded when an explicit expression for $\kappa(\eta)$ is sought once ξ is explicitly known as $\xi(\eta)$, i.e. with its components written as explicit functions of the local coordinates $\eta = (v, x, t)$ of N .

Definition 2.5 A given $n+1$ dimensional manifold S in N , characterized by the smooth graph of the function $v = \phi(x, t)$ in N , is said to be locally invariant with respect to the dynamical system (2.1) if for some noncharacteristic set of initial data (ϕ, γ) , whose graph lies on S , the solution of (2.1) locally coincides with S .

An equivalent definition of invariance can be given in terms of the field of jet-characteristic directions or, alternatively, in terms of the characteristic vector field associated to (2.1).

Definition 2.6 An $n+1$ dimensional manifold S in N is locally invariant with respect to the system (2.1) if the jet-characteristics of (2.1), i.e., the integral curves of $\xi(z)$ in E , locally belong to the prolongation $S^{(1)}$ of S to $J^1(M, R)$. Alternatively, S is locally invariant with respect to (2.1) if the characteristic vector field $\kappa(\eta)$ in TN locally belongs to the distribution tangent to S .

3. DISTRIBUTED SLIDING MODE CHARACTERIZATION IN VARIABLE STRUCTURE CONTROLLED NFOPDE

3.1 Distributed Variable Structure Controlled Systems described by NFOPDE and their associated Sliding Regimes.

Consider a dynamical distributed system described by a controlled NFOPDE:

$$\partial v/\partial t + \Phi(v, x, t, u, p) = 0 \quad (3.1)$$

$$y = h(v, x, t) \quad (3.2)$$

where y is the scalar-valued smooth output function defined on the open set N of R^{n+2} , x represents the vector of local spatial coordinate functions x_i ($i = 1, \dots, n$), t denotes time, while $u = u(v, x, t)$ is a distributed smooth time-varying feedback control law taking

values in R . The function v is regarded as the distributed "state" of the controlled system. p is an n dimensional vector with components p_i representing the partial derivatives, $\partial v/\partial x_i$ ($i = 1, \dots, n$). As before, q denotes $\partial v/\partial t$. Φ and h are smooth functions of all their arguments.

Available to the controller is a distributed variable structure feedback switching law

$$u = \begin{cases} u^+(v, x, t) & \text{for } y > 0 \\ u^-(v, x, t) & \text{for } y < 0 \end{cases} \quad (3.3)$$

with $u^+(v, x, t) > u^-(v, x, t)$, locally on N .

The condition $y = 0$ is assumed to locally define an isolated smooth manifold solution $v = \phi(x, t)$ in N , i.e., $h(\phi(x, t), x, t) = 0$. The graph of v is assumed to be a smooth time-varying surface with locally nonzero gradient except possibly on a set of measure zero. The zero level set of h is addressed as the sliding manifold, or the sliding surface, and is locally defined as

$$S = \{ \eta = (v, x, t) \in N \subset R^{n+2} : v = \phi(x, t) \} \quad (3.4)$$

The manifold S can be prolonged (See Olver, 1986, pp. 97-119) to the space of 1-jets of functions, $J^1(M, R)$, defined on M . For this, we simply complete the set of coordinate values at each point of S with those of the partial derivatives $s_x(x, t)$, $s_t(x, t)$ at that point of S . In this manner we obtain the 1-graph of the sliding surface in $J^1(M, R)$ as:

$$S^{(1)}(z) = \{ z \in J^1(M, R) : z = (\phi(x, t), x, t, s_x(x, t), s_t(x, t)) \} \quad (3.5)$$

