NONLINEAR DYNAMICAL FEEDBACK STRATEGIES IN AERO-SPACE SYSTEMS CONTROL : A DIFFERENTIAL ALGEBRAIC APPROACH.

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Abstract. Dynamical controller schemes, based on differential algebraic results, are proposed for the feedback regulation of some typical aerospace systems. The nonlinear dynamical feedback controllers are synthesized on the basis of Fliess' Generalized Observability Canonical Form (GOCC) for the regulated system. The synthesis approach is also applicable to Nonlinear Pusle-Width-Modulation (PWM) controlled systems, commonly encountered in aerospace control problems.

Key Words. Nonlinear Controller Design, Pulse-Width-Modulation Control, Differential Algebraic Systems.

1. INTRODUCTION

A new and general approach for the study of controlled dynamical systems, based on *Differential Algebra*, has been recently introduced by M. Fliess in a series of remarkable articles [1]-[3]. This approach was shown to be suitable for the unified treatment of linear and nonlinear, lumped, or distributed, controlled dynamical plants. Feedback decoupling, invertibility, model matching and realization, have also been conceptually clarified and generalized by Fliess via this powerful and most elegant approach. Crucially based on the extension to *differential fields* of the *Theorem of the Primitive Element* [4], any controlled dynamical system, described by a set of forced ordinary differential equations, was shown to posses a *Generalized Controller Canonical Form* (GCCF) exhibiting the input and a finite number of its time derivatives. By obtaining an input-output description of the dynamical system (see also Conte *et al* [5]), one may directly derive the *Generalized Observability Canonical Form* (GOCF). The GOCF coincides with the GCCF when the given state realization is minimal and the output variable qualifies as a *differential primitive element*. Such canonical forms are obtainable, in general, by means of control-dependent state coordinate transformations. As a direct consequence of this result, the problem of *feedback linearization* and that of *input output linearization* of a controlled dynamical system is always trivially solvable, in a local manner, using nonlinear, possibly time-varying, dynamical feedback. The linearizing dynamical compensators are clearly suggested by the canonical forms themselves. However, for systems with constant operating points, the asymptotic stability of the linearized closed loop dynamics, around such an equilibrium point, crucially depends on the *minimum phase* character of the nonlinear GOCF about such an equilibrium.

In this article dynamical feedback controllers, based on Fliess' GOCF, are proposed for three typical examples of aerospace control problems. As a first example, a variation of Zermelo's problem is presented for an aircraft attempting a perfectly circular (tracking) maneuver in a region of strong air currents (Bryson and Ho [6]). In this example, the control actions related to the heading of the aircraft are assumed to be continuous. A second example, already reported in Sira-Ramirez [7], is also briefly summarized here. This example deals with the problem of robust soft controlled landing, for a thrusted spacecraft, on the surface of a planet provided with atmospheric resistance, using a dynamical ON-OFF PWM control strategy. The third example is constituted by a controlled orbital transfer for a multivariable jet-controlled space vehicle (Brockett [8]) using an ON-OFF-ON PWM control strategy. In the last two cases, the dynamical feedback regulator design is based on a suitable *average* model for the PWM controlled system and the smooth designed behavior is then closely approximated by the actual PWM controlled behavior.

Section 2 presents some general results and derivations about dynamical feedback linearizing controllers using Fliess' GOCF. Section 3 presents the three application examples, described above. Simulations are included to illustrate the performance of the proposed controllers. The Appendix contains some background material on the ON-OFF, as well as on the ON-OFF-ON, PWM control of nonlinear systems and their design oriented average models.

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2. FLIESS' GENERALIZED OBSERVABILITY CANONICAL FORM OF NONLINEAR SYSTEMS

In this section, Fliess' GOCF for nonlinear dynamical systems is presented for the sake of self-containment. The results are also found in Fliess [3].

2.1. Fliess' Generalized Observability Canonical Form for Nonlinear Systems and Exact Dynamical Feedback Linearization.

An n-dimensional, minimal, realization of a nonlinear dynamical system, given in state space form :

$$\dot{x} = F(x,u) \qquad y = h(x) \tag{2.1}$$

can be locally placed, under rather mild conditions, in *Generalized Observability Canonical Form* (GOCF):

$$\begin{aligned} &\frac{d}{dt} q_i = q_{i+1} \ ; \ i = 1,2,\cdots, n-1 \\ &\frac{d}{dt} q_n = c(q\, , u\, , \dot{u}\, , \ddot{u}\, , \cdots , u^{(\nu)}) \\ &y = q_1 \end{aligned} \tag{2.2}$$

by means of an input-dependent locally diffeomorphic state coordinate transformation [5]. In the language of differential algebra, the state representation (2.2), in *generalized phase coordinates*, is always obtainable when the output variable y qualifies as a *differential primitive element.*

Exact input-output dynamic feedback linearization is simply achieved by equating the expression in the last differential equation to a (stable) linear equation in the components of the vector q, possibly including an external reference input signal ν, as follows:

$$c(q,u,\dot{u},\ddot{u},\cdots,u^{(\nu)}) = -\alpha_1 q_1 - \alpha_2 q_2 - \cdots - \alpha_n q_n + k\,\nu \tag{2.3}$$

The last equation implicitly defines a dynamical nonlinear state feedback control law which accomplishes an exact linearization of the non-redundant dynamics. The obtained linear system has preestablished asymptotic stability properties chosen by means of the α's.

