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A DYNAMICAL VARIABLE STRUCTURE CONTROL STRATEGY IN ASYMPTOTIC OUTPUT TRACKING PROBLEMS

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Abstract In this article, using Fliess' Generalized Observability Canonical Form (GOCF), a dynamical discontinuous feedback strategy of the sliding mode type is presented for asymptotic output tracking problems in nonlinear dynamical systems. A truly effective smoothing of the sliding mode controlled responses is possible, while substantially reducing the chattering for the control input.

1. INTRODUCTION

Asymptotic Output Tracking problems, in nonlinear dynamical systems, have been extensively studied in the Control Systems literature. Contributions from a *differential geometric* viewpoint are summarized in Isidori's outstanding book [1], where clear connections are established with the concept of the *Inverse System,* and the *Zero Dynamics.*

Recently, *Differential Algebra* has been proposed by Prof. M. Fliess for the study of nonlinear controlled systems (See Fliess [2]). Among many other deep contributions, Fliess' remarkable studies have found that *implicit ordinary differential equations* account for a more general, and enlightening, setting from which a unified treatment is possible for basic control theoretic concepts. Within this viewpoint, *generalized canonical forms,* for linear and nonlinear controlled systems, are introduced which explicitly exhibit time derivatives of the control input functions on the state and output equations.

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Section 2 of this article presents the asymptotic output tracking problem from the perspective of *Dynamical Variable Structure Feedback Control.* The proposed control scheme is based on Fliess' *Generalized Observability Canonical Form* (GOCF) [2]. The approach represents a viable feedback alternative exhibiting attractive features such as robustness and, more importantly, certain degree of input-output smoothness, dependent upon the relative degree of the system. The approach is especially suitable for controlling some electro-mechanical devices (See Sira-Ramirez *et al* [3]). In section 3, we present, along with computer simulations, an application example that illustrates the advantages of the proposed controller for a DC-motor angular velocity tracking task. The concluding remarks, and proposals for further work, are collected in Section 4.

A different approach to dynamically generated sliding regimes, has been presented by Fliess and Messager in [4].

2. ASYMPTOTIC OUTPUT TRACKING VIA DYNAMICAL VARIABLE STRUCTURE FEEDBACK CONTROL

The following proposition is quite basic in the developments presented in this section:

Proposition Let μ and W represent strictly positive quantities and let "sgn" stand for the *signum* function. Then, the scalar discontinuous system:

$$\dot{w} = -\mu(w + W \operatorname{sgn} w) \qquad (2.1)$$

globally exhibits a sliding regime on $w = 0$. Furthermore, any trajectory starting on the initial value $w = w(0)$, at time $t = 0$, reaches the condition $w = 0$ in finite time T, given by:

$$T = \mu^{-1} \ln\left(1 + \frac{|w(0)|}{W}\right)$$

Proof Immediate upon checking that globally: $w \frac{dw}{dt} < 0$ for $w \neq 0$, which is well known condition for sliding mode existence (See Utkin [5]). The second part follows easily from the linearity of the two intervening system "structures".

Let α be a strictly positive integer. Consider a nonlinear dynamical system expressed in Fliess' GOCF [2]:

$$\begin{aligned} \dot{\eta}_1 &= \eta_2 \\ \dot{\eta}_2 &= \eta_3 \\ &\cdots \\ \dot{\eta}_{n-1} &= \eta_n \\ \dot{\eta}_n &= c(\eta, u, \dot{u}, \ldots, u^{(\alpha)}) \\ y &= \eta_1 \end{aligned} \tag{2.2}$$

Under rather mild conditions, any analytic nonlinear system, given in the traditional Kalman state variable representation, $\frac{dx}{dt} = F(x,u)$, $y = h(x)$, can be transformed to Fliess' GOCF by means of a suitable input-dependent state coordinate transformation (see Conte *et al* [6]). Notice that the integer α in (2.2) is intimately related to the *relative degree* r of the system [1] by the relation: $\alpha = n - r$. Hence, α coincides with the dimension of the *zero dynamics*.