For an unspecified control u , system (3.1): $\partial v/\partial t + \Phi(v, x, t, u, p) = q + \Phi(v, x, t, u, p) = 0$, can be interpreted as an hypersurface E^u , defined on the manifold $J^1(M, R)$, parametrized by the control function u . If $u = u(v, x, t)$ is a fixed smooth distributed feedback control function, the closed loop system (2.1), $\partial v/\partial t + \Phi(v, x, t, u(v, x, t), p) = q + \mathcal{F}(v, x, t, p) = 0$, corresponds to an hypersurface, denoted by E , in the manifold $J^1(M, R)$. In order to introduce a suitable parametrization of the hypersurfaces representing the variable structure controlled system (3.1), (3.2) we rewrite the controlled system (3.1), (3.3) in an equivalent form using a distributed switch position function ν , taking values in the discrete set $\{0, 1\}$ and defined on each point of coordinates $\eta = (v, x, t)$, according to the value of the scalar output function $y = h(v, x, t)$. The switch function ν acts then as a binary-valued distributed control parameter:

$$\partial v/\partial t + \nu \Phi(v, x, t, u^+(v, x, t), p) + (1-\nu) \Phi(v, x, t, u^-(v, x, t), p) = 0 \quad (3.6)$$

with :

$$\nu \in \{0, 1\} \quad (3.7)$$

We will simply write the obtained switch-controlled system, with the obvious indentifications, as :

$$\partial v/\partial t + F(v, x, t, p) + \nu G(v, x, t, p) = 0 ; y = h(v, x, t) \quad (3.8)$$

$$\nu = \begin{cases} 1 & \text{for } y > 0 \\ 0 & \text{for } y < 0 \end{cases} \quad (3.9)$$

Thus, corresponding to the controlled system (3.8)-(3.9) one has, respectively, two hypersurfaces, E^+ and E^- , defined in $J^1(M, R)$ as :

$$E^+ = \{ z \in J^1(M, R) : q + F(v, x, t, p) + G(v, x, t, p) = 0 \} \quad (3.10)$$

$$E^- = \{ z \in J^1(M, R) : q + F(v, x, t, p) = 0 \} \quad (3.11)$$

The set of corresponding controlled jet-characteristics in E^+ and E^- are, according to the notation of Section

2.1. generated by the vector fields $\xi^+(z)$ and $\xi^-(z)$, which we notationally unify under the parametrized vector field: $\xi(z, \nu)$, with $\nu \in [0, 1]$. We thus denote by $\xi(z, 1)$ the vector field $\xi^+(z)$ and by $\xi(z, 0)$ the vector field $\xi^-(z)$, i.e., $\xi(z, \nu) = \nu \xi^+(z) + (1-\nu) \xi^-(z)$. The components of the vector fields $\xi^+(z)$, $\xi^-(z)$ are described, respectively, by the right hand sides of the following set of ordinary differential equations:

$$\begin{aligned} \dot{v} &= p(F_p + G_p) + q & \dot{v} &= pF_p + q \\ \dot{x} &= F_p + G_p & \dot{x} &= F_p \\ t=1 & & t=1 & \\ \dot{p} &= -(F_v + G_v)p - (F_x + G_x) & \dot{p} &= -F_v p - F_x \\ \dot{q} &= -(F_v + G_v)q - (F_t + G_t) & \dot{q} &= -F_v q - F_t \end{aligned} \quad (3.12)$$

or, briefly, in the local coordinates z of $J^1(M, R)$ by:

$$\dot{z} = \xi^+(z); \quad \dot{z} = \xi^-(z) \quad (3.13)$$

It follows, from (3.12), that the discontinuously controlled vector field $\xi(z, \nu)$, with $\nu \in [0, 1]$, is described by:

$$\begin{aligned} \dot{v} &= p(F_p + \nu G_p) + q, \quad \dot{x} = F_p + \nu G_p \\ t=1 & \quad \dot{p} = -(F_v + \nu G_v)p - (F_x + \nu G_x) \\ \dot{q} &= -(F_v + \nu G_v)q - (F_t + \nu G_t) \end{aligned} \quad (3.14)$$

i.e.,

$$\begin{aligned} \dot{z} &= \xi(z, \nu) = \nu \xi^+(z) + (1-\nu) \xi^-(z) \\ &= \xi^+(z) + \nu (\xi^+(z) - \xi^-(z)) \end{aligned} \quad (3.15)$$

The contact distribution Π intersects the field of planes tangent to E^+ , and E^- and creates on such tangent distributions, Θ^+ and Θ^- , fields of $2n+1$ -dimensional planes called the *extreme characteristic planes*. In the manner indicated in Section 2, such fields of characteristic planes uniquely determine the corresponding fields of jet-characteristic directions specified, in $TJ^1(M, R)$, by the vector fields: $\xi^+(z)$ and $\xi^-(z)$. There exist integral submanifolds (curves) of these fields of directions in E^+ and E^- which correspond to 1-graphs of functions on M . Such submanifolds will be termed, in E^+ and E^- , the *extreme controlled jet-characteristics*.

