SLIDING MODE CONTROLLER DESIGN FOR NONLINEAR SYSTEMS: AN EXTENDED LINEARIZATION APPROACH

Hebertt Sira-Ramirez and Miguel Rios-Bolívar

Departamento Sistemas de Control, Universidad de Los Andes
Mérida, Venezuela.

Abstract The method of Extended Linearization is proposed for the systematic solution of sliding mode controller design in nonlinear dynamical systems. The approach specifies discontinuous feedback strategies leading to asymptotically stable Sliding Modes around desired constant equilibrium points.

Keywords Sliding Regimes, Extended Linearization, Nonlinear Systems.

1. INTRODUCTION

In this article, a new method is proposed for the synthesis of stabilizing Sliding Modes (Utkin, 1978) in nonlinear controlled dynamical systems. The nonlinear sliding surface design method is based, entirely, on the *Extended Linearization* approach for nonlinear systems, developed by Rugh (1986,1987) and Baumann and Rugh (1986). We propose to specify a nonlinear sliding mode controller by first resorting to parametrized linearization, about a general constant equilibrium point, of the given nonlinear system. Using well known results (Utkin 1978, Itkis , 1976, a standard stabilizing sliding hyperplane design is then carried out on the basis of the parametrized family of linear systems, possibly transformed to *controllable canonical form*. The *ideal sliding dynamics*, corresponding to the linear design, is pur- posefully characterized by a set of stable eigenvalues which are independent of the constant operating point. A suitable *extension* of the sliding hyperplane design yields a nonlinear switching manifold which is tangent to the prescribed hyperplane. The designed manifold contains the equilibrium point and it is parametrizable in terms of the nominal operating conditions. Moreover, the corresponding ideal sliding dynamics can always be made locally linear (possibly, modulo a suitable local diffeomorphic state coordinate transformation derivable from the linearized system). The nonlinear sliding manifold is obtained, in a nonunique fashion, by direct integration of the synthesized linear sliding hyperplane. The nonlinear sliding mode *switching logic* is synthesized on the basis of the obtained nonlinear *sliding surface coordinate function* and the corresponding nonlinear *equivalent control*.

An important property of the proposed sliding mode controller, aside from those already mentioned, lies in the fact that if a sudden change of the nominal operating conditions takes place, the control scheme exhibits self-scheduling properties by means of which a sliding regime is automatically formed which stabilizes the system trajectories to the new equilibrium point. This last property is clearly inherited from well known merits of the extended linearization technique and it makes the "scheduling" process of the sliding manifold and of the switching "gains" totally unnecessary.

In this article only single input nonlinear systems are treated. The multi-input case will be presented elsewhere.

Section 2 of this article presents a general procedure for synthesizing stable nonlinear sliding manifolds for single-input nonlinear systems via Extended Linearization. Section 3 presents several illustrative examples -some of them of physical nature- accompanied by simulation experiments. The conclusions and suggestions for further research are collected in Section 4.

This research was supported by the Consejo de Desarrollo Científico, Humanístico y Tecnológico of the Universidad de Los Andes under Research Grant I-325-90.

2. SLIDING MODE CONTROLLERS VIA EXTENDED LINEARIZATION

2.1 Problem Formulation

Consider the n-dimensional nonlinear dynamical system:

$$\frac{dx(t)}{dt} = f(x(t),u(t)) \tag{2.1}$$

where $f(\cdot,\cdot) : R^n \times R \rightarrow R^n$ is a continuously differentiable function of its arguments. The controlled system (2.1) is assumed to have a continuous family of constant state equilibrium points, $X(U)$, corresponding to nonzero constant inputs, $u = U$. In other words : $f(X(U),U) = 0$. The pair $[\partial f/\partial x(X(U),U), \partial f/\partial u(X(U),U)]$ is assumed to be *controllable*.