Define a tracking error function $e(t)$ as the difference between the actual system output $y(t)$ and a desired output reference signal $y_R(t)$:

$$e(t) = y(t) - y_R(t) \tag{2.3}$$

We then have:

$$e^{(i)}(t) = \eta_{i+1} - y_R^{(i)}(t)\,; \quad 0 \leq i \leq n-1 \tag{2.4}$$

$$e^{(n)}(t) = \dot{\eta}_n - y_R^{(n)}(t) = c(\eta, u, \dot{u}, \ldots, u^{(\alpha)}) - y_R^{(n)} \tag{2.5}$$

Defining $e_i = e^{(i-1)}$ $(i = 1,2,\ldots,n)$, as components of an error vector **e**, we may also express the tracking error system (2.4)-(2.5) in GOCF as:

$$\begin{aligned} \dot{e}_1 &= e_2 \\ \dot{e}_2 &= e_3 \\ &\cdots \\ \dot{e}_{n-1} &= e_n \\ \dot{e}_n &= c(\xi_R(t) + e, u, \dot{u}, \ldots, u^{(\alpha)}) - y_R^{(n)}(t) \\ e &= e_1 \end{aligned} \tag{26}$$

with:

$$\begin{aligned} \xi_R(t) &= \mathrm{col}\left(y_R(t), y_R^{(1)}(t), \ldots, y_R^{(n-1)}(t)\right) \\ e &= \mathrm{col}(e_1, e_2, \ldots, e_n) \end{aligned} \tag{2.7}$$

Suppose that the asymptotic equilibrium point of the controlled tracking error system (2.6), for some suitable control input strategy, is given by $e_1 = e_2 = \cdots = e_n = 0$. Hence, under such an equilibrium condition, i.e., under perfect tracking, the system exhibits the following "remaining dynamics" or "***inverse dynamics***":

$$c(\xi_R(t), u, \dot{u}, \ldots, u^{(\alpha)}) = y_R^{(n)}(t) \tag{2.8}$$

The stability features of (2.8) for reference signals $y_R(t)$ which are bounded with bounded derivatives, $y_R^{(i)}(t)$ $(i = 1,2,\ldots,n)$ also determine, to a large extent, the physical realizability of any tracking control strategy which asymptotically achieves the perfect tracking condition $e = 0$. We assume that the solution u of (2.8) is defined for all times, and is bounded for all bounded input functions $y_R(t)$ which also exhibit bounded derivatives.

Let the set of real coefficients $\{m_0, \ldots, m_{n-2}\}$ be such that the following polynomial, in the complex variable "s", is Hurwitz;

$$s^{n-1} + m_{n-2}s^{n-2} + \cdots + m_1 s + m_0 \tag{2.9}$$

Consider now an auxiliary scalar output variable w, defined in terms of the output tracking error coordinates e_i $(i = 1,\ldots,n)$ as:

$$w = \sum_{i=1}^{n} m_{i-1} e^{(i-1)} = \sum_{i=1}^{n} m_{i-1} e_i\,; \quad \text{with } m_{n-1} = 1$$

If we impose on the evolution of the auxiliary output variable w, the discontinuous dynamics considered in (2.1), one obtains, from (2.6) and (2.10):

$$\dot{w} = \dot{e}_n + \sum_{i=1}^{n-1} m_{i-1} e_{i+1}$$

$$= -\mu\left[\sum_{i=1}^{n} m_{i-1} e_i + W\,\mathrm{sgn}\left(\sum_{i=1}^{n} m_{i-1} e_i\right)\right] \quad (2.11)$$

Using (2.5) one obtains the following dynamical feedback controller in terms of an *implicit ordinary differential equation* with discontinuous right hand side:

$$c(\xi_R + e,\ u, \dot{u}, \ldots, u^{(\alpha)}) = y_R^{(n)} - \sum_{i=1}^{n-1} m_{i-1} e_{i+1}$$

$$-\mu\left[\sum_{i=1}^{n} m_{i-1} e_i + W\,\mathrm{sgn}\left(\sum_{i=1}^{n} m_{i-1} e_i\right)\right] \quad (2.12)$$

On each one of the regions $w > 0$, and $w < 0$, a different feedback control "structure" is generated by (2.12) and the corresponding implicit differential equation is to be independently solved for the controller u, on the basis of knowledge of the error vector e and the vector of functions $\xi_R(t)$. Under the additional assumption that, locally, $\partial c/\partial u^{(\alpha)}$ is non zero in (2.12), then no singularities, of the *impasse* points type (Fliess and Hasler [7]), need be considered. Moreover, by virtue of the implicit function theorem, controller equation (2.12) is then locally equivalent to an *explicit system of first order discontinuous differential equations* which can be solved on line with no further difficulties than those involved in, say, a dynamical sliding mode observer system acting in a closed loop scheme.