It is evident that the nonlinear dynamical feedback linearization scheme presented above is based on exact cancellation of the nonlinear plant dynamics by means of the proposed controller. By means of a straightforward linearization around a constant equilibrium point -if any- of the combined GOCF and of the proposed dynamical feedback controller, one can easily demonstrate, in the single-input single output case, that the closed loop system is locally asymptotically stable if and only if the linearized transfer function of the given plant is *minimum phase* i.e., if the linearized *zero dynamics* is asymptotically stable. The corresponding statement for multi-input systems is also valid. However, in this case, it has been shown that there exists three possible, nonequivalent, extensions of the concept of *zero dynamics* (Isidori and Moog [17]).

3. SOME AEROSPACE APPLICATIONS.

3.1. A Dynamically Controlled Circular Path Maneuver for an Aircraft flying Through a Region of Strong Winds

An aircraft must fly through a region of strong air currents whose magnitude and direction are only nominally known as functions of position : $h = h(x,y)$, $v = v(x,y)$, where (x,y) are rectangular coordinates and (h,v) are the velocity components for the current in the x and y directions, respectively. The magnitude of the aircraft's velocity, relative to the air, is assumed to be a constant V. The problem consists on maneuvering the aircraft along a circular path of given radius R. The nonlinear model describing the controlled motions is (see Bryson and Ho [6, pp. 96]):

$$\dot{x} = V\cos u + h(x,y) \quad ; \quad \dot{y} = V\sin u + v(x,y) \tag{3.1}$$

where the scalar parameter u, acting as the control input, is the heading angle of the aircraft's axis relative to the fixed coordinate axes and (x,y) represents the position of the aircraft. The control angle u takes values in the real line. This unlimited control action is to regulate, in a smooth manner, the motions of the spacecraft toward the required path and then sustain the circular trajectory.

Defining a tracking error as: $s(x,y) = x^2 + y^2 - R^2$, then, the problem of sustaining a circular trajectory in the (x,y) plane is translated into the problem of zeroing the nonlinear output function s(x,y) associated to the nonlinear dynamical system (3.1).

We proceed to derive a nonlinear dynamical feedback controller for the required maneuver.

It is easy to verify that $q := s(x,y)$ is a *differential primitive element* that allows one to write the model (3.1) in a GOCF of the form (2.2) with $\nu = 1$. Thus, denoting by h_x, h_y and v_x, v_y the partial derivatives with respect to x and y, of h and v, respectively, one computes the time derivatives of the primitive element q as:

$$q = s(x,y) = x^2 + y^2 - R^2\,;\ \dot{q} = V(x\cos u + y\sin u) + x\,h + y\,v$$
$$\ddot{q} = V^2 + 2V(h\cos u + v\sin u) + h^2 + v^2 + \dot{u}\,V(y\cos u - x\sin u)$$
$$+ (xh_x + yv_x)(V\cos u + h) + (xh_y + yv_y)(V\sin u + v) \tag{3.2}$$

where the arguments (x,y) have been dropped in h and v just for notational convenience.

The controller design is based on imposing, for the exactly linearized GOCF, an asymptotically stable dynamics, with suitably chosen eigenvalues (say, real and negative) $a_1 < 0$, $a_2 < 0$, as follows:

$$\ddot{q} - (a_1 + a_2)\dot{q} + (a_1 a_2)\,q = 0 \tag{3.3}$$

The nonlinear dynamical feedback controller accomplishes any desirable exponential rate of decay on the tracking error q, as well as on its first time derivative dq/dt. Such a nonlinear dynamical feedback controller, is immediately obtained from (3.2) and (3.3) as:

$$\frac{du}{dt} = -\frac{1}{V(y\cos u - x\sin u)}\Big\{V^2 + 2V(h\cos u + v\sin u) + h^2 + v^2 + (xh_x + yv_x)(V\cos u + h) - (a_1+a_2)[V(x\cos u + y\sin u) + x\,h + y\,v] + a_1a_2(x^2 + y^2 - R^2) + (xh_y + yv_y)(V\sin u + v)\Big\} \tag{3.4}$$

Computer simulations were carried out on the system (3.1),(3.4). The radius of the circular path was chosen as R = 300 m, while the aircraft velocity was set to V = 110 m/s. The poles for the linear system were both located at -1 + j0. The functions h(x,y) and v(x,y) were nominally chosen as constants of values h_0= 10 m/s and v_0 = 40 m/s (i.e., a wind current with fixed magnitude of 41.23 m/s and direction of 75.96 ° with respect to the x axis). However, a large unmodelled wind gust perturbation, distributed around the point (x,y) = (0,-R) of the circular path, was included in the simulation of the system model (3.1), but the corresponding expression for the perturbation function was never substituted on the designed controller (3.4). Such perturbation was assumed to be of the form :

$$v(x,y) = v_0\left\{1 + \exp\left(-\left[\left(\frac{x}{100}\right)^2 + \left(\frac{y+R}{100}\right)^2\right]\right)\right\} \tag{3.5}$$

Figure 1 shows a computer simulated trajectory in the plane x,y , depicting the response of the dynamical feedback controlled maneuver under nominal and perturbed conditions for v(x,y). Figure 2 compares the time responses of the distance error to the required circular trajectory for the perturbed and unperturbed controlled systems. This error was defined, just for numerical convenience as compared to s(x,y), as: $e(x,y) = (x^2+y^2)^{1/2} - R$.