It is desired to locally maintain, in a stable fashion, the trajectories of the nonlinear system (2.1) at the constant nominal equilibrium trajectory, $X(U)$, by means of a sliding motion suitably induced on a manifold S which contains such an equilibrium point. In other words, it is required to synthesize 1) a nonlinear sliding surface S, parametrized by the nominal control input U, of the form:

$$S = \{ x \in R^n, s(x,U) = 0 \} \tag{2.2}$$

such that $s(X(U),U) = 0$, and 2) an associated variable structure control law :

$$u(x,U) = \begin{cases} u^+(x,U) & \text{for } s(x, U) > 0 \\ u^-(x,U) & \text{for } s(x, U) < 0 \end{cases} \tag{2.3}$$

which automatically forces every small state deviation, from the nominal operating conditions, to zero, via the local creation of a stable sliding regime taking place on S and leading the state trajectory to $X(U)$.

In order to specify such a sliding manifold we propose to resort to the method of *Extended Linearization* (See Rugh) as indicated in the following paragraphs.

2.2 A Nonlinear Sliding Mode Controller Design based on Extended Linearization.

1) Linearize the dynamical system about each point in the family of constant operating trajectories, $[U,X(U)]$, obtaining the following parametrized family of linear systems:

$$\dot{x}_\delta = A(U)x_\delta + b(U)u_\delta \tag{2.4}$$

where, for fixed U, the input and state perturbation variables are defined, respectively, as: $u_\delta = u(t) - U$, $x_\delta(t) = x(t) - x(U)$, while the $n \times n$ matrix $A(U)$ and the n-vector $b(U)$ are defined as :

$$A(U) := \frac{\partial f}{\partial x}(X(U),U) \;\; ; \;\; b(U) := \frac{\partial f}{\partial u}(X(U),U) \qquad (2.5)$$

Since the pair $[A(U),b(U)]$ is assumed to be controllable, a similarity transformation exists of the form:

$$z_\delta = P(U)x_\delta =: [p_1(U), p_2(U), \cdots, p_n(U)]x_\delta \qquad (2.6)$$

such that (2.4) may be represented as a *controllable canonical realization*. The nonsingular matrix $P(U)$ is obtained from the well known expression :

$$P^{-1}(U) = \left[b(U), A(U)b(U), \dots, A^{n-1}(U)b(U)\right] \begin{bmatrix} \alpha_1(U) & \alpha_2(U) & \cdots & 1 \\ \alpha_2(U) & \alpha_3(U) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{n-1}(U) & 1 & \cdots & \vdots \\ 1 & 0 & \cdots & 0 \end{bmatrix} \qquad (2.7)$$

where:
$\det[\lambda I - A(U)] = \lambda^n + \alpha_{n-1}(U)\,\lambda^{n-1} + \alpha_{n-2}(U)\,\lambda^{n-2} + \dots + \alpha_0(U)$.

2) Obtain the transformed system in *controllable canonical form* as:

$$\begin{aligned} \dot z_{1\delta} &= z_{2\delta} \\ \dot z_{2\delta} &= z_{3\delta} \\ &\vdots \\ \dot z_{(n-1)\delta} &= z_{n\delta} \\ \dot z_{n\delta} &= -\alpha_{n-1}(U)z_{n\delta} - \alpha_{n-2}(U)z_{(n-1)\delta} - \cdots - \alpha_0(U)z_{1\delta} + u_\delta \end{aligned} \qquad (2.8)$$

3) Use as a *sliding surface* the linear manifold :

$$\Sigma_\delta = \{ z_\delta \in R^n : \sigma_\delta(z_\delta) = \sum_{i=1}^{n} c_i z_{i\delta} = c^T z_\delta = 0 \;\; ; \;\; c_n = 1 \} \qquad (2.9)$$

and choose the coefficients c_i, independently of the operating point $[X(U),U]$, such that the roots of the *characteristic polynomial* :

$$\sum_{i=1}^{n} c_i \lambda^{i-1} = 0 \qquad (2.10)$$

for the (reduced) *linear ideal sliding dynamics* are specified at convenient locations in the open left half of the complex plane, i.e., so that the autonomous ideal sliding mode dynamical system:

$$\begin{aligned} \dot z_{1\delta} &= z_{2\delta} \\ \dot z_{2\delta} &= z_{3\delta} \\ &\vdots \\ \dot z_{(n-1)\delta} &= -c_{n-1} z_{(n-1)\delta} - c_{n-2} z_{(n-2)\delta} - \cdots - c_1 z_{1\delta} \end{aligned} \qquad (2.11)$$

is asymptotically stable toward the origin of transformed coordinates.