The extreme jet-characteristics in $J^1(M, R)$ uniquely define *extreme characteristics* in the open set N by simple projection. These are obtained by identification of integral submanifolds of the jet-characteristics with 1-graphs of functions defined on M which constitute solutions of the corresponding NFOPDE. Associated to such characteristics one defines *extreme characteristic vector fields* $\kappa^+(\eta)$ and $\kappa^-(\eta)$ in TN whose prolongations to $TJ^1(M, R)$ (See Olver, 1986, p.104) coincide, respectively, with the jet-characteristic vector fields $\xi^+(z)$ and $\xi^-(z)$. It is evident that the vector fields $\kappa^+(\eta)$ and $\kappa^-(\eta)$, defined in TN , are uniquely determined. It is easy to see that the parametrization previously adopted for the extreme jet-characteristic fields $\xi^+(z)$ and $\xi^-(z)$ in terms of the switching control function ν , denoted as $\xi(z, \nu)$, is trivially inherited by the corresponding extreme characteristic vector fields $\kappa^+(\eta)$ and $\kappa^-(\eta)$ (the argument being that the projection from $TJ^1(M, R)$ to TN does not destroy the adopted parametrization since such an operation simply amounts to deleting the prolongation components in $\xi^+(z)$ and $\xi^-(z)$). In correspondence with such an inherited parametrization, we also denote by $\kappa(\eta, 1)$ the vector field $\kappa^+(\eta)$ and by $\kappa(\eta, 0)$ the vector field $\kappa^-(\eta)$. One trivially has $\kappa(\eta, \nu) = \nu \kappa^+(\eta) + (1-\nu) \kappa^-(\eta)$. It is easy to see from the prolongation formula for vector fields (Olver 1986, p.104) that, conversely, such a parametrization is preserved, via prolongation, in the controlled jet-characteristic vector fields, i.e., denoting by "pr κ " the prolongation of the vector field κ we have: pr $\kappa(\eta, \nu) = \text{pr}[\nu \kappa^+(\eta) + (1-\nu) \kappa^-(\eta)] =$

$$\nu \text{pr} \kappa^+(\eta) + (1-\nu) \text{pr} \kappa^-(\eta) = \nu \xi^+(z) + (1-\nu) \xi^-(z) = \xi(z, \nu)$$

Definition 3.1. A distributed sliding regime is said to locally exist on an open set \mathcal{M} ($= N \cap S$) of the sliding manifold S if the total time derivative of the output function of the controlled system (3.8)-(3.9) satisfies:

$$\lim_{y \rightarrow +0} dy/dt < 0 \quad \text{and} \quad \lim_{y \rightarrow -0} dy/dt > 0 \quad (3.16)$$

Theorem 3.2. Given Cauchy data, (ϕ, γ) , defining an initial submanifold \mathcal{G} of noncharacteristic points in N , there locally exists a distributed sliding regime for the solutions of system (3.8), (3.9) on an open set \mathcal{M} of S if and only if the extreme characteristics (or phase flows corresponding to the extreme controlled characteristic direction fields $\kappa^+(\eta)$, $\kappa^-(\eta)$) which arise from the initial submanifold \mathcal{G} exhibit such a local sliding regime on \mathcal{M} , in correspondence with the distributed control policy (3.9).