3.2 Dynamically Controlled Soft Landing Maneuver on a Planet Including Atmospheric Resistance.

The nonlinear dynamical model describing the vertical descent, of a thrust controlled vehicle, on the surface of a planet of gravity acceleration g and non-negligible atmospheric resistance force opposing the vertical downwards motion (see Arnol'd [9, p.4]) is given by.

$$\frac{dx_1}{dt} = x_2 \quad ; \quad \frac{dx_2}{dt} = g - \left(\frac{\gamma}{x_3}\right)x_2^2 - \left(\frac{\sigma\alpha}{x_3}\right)u \quad ; \quad \frac{dx_3}{dt} = -\alpha u \tag{3.6}$$

where x_1 is the position (height) on the vertical axis, chosen here to be positively oriented downwards (i.e., $x_1 < 0$, for actual positive height), x_2 is the downwards velocity and x_3 represents the combined mass of the vehicle and the residual fuel. The function u is a binary-valued control function with values in the set { 0,1 } regulating, in a pulsed or bang-bang manner, the constant rate of ejection per unit time α and effectively acting as a control parameter. The constant σ represents the relative ejection velocity of the gases in the thruster. Thus, $\sigma\alpha$ is the maximum thrust of the braking engine, while γ is a positive quantity representing the atmospheric resistance coefficient.

The binary-valued control signal u is assumed to be synthesized on the basis of a PWM control strategy specified by :

$$u = \begin{cases} 1 & \text{for } t_k < t \le t_k + \mu[x(t_k)]T \\ 0 & \text{for } t_k + \mu[x(t_k)]T < t \le t_k + T \end{cases} \quad ; \quad k = 0,1,2,\ldots \tag{3.7}$$

where $\mu(x(t))$ is the *duty ratio* function generated in a feedback manner from knowledge of the sampled state vector x(t) at time t_k. The feedback synthesis problem is then defined as the problem of specifying a suitable duty ratio function μ, in a feedback manner. We shall base our design on the *average model* of continuous nature for the PWM feedback controlled system (3.6),(3.7) as developed in the Appendix.

A soft landing on the surface $x_1 = 0$ may be seen as a particular case of a controlled descent toward a sustained hovering about certain preestablished height $x_1 = K$. Usually, the landing maneuver entitles a regulated descent toward a small height (typically 1 m, or so, i.e. K = – 1) on which a short hovering takes place before the main thruster is safely shut off. The final touchdown stage is actually a free fall toward the surface from the small hovering height. Taking the output function of the system as $y = h(x) = x_1 - K$, the problem of sustained hovering is translated into the problem of zeroing the output y associated to the nonlinear system (3.6).

According to the results of the Appendix, the *average PWM controlled model* of the vertical descent of the controlled spacecraft is given by:

$$\frac{dz_1}{dt} = z_2 \quad ; \quad \frac{dz_2}{dt} = g - \left(\frac{\gamma}{z_3}\right)z_2^2 - \left(\frac{\sigma\alpha}{z_3}\right)\mu \quad ; \quad \frac{dz_3}{dt} = -\alpha\mu$$
$$y = z_1 - K \tag{3.8}$$

where μ is the *duty ratio* function, satisfying the limiting constraints $0 < \mu < 1$, acting as the piece-wise smooth control parameter to be designed in a dynamical feedback manner.

Remark 1 It is easy to see that, during the controlled descent, the downwards acceleration is always bounded above by zero (see Sira-Ramirez [7]).

It may also be easily verified that $q_1 = z_1 - K$ is a *differential primitive element* of system (3.8). Thus, a control-dependent state coordinate transformation of the average PWM controlled system (3.8) which leads to a GOCF of (3.9) is given by :

$$q_1 = z_1 - K\ ,\ q_2 = z_2\ ,\ q_3 = g - \frac{\gamma z_2^2 + \sigma\alpha\mu}{z_3}$$
$$z_1 = q_1 + K\ ,\ z_2 = q_2\ ,\ z_3 = \frac{\gamma q_2^2 + \sigma\alpha\mu}{g - q_3} \tag{3.9}$$

from where:

$$\begin{aligned} \dot q_1 &= q_2 \\ \dot q_2 &= q_3 \\ \dot q_3 &= -(g-q_3)\left[\frac{2\gamma q_2 g + \sigma\alpha\dot\mu}{\gamma q_2^2 + \sigma\alpha\mu}\right] + (2\gamma q_2 - \alpha\mu)\left[\frac{(g-q_3)^2}{\gamma q_2^2 + \sigma\alpha\mu}\right] \\ y &= q_1 \end{aligned} \tag{3.10}$$