4) Obtain, on the basis of the previously described design steps, the parametrized sliding hyperplane specification in terms of the original perturbed state coordinates x_δ, as follows:

$$S_\delta = \{ x_\delta \in R^n : s_\delta(x_\delta, U) := \sigma_\delta(P(U)x_\delta) = c^T P(U) x_\delta = 0 \} \qquad (2.12)$$

5) Obtain a nonlinear sliding manifold S such that its corresponding linearization about the operating point $[X(U),U]$, yields back the sliding hyperplane (2.12). In other words, find a nonlinear switching surface which is tangent to the sliding hyperplane (2.12) at the equilibrium point.

5a) Sliding Manifold We must, thus, find a nonlinear sliding surface coordinate function $s(x,U)$, parametrized by the constant operating point U, such that the following relations are satisfied:

$$\left.\frac{\partial s(x,U)}{\partial x}\right|_{x=X(U)} = c^T P(U) = \left[c^T p_1(U), c^T p_2(U), \cdots, c^T p_n(U)\right] \qquad (2.13)$$

or, componentwise :

$$\left.\frac{\partial s(x,U)}{\partial x_i}\right|_{x=X(U)} = c^T p_i(U) \;\; ; \;\; i = 1,2,\dots,n \qquad (2.14)$$

with the additional (boundary) condition : $s(X(U),U) = 0$.

Remark In general, there are many parametrized sliding surface coordinate functions, $s(x,U)$, which satisfy relations (2.14) and the boundary condition. Such a lack of uniqueness of solution may not be totally inconvenient. However, the following *direct integration procedure*, inspired by the results in Rugh (1986a), allows one to obtain a nonlinear sliding manifold in a systematic manner :

1) Assume, without loss of generality, that the first component $X_1(U)$ of the vector $X(U)$ is invertible, i.e., let there exist a unique solution, $X_1^{-1}(x_1)$, for U in the equation $x_1 = X_1(U)$.

2) It can be verified, after partial differentiation with respect to the components of the vector x and substitution of the equilibrium point, that the following manifold is one possible solution for the required parametrized nonlinear sliding manifold:

$$S = \{ x \in R^n : s(x,U) = \int_U^{X_1^{-1}(x_1)} c^T P(\upsilon) \frac{dX(\vartheta)}{d\vartheta}\, d\vartheta + \sum_{j=2}^{n} c^T p_j(X_1^{-1}(x_1))\left[x_j - X_j(X_1^{-1}(x_1))\right] = 0 \} \qquad (2.15)$$

■

5b) Equivalent Control Once the nonlinear sliding surface coordinate function $s(x,U)$ is known, computation of the equivalent control follows by imposing the well known (ideal) invariance conditions, which make of the switching manifold a *local integral manifold* of the smoothly controlled system:

$$s(x,U) = 0 \;\; , \;\; \frac{d}{dt} s(x,U) = 0 \qquad (2.16)$$

5c) Sliding Mode Switching Logic A nonlinear sliding mode switching strategy is usually synthesized such that the sliding mode existence conditions (Utkin,1978) are satisfied, at least, in a local fashion. Such well known conditions are given by:

$$\lim_{s \to 0^+} \frac{ds(x,U)}{dt} < 0 \;\; ; \;\; \lim_{s \to 0^-} \frac{ds(x,U)}{dt} > 0 \qquad (2.17)$$

It has been shown that, for nonlinear systems which are linear in the scalar control input, a necessary and sufficient conditions for the local existence of a sliding mode is that the equivalent control locally exhibits values which are *intermediate* between the extreme values of feedback laws among which the switching take place (i.e., $u^+(x,U) < u^{EQ}(x,U) < u^-(x,U)$). The region of existence of such a sliding regime coincides, precisely, with the region where such an *intermediacy* condition is satisfied by the equivalent control. One may, therefore, synthesize the nonlinear sliding mode switching logic, for such a large class of nonlinear systems, from knowledge of the equivalent control function, $u^{EQ}(x,U)$, and of the sliding manifold coordinate function, $s(x,U)$, (See Sira- Ramirez, 1988) as follows :

$$u(x,U) = -k\left|u^{EQ}(x,U)\right| \operatorname{sgn} s(x,U) \;\; ; \;\; k > 1 \qquad (2.18)$$

In more general cases, where there is no special input structure to the system, the above switching logic, or any one satisfying the equivalent control intermediacy condition, may still locally create a sliding regime provided the system exhibits a *control foliation property* (See Sira-Ramirez, 1989a,1989b). For the class of application examples we will be presenting in the next section, a switching control law of the form (2.18) suffices.