It follows from (2.6), (2.10) and the invariance conditions $w = 0$; $\frac{dw}{dt} = 0$, that the *ideal sliding dynamics* [5] is non-redundantly given by:

$$\begin{aligned} \dot{e}_1 &= e_2 \\ \dot{e}_2 &= e_3 \\ &\cdots \\ \dot{e}_{n-1} &= -\sum_{i=1}^{n-1} m_{i-1} e_i \end{aligned} \quad (2.13)$$

which exhibits an asymptotically stable motion toward the origin of the error vector coordinates, with eigenvalues uniquely specified by the prescribed set of constant coefficients $\{m_0, \ldots, m_{n-2}\}$. In particular, the output tracking error function $e_1 = \eta_1 - y_R(t)$ asymptotically converges to zero, as desired.

Remark Two important advantages can be readily established about the dynamical variable structure controller represented by (2.12). The first one is the fact that the output tracking error function e(t) asymptotically approaches zero with substantially reduced, or smoothed out, "chattering". Secondly, and this is possibly the most important advantage of the approach, a traditional explicit *canonical phase variable* representation for the dynamical controller (2.12) indicates that the control input u is the outcome of α integrations, performed on a nonlinear function of the discontinuous actions that lead the auxiliary output w to zero. This implies substantially smoothed control inputs which do not result in a "bang-bang" behavior for the actuator.

3. AN APPLICATION EXAMPLE

The following nonlinear dynamical model of a field controlled DC-motor is taken from Rugh [8, pp. 98].

$$\begin{aligned} \dot{x}_1 &= -\frac{R_a}{L_a} x_1 - \frac{K}{L_a} x_2 u + \frac{V_a}{L_a} \\ \dot{x}_2 &= -\frac{B}{J} x_2 + \frac{K}{J} x_1 u \end{aligned} \quad (3.1)$$

Where x_1 represents the armature circuit current, x_2 is the angular velocity of the rotating axis. V_a is a fixed voltage applied to the armature circuit, while u is the field winding input voltage, acting as the control variable.

Suppose $y_R(t)$ is a known, desired, bounded reference trajectory for the angular velocity x_2 considered as the output function. One can obtain a GOCF for the dynamics of the tracking error $e = x_2 - y_R(t)$, by defining a time-varying input dependent state coordinate transformation of the form:

$$\begin{aligned} e_1 &= x_2 - y_R(t) \\ e_2 &= -\frac{B}{J} x_2 + \frac{K}{J} x_1 u - \dot{y}_R(t) \end{aligned}$$

$$\begin{aligned} x_2 &= e_1 + y_R(t) \\ x_1 &= \frac{J}{Ku}\left[e_2 + \frac{B}{J}(e_1 + y_R(t)) + \dot{y}_R(t)\right] \end{aligned} \quad (3.2)$$

yielding:

$$\dot{e}_1 = e_2$$

$$\dot{e}_2 = -\frac{R_a B}{L_a J}(e_1 + y_R(t)) - (\frac{B}{J} + \frac{R_a}{L_a})(e_2 + \dot{y}_R(t)) + \frac{KV_a}{L_a J}u - \frac{K^2}{L_a J}(e_1 + y_R(t))u^2 + \frac{\dot{u}}{u}\left[e_2 + \frac{B}{J}(e_1 + y_R(t)) + \dot{y}_R(t)\right] - \ddot{y}_R(t) \quad (3.3)$$

$$e = e_1$$

Notice that $u = 0$ corresponds to a singularity of the transformation (3.2) and, hence, stabilization or tracking tasks that imply polarity reversals in the field winding input voltage must be treated by different techniques which imply inducing appropriate "jumps", or discontinuities, in the input variable u or in some of its time derivatives (see Fliess *et al* [9]).

Since the problem of smoothly transferring the constant operating angular velocity, $y_R(t) = \Omega$ to a new constant reference value, $y_R(t) = \Omega^*$, eventually entitles the need for a controlled stable steady-state operation, we first study the stability features associated to the zero dynamics of system (3.3) when $y_R(t)$ is a nonzero constant of value, say, Ω.