The following input-derivative-dependent control space coordinate transformation exactly feedback linearizes to *Brunovsky's Observable Canonical Form* the transformed system (3.10):

$$v = -(g-q_3)\left[\frac{2\gamma q_2 g + \sigma\alpha\dot\mu}{\gamma q_2^2 + \sigma\alpha\mu}\right] + (2\gamma q_2 - \alpha\mu)\left[\frac{(g-q_3)^2}{\gamma q_2^2 + \sigma\alpha\mu}\right] \tag{3.11}$$

The exactly linearized system is now easily stabilized around the origin of transformed coordinates by a standard linear state-feedback controller of the form $v = -\alpha_1 q_1 - \alpha_2 q_2 - \alpha_3 q_3$, with suitably chosen coefficients α_1, α_2, and α_3. The dynamical feedback controller synthesizing the *computed duty ratio* (henceforth denoted by μ) accomplishes, within non saturating conditions for the actuator's duty ratio values, any desirable exponential rate of decay on the height, vertical velocity and vertical acceleration variables. Such a dynamical feedback controller, yielding the computed duty ratio μ is immediately obtained from (3.11) and the linear expression for v:

$$\frac{d}{dt}\hat\mu = \frac{\gamma q_2^2 + \sigma\alpha\hat\mu}{\sigma\alpha(g-q_3)}\left\{\alpha_1 q_1 + \alpha_2 q_2 + \alpha_3 q_3 + (2\gamma q_2 - \alpha\hat\mu)\left[\frac{(g-q_3)^2}{\gamma q_2^2 + \sigma\alpha\hat\mu}\right]\right\} - \frac{2\gamma q_2 g}{\sigma\alpha} \tag{3.12}$$

Notice that no singularity is implied by the presence of the factor $(g-q_3)^{-1}$ in (3.12) due to the established negativity of the vertical acceleration q_3 during the descent maneuver (see Remark 1). In original average coordinates, the dynamical feedback controller is given by :

$$\begin{aligned} \frac{d}{dt}\hat\mu = \frac{z_3}{\sigma\alpha}\Bigg[&\alpha_1(z_1 - K) + \alpha_2 z_2 + \alpha_3\left(g - \frac{\gamma z_2^2 + \sigma\alpha\hat\mu}{z_3}\right) \\ &+ (2\gamma z_2 - \alpha\hat\mu)\left(\frac{\gamma q_2^2 + \sigma\alpha\hat\mu}{z_3^2}\right)\Bigg] - \frac{2\gamma z_2 g}{\sigma\alpha} \end{aligned} \tag{3.13}$$

The actual duty ratio function μ is obtained by properly limiting between 0 and 1 the values of the computed duty ratio function μ , obtained as a solution of the nonlinear time-varying differential equation (3.13).

A hovering condition on y = 0 implies a zero equilibrium point for the position, vertical velocity and vertical acceleration in (3.10). As it can be easily seen, from (3.13), the (zero dynamics) hovering condition: $q_1 = q_2 = q_3 = 0$, entitles an exponentially asymptotically stable autonomous trajectory to zero for the total mass behavior. Thus, the equilibrium point of the zero dynamics is not physically meaningful. As a matter of fact, since the fuel mass is depleted in finite time, the average model (3.8) becomes unrealistic after the fuel mass has been exhausted. In spite of this fact, the controlled descent toward the surface can still be practically performed at the expense of sustained fuel mass expenditure within an allowable safety limit in the hovering condition. The final free fall descent, from the hovering position, via switching off of the main engine, must be performed so as to guarantee enough residual fuel for the ascending maneuver, if any, later on.

Simulations were performed for both the average and the discontinuous controlled landing models discussed above, with the following constant parameters:

$$\sigma = 200\ [\text{m/s}]\ ;\ \alpha = 50\ [\text{Kg/s}]\ ;$$
$$g = 3.72\ [\text{m/s}^2]\ ;\ \gamma = 1\ [\text{Kg/m}]\ ;\ K = -1\ [\text{m}]$$

The three poles of the exactly linearized closed loop system were located at $-1.2\ s^{-1}$. The sampling frequency for the PWM actuator was set at 5 samples per second, i.e., T = 0.2 s. On a planet with the given physical constants, the free fall limit velocity is 51.03 [m/s] which was taken as the downward velocity initial condition for the simulation. Figure 3 shows the evolution of the actual PWM controlled state variables x_1, x_2 and x_3 (height, vertical velocity and total mass). Initial conditions were chosen, from a free fall condition, at:

$$x_1(0) = -500\ [\text{m}]\ ,\ \ x_2(0) = 51.03\ [\text{m/sec}]\ ,\ \ x_3(0) = 700\ [\text{Kg}]$$

The obtained actual PWM trajectories exhibit a negligible difference with respect to the corresponding average PWM responses. Robustness of the controller performance, in the presence of unmodelled spatial perturbations in the coefficient of atmospheric resistance, was evaluated in Sira-Ramirez [8] by performing an experiment similar to the one presented in the previous example.