3. SOME APPLICATION EXAMPLES

In this section we present some illustrative examples of sliding mode control synthesis, for nonlinear plants, using the method of Extended Linearization. We begin by a somewhat general second order example in which the synthesized sliding surface is seen to entirely coincide with the intuitive solution that one would propose, in general, for achieving a linear ideal sliding dynamics. The proposed sliding mode design process, thus, appears as a natural synthesis procedure. The rest of the examples in this section represent simple applications of physical nature.

3.1 A General Second Order Example

Consider the nonlinear controlled system, defined in R^2, expressed in *regular canonical form* (See Luk'yanov and Utkin, 1981):

$$\begin{aligned} \dot{x}_1 &= \varphi(x_1, x_2) \\ \dot{x}_2 &= \gamma(x_1, x_2) + g(x_1, x_2)u \end{aligned} \tag{3.1}$$

We assume the existence of a continuum of constant equilibrium points, parametrized by the corresponding constant value U of the control input u (parametrization with respect to equilibrium values of the state variables is also possible in cases where $U = 0$, See Rugh (1986a) and Example 3.2 below):

$$u = U \; ; \; x_1(U) = X_1(U) \; ; \; x_2(U) = X_2(U) \tag{3.2}$$

such that : $\partial\varphi/\partial x_2(X_1(U),X_2(U)) \neq 0$, and $g(X_1(U),X_2(U)) \neq 0$.

Design of the Stabilizing Switching Line for the Family of Linearized Systems

Linearization of system (3.1) about an equilibrium point of the form (3.2) yields:

$$\begin{aligned} \dot{x}_{1\delta} &= \varphi_1(X_1(U),X_2(U))x_{1\delta} + \varphi_2(X_1(U),X_2(U))x_{2\delta} \\ \dot{x}_{2\delta} &= [\gamma_1(X_1(U),X_2(U)) + g_1(X_1(U),X_2(U))U]\, x_{1\delta} + \\ &\quad [\gamma_2(X_1(U),X_2(U)) + g_2(X_1(U),X_2(U))U]x_{2\delta} + g(X_1(U),X_2(U))u_\delta \end{aligned} \tag{3.3}$$

where $\varphi_i := \partial\varphi/\partial x_i$; $\gamma_i := \partial\gamma/\partial x_i$; $g_i := \partial g/\partial x_i$; $i = 1,2$.

We briefly express such a linearized system by :

$$\begin{aligned} \dot{x}_{1\delta} &= \varphi_1 x_{1\delta} + \varphi_2 x_{2\delta} \\ \dot{x}_{2\delta} &= [\gamma_1 + g_1 U]\, x_{1\delta} + [\gamma_2 + g_2 U]\, x_{2\delta} + g u_\delta \end{aligned} \tag{3.4}$$

As it can be easily seen, the linearized system (3.4) may be placed in *controllable canonical form* by means of the following similarity transformation, parametrized by the equilibrium point :

$$\begin{aligned} z_{1\delta} &= \frac{1}{g\varphi_2}\, x_{1\delta} \\ z_{2\delta} &= \frac{1}{g\varphi_2}[\varphi_1 x_{1\delta} + \varphi_2 x_{2\delta}] \end{aligned} \tag{3.5}$$

The previous assumption about the nonvanishing of g and φ_2 at the equilibrium point $[X_1(U),X_2(U)]$, locally guarantees the nonsingularity of such a linear coordinate transformation. Evidently, a sliding line rendering an asymptotically stable *ideal sliding dynamics*, for the transformed system, is given by $\sigma_\delta(z_\delta) = c_1 z_{1\delta} + z_{2\delta} = [\, c_1 \;\; 1]^T z_\delta =: c^T z_\delta$. Using (3.5), the sliding line equation in original coordinates is obtained, after multiplication by the nonzero factor $g\varphi_2$, as :

$$S_\delta = \left\{ x_\delta \in R^2 : s_\delta(x_\delta,U) = (\varphi_1 + c_1)\, x_{1\delta} + \varphi_2 x_{2\delta} = 0 \; ; \; c_1 > 0 \right\} \tag{3.6}$$

Indeed, if the system is ideally maintained on such a sliding line, the resulting dynamics (known as the *ideal sliding dynamics*) is simply governed, according to (3.4), by :

$$\dot{x}_{1\delta} = -c_1 x_{1\delta} \quad ; \quad c_1 > 0 \tag{3.7}$$

which is asymptotically stable to zero, and independent of the operating point.