The *zero dynamics*, is easily obtained from (3.3) as:

$$-\frac{R_a}{L_a}u + \frac{KV_a}{\Omega L_a B}u^2 - \frac{K^2}{L_a B}u^3 + \dot{u} = 0 \quad (3.4)$$

The constant equilibrium points $u = U$ of (3.4) are obtained from the solutions of the following third order algebraic equation:

$$-R_a B\, u + \frac{KV_a}{\Omega}u^2 - K^2 u^3 = 0 \quad (3.5)$$

One of the possible solutions of (3.5) corresponds to the singular equilibrium solution $u = 0$, which is, hence, discarded. The two other solutions of (3.5) are given by:

$$U = \frac{V_a K}{2\Omega}\left(1 \pm \sqrt{1 - 4\frac{R_a B \Omega^2}{V_a^2}}\right) \quad (3.6)$$

If the discriminant $D := V_a^2 - 4R_a B\Omega^2$ is negative, then there is no real solution to the stabilization problem. If, on the other hand, D is zero, or positive, then there are two real, positive, roots for u in (3.5). The stability properties of these equilibria may be directly determined via approximate linearization of (3.4).

Linearization of the zero dynamics (3.4) around the equilibrium point $u = U$ yields:

$$\dot{u}_\delta + \frac{1}{BL_a}(R_a B - K^2U^2)u_\delta = 0 \quad (3.7)$$

where $u_\delta := u - U$ represents the incremental field circuit input voltage. The linearized zero dynamics (3.7) is evidently asymptotically stable to zero, provided the constant equilibrium input voltage U satisfies the condition:

$$R_a B > K^2 U^2 \quad (3.8)$$

which identifies, together with the condition: $V_a^2 \geq 4R_a B\Omega^2$, the minimum phase region in the input-output space for the given system (3.1). The operating equilibrium points, Ω and Ω^*, associated to the smooth angular velocity transfer maneuver, defined via a suitably proposed tracking problem, must then be tested, via the corresponding values U and U^* obtained from (3.6), against condition (3.8). This will asses the stability of the corresponding zero dynamics on the involved equilibria.

Consider the auxiliary output variable w, written now in terms of the tracking error velocity and acceleration error variables, as it was defined in (2.10):

$$w = e_2 + m_0 e_1 \quad (3.9)$$

Imposing on w the discontinuous dynamics given in (2.1), a time-varying differential equation for the dynamical controller is obtained which synthesizes the control signal u, in terms of the reference input signal $y_R(t)$, its time derivative $\frac{dy_R(t)}{dt}$ and the tracking error variables e_1 and e_2. Writing, however, the variable structure dynamical feedback controller in terms of the original state coordinates x_1 and x_2 of the controlled system (3.1), one obtains

$$\dot{u} + \frac{J}{Kx_1}\left[\left(-\frac{B^2}{J^2} + \frac{B}{J}(\mu + m_0) - \mu m_0\right)x_2 + \frac{K}{J}\left(\frac{B}{J} + \frac{R_a}{L_a} - \mu - m_0\right)x_1 u - \frac{V_a K}{JL_a}u + \frac{K^2}{JL_a}x_2 u^2\right.$$

$$+ \mu m_0 y_R(t) + (\mu + m_0)\dot{y}_R(t) + \ddot{y}_R(t)$$

$$- \mu W \,\text{sign}\left[(-\frac{B}{J} + m_0)x_2 + \frac{K}{J}x_1 u - m_0 y_R(t) - \dot{y}_R(t)\right] \quad (3.10)$$

Simulation Results

Simulations of a tracking task were performed for a DC motor with the following parameter values:

$R_a = 7$ Ohm; $L_a = 120$ mH; $V_a = 5$ V;

A desired output reference trajectory $y_R(t)$ was considered which allowed for a smooth transition from a nominal (equilibrium) angular velocity Ω, to a new chosen operating angular velocity Ω^*. Such reference function was set to be:

$$y_R(t) = \begin{cases} \Omega \text{ for } 0 < t < t_1 \\ \Omega^* + (\Omega - \Omega^*)\exp(-kt^2) \text{ for } t > t_1;\ k > 0 \end{cases} \quad (3.11)$$