3.3. Dynamical PWM Feedback Controlled Orbital Transfer Maneuver

A well known model for a normalized (unit) mass spacecraft performing a controlled orbital transfer is given by (see Brockett [7, pp.14], and, also, Bryson and Ho [6, pp.66]) :

$$\dot r = v\ ;\ \dot v = r\omega^2 - \frac{k}{r^2} + u_1\ ;\ \dot\omega = -\frac{2v\omega}{r} + \frac{1}{r}u_2 \tag{3.14}$$

with r being the radial distance of the spacecraft from the center of the earth, v is the radial component of the velocity, ω is the angular velocity, k is the earth gravitational constant, and u_1 and u_2 represent, respectively, thrust in the radial and tangential directions. The orbital parameters are constrained to satisfy, in steady state equilibrium conditions: $r = R$, $\omega = \Omega$, $u_1 = u_2 = 0$, the relation $R^3\Omega^2 = k$. A PWM ON-OFF-ON control policy, such as that described in (A.6),(A.7), is assumed for each thruster. The control inputs u_1 and u_2 take values, respectively in the discrete sets $\{+U_1, 0, -U_1\}$ and $\{+U_2, 0, -U_2\}$. We summarize below the steps leading to a PWM dynamical controller design based on the average PWM model of the given multivariable system.

<u>Average PWM Model</u>

$$\dot r = v\ ;\ \dot v = r\omega^2 - \frac{k}{r^2} + U_1\mu_1\ ;\ \dot\omega = -\frac{2v\omega}{r} + \frac{1}{r}U_2\mu_2 \tag{3.15}$$

<u>Generalized Observability Canonical Form for the Average PWM Model</u>

Taking as the output variable $y = r - R$, it is easy to see that $q_1 = y$ qualifies as a *differential primitive element* on the basis of which we can generate a GOCF for (3.15):

$$\begin{aligned} \dot q_i &= q_{i+1}\ ;\ i = 1,2 \\ \dot q_3 &= -3q_2\left[\frac{q_3 - U_1\mu_1}{q_1 + R} + \frac{k}{(q_1+R)^3}\right] + 2U_2\mu_2\sqrt{\frac{q_3 - U_1\mu_1}{q_1+R} + \frac{k}{(q_1+R)^3}} \\ &\quad + \frac{2k\,q_2}{(q_1+R)^3} + U_1\dot\mu_1 \\ y &= q_1 \end{aligned} \tag{3.16}$$

<u>Desired Linearized PWM Average Dynamics</u>

$$\dot q_1 = q_2\ ;\ \dot q_2 = q_3\ ;\ \dot q_3 = -\alpha_1 q_1 - \alpha_2 q_2 - \alpha_3 q_3\ ;\ y = q_1 \tag{3.17}$$

<u>Dynamical Average Feedback Controller Synthesizing the Unrestricted Computed Duty Ratios</u>

Equating the last state equations in (3.16) and (3.17) leads to a time-varying differential equation for μ_1 with an indeterminate quantity represented by the duty ratio function μ_2 . This duty ratio function can thus be independently chosen. We propose to use a μ_2 which exactly linearizes, to an asymptotically exponentially stable dynamics, the last equation in (3.15).

$$\frac{d}{dt}\hat{\mu}_1 = -\frac{\alpha_1 q_1 + \alpha_2 q_2 + \alpha_3 q_3}{U_1} + \frac{3q_2}{U_1}\left[\frac{q_3 - U_1\hat{\mu}_1}{q_1+R} + \frac{k}{(q_1+R)^3}\right]$$

$$-\frac{2U_2\hat{\mu}_2}{U_1}\sqrt{\frac{q_3 - U_1\hat{\mu}_1}{q_1+R} + \frac{k}{(q_1+R)^3}} - \frac{2k\,q_2}{U_1(q_1+R)^3}$$

$$\hat{\mu}_2 = \frac{\alpha_4}{U_2}\sqrt{\frac{k}{R^3}}\,(q_1+R) + \frac{[2q_2 - \alpha_4(q_1+R)]}{U_2}\sqrt{\frac{q_3 - U_1\hat{\mu}_1}{q_1+R} + \frac{k}{(q_1+R)^3}} \tag{3.18}$$

The dynamical controller specifying the computed duty ratio μ_1 is synthesized on the basis of achieving an average asymptotically stable behavior of the orbital radius error $q_1 = r - R$. The non-dynamic feedback controller specifying the computed μ_2 yields an asymptotically exponentially stable motion toward the corresponding constant angular velocity : $\Omega = (k/R^3)^{1/2}$. (i.e., the desired dynamics for ω is imposed as : $d\omega/dt = -\alpha_4(\omega - \Omega)$. The exponential rate of decay of ω is thus specified by α_4. The actual duty ratios μ_1 and μ_2 are obtained constraining the solutions μ_1, μ_2 of (3.18) to the closed interval [-1,1].