Proof. Suppose a distributed sliding mode locally exists for (3.8), (3.9) on an open set \mathcal{M} of S . Then, the total time derivatives of y -computed on any point η of an open $n+2$ dimensional neighborhood of S with nonempty intersection with \mathcal{M} - satisfy conditions (3.16) respectively on the regions $y < 0$ and $y > 0$. The total time derivative at any point η in N can be computed in terms of the directional derivative of the scalar function η along the controlled characteristic direction fields $\kappa^+(\eta)$, $\kappa^-(\eta)$. The directional derivative depends on the location of the point η with respect to the sliding surface S and, hence, it is given by:

for $y > 0$:

$$\begin{aligned} dy/dt &= [\partial h / \partial v] dv/dt + [\partial h / \partial x] dx/dt + [\partial h / \partial t] \\ &= [\partial h / \partial \eta] \kappa^+(\eta) = L \kappa^+(\eta) \quad h = L \kappa(\eta, 1) \quad h < 0. \end{aligned}$$

for $y < 0$:

$$\begin{aligned} dy/dt &= [\partial h / \partial v] dv/dt + [\partial h / \partial x] dx/dt + [\partial h / \partial t] \\ &= [\partial h / \partial \eta] \kappa^-(\eta) = L \kappa^-(\eta) \quad h = L \kappa(\eta, 0) \quad h > 0. \end{aligned}$$

Hence, the flows corresponding to the characteristic vector fields $\kappa^+(\eta)$ and $\kappa^-(\eta)$ satisfy the conditions for the existence of a sliding regime on S (See Sira-Ramirez, 1989b).

Sufficiency is easily obtained by assuming that a sliding regime locally exists for the flows corresponding to the extreme controlled characteristic direction fields $\kappa^+(\eta)$ and $\kappa^-(\eta)$, about an open set N of the sliding manifold $y = 0$, while hypothesizing that a distributed sliding mode *does not* exist on such a set. By reversing the above arguments, a contradiction is readily established. \square

Remark. Existence of local distributed sliding regimes for (3.8)-(3.9) on open subsets \mathcal{M} of the sliding manifold S are completely characterized in terms of the existence of local sliding regimes -on the same sliding manifold S - for the $(n+2)$ -dimensional time-varying dynamical system generating the control parametrized characteristics of (3.8) in N :

$$\begin{aligned} d\eta/dt &= \kappa(\eta, \nu) = \nu \kappa^+(\eta) + (1-\nu) \kappa^-(\eta) \\ &= \kappa^-(\eta) + \nu [\kappa^+(\eta) - \kappa^-(\eta)] \quad y = h(\eta) \end{aligned} \quad (3.17)$$

with ν given by (3.9), according to the sign of y

We refer to (3.17)-(3.9) as the *controlled characteristic system*.

The problem of characterizing distributed sliding regimes in controlled dynamical systems described by NFOPDE is thus reduced to the problem of characterizing sliding regimes for an associated

nonlinear time-varying dynamical system described by a finite set of discontinuously controlled ordinary differential equations representing the controlled characteristic system. All known results for the description of sliding motions in finite dimensional nonlinear controlled dynamical systems thus become immediately available for the above formulated distributed control problem. The following theorems characterize the existence of distributed sliding motions for (3.8)-(3.9) in terms of the associated controlled characteristic system (3.17)-(3.9). Some additional results can also be found in Sira-Ramirez (1988 and 1989b).

Theorem 3.3. Let $L_{\mathcal{X}}h$ denote the directional (Lie) derivative of the scalar function h with respect to the vector field \mathcal{X} . A distributed sliding regime locally exists on an open set \mathcal{M}^* of S for the distributed system (3.8)-(3.9) only if the extreme characteristic vector fields $\mathcal{X}^+(\eta)$, $\mathcal{X}^-(\eta)$, associated to the system (3.17)-(3.9), satisfy, on the open set \mathcal{M}^* , the following distributed transversality condition:

$$L[\mathcal{X}^+(\eta) - \mathcal{X}^-(\eta)]h < 0 \quad (3.18)$$

Proof. Let there exist a distributed sliding regime on an open set \mathcal{M}^* of S . Then, according to the result of theorem 3.2 one has, on any point η of S : $L_{\mathcal{X}^+(\eta)}h < 0$ and $L_{\mathcal{X}^-(\eta)}h > 0$. In other words, $L_{\mathcal{X}^+(\eta)}h - L_{\mathcal{X}^-(\eta)}h = L[\mathcal{X}^+(\eta) - \mathcal{X}^-(\eta)]h < 0$. \square