Figure 1 portrays the time response of the dynamical sliding mode controlled angular velocity. The dynamical variable structure controller smoothly leads the angular velocity from $\Omega = 300$ rad/s to a new operating value $\Omega^* = 200$ rad/s. The parameters of the induced dynamics (2.1) were set as: $\mu = 100$, $W = 10$, $m_0 = 20$. It may be verified that, according to the chosen values of the parameters, the initial and final angular velocities are located on the *minimum phase* region of the system. Time t_1, and the constant k, in (3.7) were set, respectively, as $t_1 = 0.5$ s and $k = 3$. Figure 2 portrays the time response of the armature circuit current for the transition maneuver, while Figure 3 shows the corresponding control input voltage trajectory exhibiting almost no chattering.

4. CONCLUSIONS

Dynamical Variable Structure Controllers accomplishing asymptotic reference output tracking are readily obtainable for nonlinear systems described in Fliess' *Local Generalized Observability Canonical Form.* Such a canonical form naturally leads to a dynamical sliding mode controller which zeroes, in finite time, an auxiliary output function defined in terms of the tracking error time derivatives. The resulting ideal sliding dynamics induces an asymptotic stabilization of the output tracking error function with eigenvalues totally prescribed at will. The obtained discontinuous controller design exhibits the advantage of effective chattering reduction for both the input and output signals, without resorting to the well-known high-gain amplifier alternative (see Slotine and Li [10]). The approach, however, requires full state feedback and it entitles dealing with the complexity of nonlinear time-varying implicit dynamical controllers, which may not be globally defined. Some of the associated difficulties include the presence of *impasse* points, or the operation of the controller in a *region of non-minimum phase* characteristics. In such pathological cases, the usual remedy indicates the use of discontinuities in the control signal. The difficulties have been properly addressed with desirable results in [9].

It should be stressed that using time-varying, input-dependent, sliding surfaces, the discontinuities associated to the proposed dynamical sliding mode control strategy take place in the *state space of the dynamical controller* and not in the state space of the system itself. This fact makes possible the application of sliding mode control techniques to areas where they were not traditionally feasible, such as: chemical process control, biological systems control, and the regulation of mechanical and electro-mechanical systems (see also [3]).

In this article a nonlinear DC motor example, dealing with smooth controlled transitions of nominal angular velocities to new constant operating values, was presented along with encouraging simulation results. As topics for further research, the dynamical variable structure feedback controller here proposed could be implemented in an actual DC-motor using nonlinear analog electronics.

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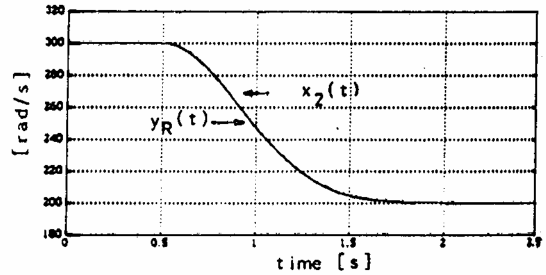

Figure 1. Angular velocity response for dynamical sliding mode controlled tracking task.

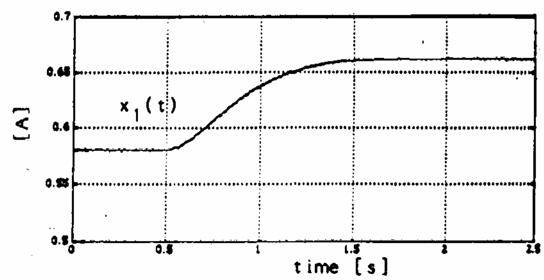

Figure 2. Time response of armature circuit current for angular velocity tracking task.

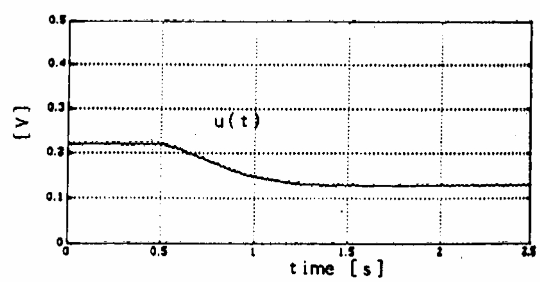

Figure 3. Control input voltage to field windings circuit for angular velocity tracking task.