A computer simulated experiment was also carried out to illustrate the quality of the response obtained with the designed discontinuous dynamical feedback controller. The radius of the earth was taken as 6371Km and the gravity constant k, for such magnitude of heights, was set to $k = 389258.1\ Km^3/s^2$. Initial conditions were taken for a circular orbit located some 150 Km high above the surface of the earth (orbit data : $r = 6521$ Km, $v = 0$ Km/s, $\omega = 1.1832 \times 10^{-3}$ rad/s). A controlled maneuver was performed which brought the spacecraft to a second orbit of 175 Km of height (orbit data : $r = 6546$ Km, $v = 0$ Km/s, $\omega = 1.1780 \times 10^{-3}$ rad/s). The three poles of the linearized system were located at $-0.1\ s^{-1}$. Figure 4 shows the time response of the radius and the angular velocity for the orbital transfer of the spacecraft.

4. CONCLUSIONS

Synthesis of nonlinear dynamical feedback regulators, based on the differential algebraic approach to controlled systems dynamics, has been carried for some typical aerospace problems. In spite of the exact linearization involved in the determination of the dynamical feedback policies, the simulation results demonstrate certain degree of robustness of the obtained control schemes with respect to unmodelled disturbances. Moreover, the synthesis approach was seen to be suitable for either continuously or discontinuously (PWM) controlled plants. In the second case, an infinite frequency average model transforms the discontinuous controller design problem into a problem of continuous nature. The average solution was shown to be approximated, in an arbitrarily close manner, by the discontinuous control policy provided a sufficiently high sampling frequency is used.

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APPENDIX

A.1 Generalities about ON-OFF Pulse-Width-Modulation Control of Nonlinear Systems

Consider a single input nonlinear dynamical system defined on an open set of R^n described by:

$$\dot{x} = f(x, \upsilon) \tag{A.1}$$

with υ a discontinuous feedback control strategy of the PWM type, given by :

$$\upsilon = \begin{cases} \upsilon^+(x) & \text{for } t_k < t \le t_k + \mu[x(t_k)]T \\ \upsilon^-(x) & \text{for } t_k + \mu[x(t_k)]T < t \le t_k + T \end{cases} \quad ; \quad k = 0,1,2,\ldots \tag{A.2}$$

where T is a fixed sampling period also known as the *duty cycle*, t_k is the k-th sampling instant and $\mu(x(t))$ is a continuous piece-wise smooth feedback function known as the *duty ratio* function determining the variable structure feedback control pulse width during the ongoing inter-sampling interval $[t_k, t_k+T]$. The pulse width $\mu(x(t_k))T$ is determined at the beginning of each sampling interval t_k on the basis of the value of the state vector at such instant (schemes on the basis of an output function or output error are also possible). The continuous piece-wise smooth duty ratio function is assumed to be bounded by $0 < \mu(x(t)) < 1$ for all t.

The effect of such a discontinuous feedback strategy on the controlled state trajectories is to produce a zig-zag motion, very much reminiscent of actual Sliding Mode Controlled trajectories (Utkin [10]) . The analysis and design of such class of hybrid systems (A.1)-(A.2) is extremely difficult and can only be carried out in an approximate manner. However, the inconveniences of the nonlinear discrete-time approximations can be eliminated if some smooth continuous average model is adopted as an approximation for the actual PWM controlled system. Such a smooth average behavior may be considered on the basis of a high sampling frequency for systems which are relatively slow as compared with such fast control changes. In the following paragraphs we justify the use of an average continuous model based on an *infinite sampling frequency* assumption for (A.1)-(A.2). The advantages of such an averaging procedure, aside from some intimate connections with Sliding Mode Control (see Sira-Ramirez [11]-[13]), lay in the possibility of using modern nonlinear feedback control design techniques for the synthesis of the duty ratio function. Furthermore, the smooth average designed behavior can be arbitrarily closely approximated by the actual discontinuous feedback controlled trajectories as the sampling frequency of the PWM actuator is suitably increased within finite bounds.

Let $f(x,v^+(x)) = X^+(x)$ and $f(x,v^-(x)) = X^-(x)$. It is easily seen that the discontinuously controlled model (A.1),(A.2) is equivalent to the following switch controlled model:

$$\frac{dx}{dt} = u\,X^+(x) + (1-u)\,X^-(x) = X^-(x) + \left[X^+(x) - X^-(x)\right]u =: f(x) + g(x)u \qquad (A.3)$$

with :

$$u = \begin{cases} 1 & \text{for } t_k < t \le t_k + \mu[x(t_k)]T \\ 0 & \text{for } t_k + \mu[x(t_k)]T < t \le t_k + T \end{cases} \quad ; \; k = 0,1,2,\ldots \qquad (A.4)$$

Definition A.1 An *average PWM model* for the discontinuously controlled system (A.1)-(A.2) (or equivalently (A.3), (A.4)) is defined by the dynamical system formally obtained by letting the sampling frequency 1/T of the PWM actuator grow to infinity, i.e., letting the duty cycle $T \to 0$. We shall shall denote the state of the averaged system by z(t) to differentiate it from the state vector x(t) of the discontinuously controlled system.