3.2 Characterization of the Distributed Ideal Sliding Dynamics and the Distributed Equivalent Control

In this section a characterization is presented for the Distributed Ideal Sliding Dynamics and the Distributed Equivalent Control (Utkin, 1978). Intuitively, under ideal distributed sliding mode conditions, the closed loop system solution must locally adopt the zero output level set S as a local solution manifold of the resulting controlled partial differential equation, provided the (noncharacteristic) initial Cauchy data is specified precisely on the sliding manifold S . The characteristics of the ideal sliding dynamics must then be invariant with respect to the sliding manifold. The smooth distributed feedback control law responsible for the idealized controlled response is known as the Distributed Equivalent Control and it formally replaces the distributed switch position function v in the system equations (3.17). Such a smooth distributed feedback control law is here denoted by $v^{EQ}(v,x,t)$ or $v^{EQ}(\eta)$.

Definition 3.4. Consider the $n+1$ dimensional distribution in TN , tangent to the sliding manifold S . Any smooth vector field $\mathcal{X}(\eta)$, locally belonging to such a distribution, annihilates the 1-form dh , i.e., $\langle dh, \mathcal{X}(\eta) \rangle = 0$. We denote such a distribution by $\text{Ker } dh$ and define it as: $\text{Ker } dh = \{ \mathcal{X}(\eta) \in TN : \langle dh, \mathcal{X}(\eta) \rangle = 0 \}$.

The following proposition is just a restatement, in simple terms of the annihilating distribution of a 1-form, of an invariance criterion for partial differential equations appearing, in terms of symmetry groups and infinitesimal generators, in Olver (1986, p.103 Thm 2.27 and p. 165, Thms. 2.71 and 2.72).

Proposition 3.5 The prolongation $\xi(z)$ of a vector field $\mathcal{X}(\eta)$ to $TJ^1(M,R)$ (denoted by $\xi = pr \mathcal{X}$), belongs to the distribution tangent to $S^{(1)}$ in $TJ^1(M,R)$ if and only if $\mathcal{X}(\eta) \in \text{Ker } dh$.

Hence, there exists a 1-form ω in the cotangent space $T^{*1}(M,R)$ such that $\langle \omega, \xi(z) \rangle = 0$. We define such a 1-form ω as the prolongation of the 1-form dh and denote it by " $pr dh$ ". In other words, Proposition 3.5 states that $\langle dh, \mathcal{X} \rangle = 0$ if and only if $\langle pr dh, pr \mathcal{X} \rangle = 0$.

The next proposition states that the smoothly controlled jet-characteristics, generated by the controlled vector field $\xi(z,v)$ in $TJ^1(M,R)$, locally adopt the prolongation $S^{(1)}$ of the sliding manifold S as their integral manifold if and only if the corresponding characteristics, generated by the controlled vector field $\mathcal{X}(\eta,v)$ in TN , locally adopt S as their integral manifold. Hence the invariance of the controlled jet-characteristics with respect to $S^{(1)}$ is equivalent to the invariance of the controlled characteristics with respect to S .

Proposition 3.6 Consider the prolongation $S^{(1)}$ of S to $TJ^1(M,R)$. The controlled jet-characteristic direction field $\xi(z,v) = v\xi^+(z) + (1-v)\xi^-(z)$ belongs, for some smooth control function $v(\eta)$, to the distribution in $TJ^1(M,R)$ tangent to $S^{(1)}$ if and only if the smoothly controlled characteristic direction field $\mathcal{X}(\eta,v) = v\mathcal{X}^+(\eta) + (1-v)\mathcal{X}^-(\eta)$, whose prolongation to $TJ^1(M,R)$ coincides with $\xi(z,v)$, belongs to the distribution tangent to S i.e., to the annihilating distribution of the 1-form dh , here denoted by $\text{Ker } dh$.

Proof. Let $\mathcal{X}(\eta,v) \in \text{Ker } dh$, locally in \mathcal{M}^* for some smooth function $v(\eta)$. Then $\langle dh, \mathcal{X}(\eta,v) \rangle = 0$ locally in \mathcal{M}^* . Hence, by the result of Proposition 3.5 above, $\mathcal{X}(\eta,v) \in \text{Ker } dh$ if and only if $\langle pr dh, pr \mathcal{X}(\eta,v) \rangle = \langle pr dh, \xi(z,v) \rangle = 0$ i.e., if and only if $\xi(z,v) \in \text{Ker } pr dh$. \square

The Distributed Ideal Sliding Dynamics is thus obtained by imposing an invariance condition, with respect to the manifold S , on the ideally smooth controlled characteristic flows arising from the controlled characteristic system (3.17)-(3.9). This amounts to constraining the corresponding smoothly controlled characteristic vector field $\mathcal{X}(\eta,v) = v\mathcal{X}^+(\eta) + (1-v)\mathcal{X}^-(\eta)$ to the distribution, $\text{Ker } dh$, tangent to the given sliding surface S . Thus,

$$\begin{aligned} \langle dh, \mathcal{X}^-(\eta) + vEQ(\eta)[\mathcal{X}^+(\eta) - \mathcal{X}^-(\eta)] \rangle \\ = L_{\mathcal{X}^-(\eta)}h + vEQ(\eta)[\mathcal{X}^+(\eta) - \mathcal{X}^-(\eta)]h = 0 \end{aligned}$$

it follows that the equivalent control is uniquely given by:

$$\begin{aligned} vEQ(\eta) &= -\langle dh, \mathcal{X}^-(\eta) \rangle / \langle dh, [\mathcal{X}^+(\eta) - \mathcal{X}^-(\eta)] \rangle \\ &= -L_{\mathcal{X}^-(\eta)}h / L[\mathcal{X}^+(\eta) - \mathcal{X}^-(\eta)]h \end{aligned} \quad (3.19)$$

Hence, by virtue of (3.18), the distributed equivalent control $v^{EQ}(\eta)$ is locally well defined on the open set \mathcal{M}^* of S where a distributed sliding regime exists. The following theorem gives a sufficient condition for the existence of a local distributed sliding regime in terms of the computable distributed equivalent control function $v^{EQ}(\eta)$.

Theorem 3.7. A local distributed sliding regime exists on an open set \mathcal{M}^* of S if and only if the distributed equivalent control, $v^{EQ}(\eta)$, satisfies:

$$0 < v^{EQ}(\eta) < 1 \quad (3.20)$$

Proof. From (3.19), it readily follows that if (3.20) holds valid on an open set \mathcal{M}^* of the sliding manifold S then,

$$0 < -\langle dh, \mathcal{X}^-(\eta) \rangle / \langle dh, [\mathcal{X}^+(\eta) - \mathcal{X}^-(\eta)] \rangle < 1 \quad (3.21)$$

By virtue of the transversality condition (3.18) and the local smoothness assumption on the involved vector fields, $\mathcal{X}^+(\eta)$, $\mathcal{X}^-(\eta)$, and the function $h(\eta)$, the left hand side inequality in (3.21) yields: $\langle dh, \mathcal{X}^-(\eta) \rangle > 0$, for points η located in \mathcal{M}^* and located on any arbitrarily small neighborhood of \mathcal{M}^* in R^{n+2} . Similarly, the right hand side inequality in (3.21) yields: $\langle dh, \mathcal{X}^+(\eta) \rangle < 0$.

The result in Theorem 3.2 completes the sufficiency part of the proof.

To prove necessity, suppose that a distributed sliding regime locally exists on the open subset \mathcal{M}^* of the manifold S . Then, from Theorem 3.3 the transversality condition: $L[\mathcal{X}^+(\eta) - \mathcal{X}^-(\eta)]h < 0$ holds locally valid.