Proposition A.2 The average PWM model obtained by formally imposing an infinitely large sampling frequency, 1/T, for the controlled system (A.3),(A.4) is given by :

$$\frac{dz}{dt} = \mu\,X^+(z) + (1-\mu)\,X^-(z) = X^-(z) + \left[X^+(z) - X^-(z)\right]\mu =: f(z) + g(z)\,\mu \qquad (A.5)$$

Proof (see [11]).

Remark The average PWM model (A.5) has a right hand side which coincides with the Filippov average vector field (see Filippov [14]) of $X^-(z)$ and $X^+(z)$ when an infinitely fast switching strategy takes place around a discontinuity surface on which the resulting controlled trajectory can be locally sustained. The switching surface is then none other than *an integral manifold* for the closed loop system (A.5) and the *equivalent control* that induces the manifold invariance is just the duty ratio function μ (see [11] for more details and connections with sliding regimes of variable structure control). Notice, furthermore, that (A.5) is a linear-in-the-control vector differential equation formally obtained from the original discontinuous model (A.3),(A.4) just by replacing the binary control parameter u by the continuous piece-wise smooth duty ratio function μ .

The following result states that under identical initial conditions, the controlled trajectories of the actual discontinuous feedback controlled system (A.3),(A.4) continuously tend toward the average PWM controlled trajectories generated by (A.5) as the sampling frequency associated to the PWM actuator (A.4) is increased without limit. Hence, to arbitrarily closely retain the qualitative and quantitative stability characteristics of the average PWM designed trajectories, a sufficiently high sampling frequency is required for the PWM actuator of the actual discontinuously controlled system. This is the key feature that allows an efficient design scheme based on the continuous average PWM model.

Theorem A.3 Let $\mu(t)$ be a given continuous piece-wise smooth duty ratio function bounded by $0 < \mu(t) < 1$. Under identical initial conditions for the actual and average PWM controlled models , the corresponding controlled state trajectories of the discontinuous PWM system (A.3),(A.4) continuously and globally converge toward those of the corresponding average PWM system (A.5) as the sampling frequency 1/T grows without bound.

Proof (see [8]).

The final step in completing a design procedure based on the average PWM model consists in translating the average continuous stabilizing feedback controller design into a suitable ON-OFF (i.e., discontinuous) feedback controller of PWM nature. Such ON-OFF controller must retain the stabilizing features of the continuous average designed controller and, at the same time, it should yield actual discontinuous responses that remain arbitrarily close to the smooth designed responses. This is primarily accomplished by specifying a sufficiently high sampling frequency for the actual PWM actuator and, secondly, by suitably smoothing of the state variables before using them in the synthesis of the average stabilizing designed controller. The smoothing action may be accomplished by introducing low pass filtering effects on the state variables measurements. One then simply relies on the high-frequency rejection characteristics of most sensing devices.

A.2. Generalities about ON-OFF-ON Pulse-Width- Modulation Control of Nonlinear Systems

In a typical ON-OFF-ON PWM control strategy for the control of the nonlinear system (A.1), the switching actions are specified according to the sign of the duty ratio function as :

For $\mu[x(t_k)] > 0$:

$$v = \begin{cases} v^+(x) & \text{for } t_k < t \le t_k + \mu[x(t_k)]T \\ 0 & \text{for } t_k + \mu[x(t_k)]T < t \le t_k + T \end{cases} \quad ; \; k = 0,1,2,\ldots \qquad (A.6)$$

For $\mu[x(t_k)] < 0$:

$$v = \begin{cases} v^-(x) & \text{for } t_k < t \le t_k + |\mu[x(t_k)]|T \\ 0 & \text{for } t_k + |\mu[x(t_k)]|T < t \le t_k + T \end{cases} \quad ; \; k = 0,1,2,\ldots \qquad (A.7)$$

where $\mu(x(t))$ is the *duty ratio* function constrained now within the bounds $-1 < \mu(x(t)) < 1$.

Let $f(x,v^+(x)) = X^+(x)$, $f(x,v^-(x)) = X^-(x)$ and $f(x,0) = X^0(x)$ It is easily seen that the discontinuously controlled model (A.1),(A.6),(A.7) is equivalent to the following switch controlled models:

For $\mu[x(t_k)] > 0$:

$$\frac{dx}{dt} = u\,X^+(x) + (1-u)\,X^0(x) = X^0(x) + \left[X^+(x) - X^0(x)\right]u =: f(x) + g^+(x)u \qquad (A.8)$$

with :

$$u = \begin{cases} 1 & \text{for } t_k < t \le t_k + \mu[x(t_k)]T \\ 0 & \text{for } t_k + \mu[x(t_k)]T < t \le t_k + T \end{cases} \quad ; \; k = 0,1,2,\ldots \qquad (A.9)$$

For $\mu[x(t_k)] < 0$:

$$\frac{dx}{dt} = u\,X^-(x) + (1-u)\,X^0(x) = X^0(x) + \left[X^-(x) - X^0(x)\right]u =: f(x) + g^-(x)u \qquad (A.10)$$

with :

$$u = \begin{cases} 1 & \text{for } t_k < t \le t_k + |\mu[x(t_k)]|T \\ 0 & \text{for } t_k + |\mu[x(t_k)]|T < t \le t_k + T \end{cases} \quad ; \; k = 0,1,2,\ldots \qquad (A.11)$$