Also, for any η in \mathcal{M}^* , the relations: $L\mathcal{X}^+(\eta)h < 0$, and $L\mathcal{X}^-(\eta)h > 0$, are locally true. It follows that there exist strictly positive smooth function $a(\eta)$ and $b(\eta)$ such that locally in \mathcal{M}^* , $a(\eta)L\mathcal{X}^+(\eta)h + b(\eta)L\mathcal{X}^-(\eta)h = L[a(\eta)\mathcal{X}^+(\eta) + b(\eta)\mathcal{X}^-(\eta)]h$
 $= L[a(\eta) + b(\eta)]\mathcal{X}^-(\eta) + a(\eta)[\mathcal{X}^+(\eta) - \mathcal{X}^-(\eta)]h$
 $= [a(\eta) + b(\eta)]L\mathcal{X}^-(\eta) + a(\eta)[a(\eta) + b(\eta)][\mathcal{X}^+(\eta) - \mathcal{X}^-(\eta)]h = 0$. From the uniqueness of the distributed equivalent control, it follows that $\nu^{EQ}(\eta) = a(\eta)/[a(\eta) + b(\eta)]$. Hence $0 < \nu^{EQ}(\eta) < 1$ as claimed. \square

A more explicit characterization of ideal distributed sliding motions and the corresponding distributed equivalent control is still possible in terms of the functions F , G and h , defining the controlled NFOPDE in (3.8). Indeed, the components of the controlled characteristic field $\mathcal{X}(\eta, \nu)$, can be obtained by projection of those of the controlled jet-characteristic direction vector $\xi(z, \nu)$ given in (3.14). However, recall that such projected components are supposed to be ultimately explicit functions of ν, x, i i.e., of η . The components of $\mathcal{X}(\eta, \nu)$ have the form:

$$\dot{\nu} = pF_p + \nu G_p, \quad \dot{x} = F_p + \nu G_p, \quad i=1 \quad (3.22)$$

with $\nu \in [0, 1]$ determining the vector fields $\mathcal{X}^-(\eta)$ and $\mathcal{X}^+(\eta)$. The distributed transversality condition (3.18) is readily interpreted as:

$$[(\partial h / \partial \nu)p + (\partial h / \partial x)]G_p = 2(\partial h / \partial x)G_p < 0 \quad (3.23)$$

and the equivalent control is just given by:

$$\nu^{EQ}(\eta) = -L\mathcal{X}^-(\eta)h / L[\mathcal{X}^+(\eta) - \mathcal{X}^-(\eta)]h =$$

$$= -\frac{\left[\frac{\partial h}{\partial \nu} (pF_p + q) + \frac{\partial h}{\partial x} F_p + \frac{\partial h}{\partial t} \right]}{2 \frac{\partial h}{\partial x} G_p} = -\frac{\left[\frac{\partial h}{\partial x} F_p + \frac{\partial h}{\partial t} \right]}{\frac{\partial h}{\partial x} G_p} \quad (3.24)$$

The ideal sliding dynamics is obtained by formally using the distributed smooth control $\nu^{EQ}(\eta)$ from (3.24) in the system equations (3.8), with initial Cauchy data ξ specified precisely on the sliding manifold S .

4. CONCLUSIONS AND SUGGESTIONS FOR FURTHER RESEARCH

The theory of Variable Structure Systems undergoing local sliding motions has been extended to distributed controlled systems described by first order nonlinear PDE's. The key property of such a class of dynamical systems is the possibility of relating properties of their controlled solutions to those of a controlled system described by a finite set of ordinary differential equations known as the characteristic equations. This property was used in this article to establish conditions for the local existence of a distributed sliding regime on a given sliding manifold for nonlinear first order distributed dynamical systems solved with respect to the time derivative. A distributed sliding mode locally exists for the infinite dimensional dynamical system whenever the corresponding finite dimensional controlled characteristic system exhibits such kind of motions on the prescribed sliding manifold. The given

sliding manifold also qualifies as a local integral manifold for the flows of a smoothly controlled (equivalent) characteristic direction field. The equivalent characteristic direction field coincides with Filippov's average direction field also prescribed by the Equivalent Control Method (Utkin, 1978) on the finite dimensional discontinuously controlled characteristic system.

The case of distributed sliding regimes in dynamical systems described by higher order nonlinear partial differential equations, of the implicit or explicit type, can be adequately treated from a Lie group theoretic viewpoint by using notions of symmetry groups, invariance, and prolongations to appropriate jet spaces, of associated functions and infinitesimal generators of such symmetry groups. This avenue is left as a topic for further research.

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