It is now easy to see that the results of the ON-OFF case previously presented directly apply to the case of system (A.1) with control of the form (A.6),(A.7). Using the same arguments as before on each case, the infinite sampling frequency average model is now constituted by two models, one valid for $\mu > 0$ and the other valid for $\mu < 0$. Such models are readily obtained as:

For $\mu[x(t)] > 0$:

$$\frac{dx}{dt} = \mu X^{+}(x)+(1-\mu) X^{0}(x) = X^{0}(x)+\left[X^{+}(x) - X^{0}(x)\right]\mu \\ =: f(x) + g^{+}(x)\mu \tag{A.12}$$

For $\mu[x(t)] < 0$:

$$\frac{dx}{dt} = |\mu| X^{-}(x) + (1-|\mu|) X^{0}(x) = X^{0}(x) + \left[X^{-}(x)-X^{0}(x)\right]|\mu| \\ =: f(x) + g^{-}(x)|\mu| \tag{A.14}$$

which can be synthesized in a single model, independently of the sign of the duty ratio function μ, as:

$$\frac{dx}{dt} = (1-|\mu|)X^{0}(x) + \frac{1}{2}\left\{\mu\left[X^{+}(x) - X^{-}(x)\right] + |\mu|\left[X^{+}(x) + X^{-}(x)\right]\right\} \\ = f(x) + \frac{\mu}{2}\left[g^{+}(x) - g^{-}(x)\right] + \frac{|\mu|}{2}\left[g^{+}(x) + g^{-}(x)\right] \tag{A.15}$$

As corollaries to this result, if $f(x,\nu)$ is of the form $f(x) + \nu g(x)$ and ν takes values in the discrete set $\{-U,0,+U\}$, the average mode is simply given by $dx/dt = f(x) + \mu U g(x)$, and finally if U happens to be equal to 1, then the average model is simply: $dx/dt = f(x) + \mu\, g(x)$.

The preceding developments straightforwardly extend to the case of multivariable nonlinear systems. The reader is referred to Sira-Ramirez *et al* [16] for a concrete application and further details about this important case..

Figures

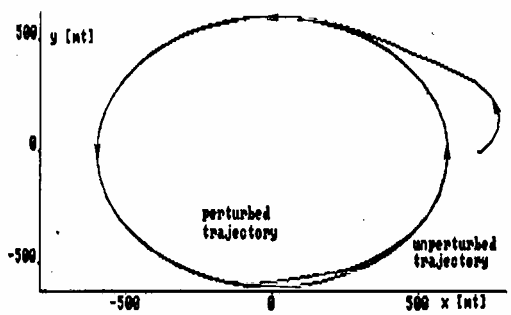

Figure 1. Perturbed Dynamical Feedback Controlled Airplane Tracking of Circular Path

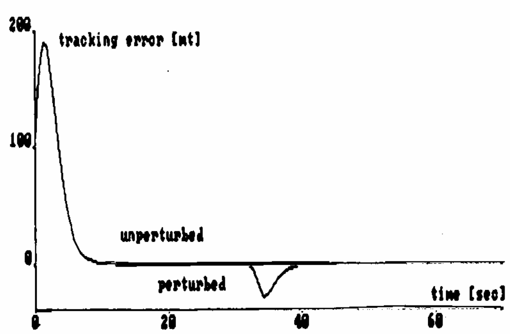

Figure 2. Tracking error of Dynamical Feedback Controlled Airplane Following a Circular Path under Strong Wind Perturbation

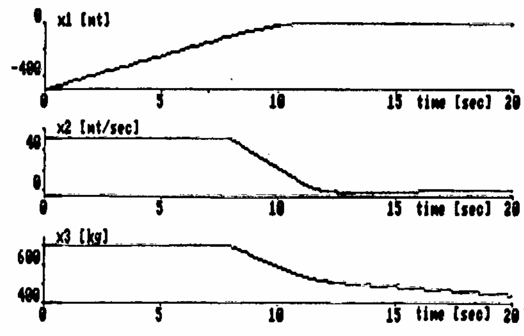

Figure 3 . State Variable Responses of Dynamical Feedback Controlled Soft Controlled Landing Maneuver.

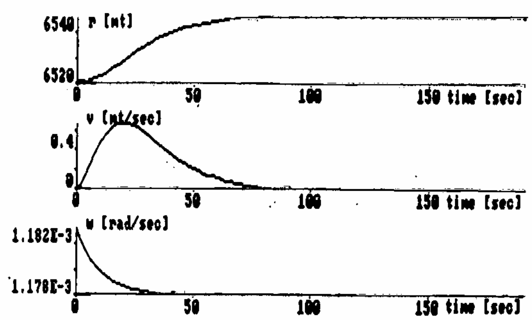

Figure 4. State Response for Dynamical Feedback Controlled Orbital Transfer.