The *equivalent control*, $u_\delta{}^{EQ}(x_\delta)$ exists and it is uniquely obtained from the ideal sliding mode conditions :

$$s_\delta(x_\delta,U) = 0 \tag{3.8}$$

$$\frac{d}{dt}\, s_\delta(x_\delta,U) = 0 \tag{3.9}$$

as:

$$u_\delta^{EQ}(x_\delta,U) = -\frac{1}{\varphi_2 g}\Big[\, (\varphi_2\gamma_1 - \gamma_2\varphi_1) + (\varphi_2 g_1 - g_2\varphi_1)\, U - c_1(\gamma_2 + \varphi_1 + c_1 + g_2 U) \Big]\, x_{1\delta} \tag{3.10}$$

Synthesis of the Sliding Mode Controller for the Nonlinear System

In general, the key idea behind the method of Extended Linearization for obtaining a nonlinear controller design, once a linear feedback stabilizing controller has been properly synthesized for any member of the parametrized family of linear systems, consists in finding a <u>nonlinear regulator</u> such that when linearized about the nominal operating trajectory yields back the designed linear regulator specified for the family of linearized systems.

In nonlinear sliding mode design, the method of Extended Linearization consists in specifying a (nonlinear) sliding manifold, and its associated equivalent control, on the basis of the linearized surface design. This manifold must be such that when linearized about the constant equilibrium point yields back the designed stabilizing sliding hyperplane corresponding to the linearized system. The linearization of the corresponding nonlinear equivalent control, about the operating point, yields the linear equivalent control previously obtained.

Expressions (2.14) yield, in this case, the following conditions :

$$\begin{aligned} \left.\frac{\partial s(x,U)}{\partial x_1}\right|_{\substack{x_1 = X_1(U) \\ x_2 = X_2(U)}} &= c_1 + \varphi_1(X_1(U),X_2(U)) \\ \left.\frac{\partial s(x,U)}{\partial x_2}\right|_{\substack{x_1 = X_1(U) \\ x_2 = X_2(U)}} &= \varphi_2(X_1(U),X_2(U)) \end{aligned} \tag{3.11}$$

Let x stand for (x_1,x_2), it is easy to verify, from the definition of φ_1 and φ_2, that the following nonlinear sliding manifold S is such that its surface coordinate function $s(x,U)$ satisfies the above conditions,

$$S = \left\{ x \in R^2 : s(x, U) = \varphi(x) + c_1(x_1 - X_1(U)) = 0 \; ; \; c_1 > 0 \right\} \tag{3.12}$$

It is immediate to verify that $s(X(U),U) = 0$, as required, i.e., S contains the equilibrium point.

The ideal sliding dynamics corresponding to the manifold (3.12), according to the first equation in (3.1), is clearly given by the linear system:

$$\dot{x}_1 = -c_1(x_1 - X_1(U)) \; ; \quad c_1 > 0 \tag{3.13}$$

which represents an asymptotically stable linear dynamics whose solution converges toward the first component of the equilibrium point. Since φ_2 is nonzero at the equilibrium point, the implicit function theorem guarantees that the local isolated solution for x_2 of $s(X_1(U),x_2,U) = \varphi(X_1(U),x_2,U) = 0$ exists and it is unique. Thus, the solution for x_2 coincides, precisely, with $X_2(U)$.

The ideal sliding conditions: $s = 0$, $ds/dt = 0$, yield, by virtue of (3.1), (3.12) and (3.13), the following equation for the equivalent control on $s = 0$ (where x stands for $(x_1,x_2)^T$):

$$\dot{s} = -c_1[\varphi_1(x)+c_1]\left(x_1-X_1(U)\right)+\varphi_2(x)\left[\gamma(x) + g(x)u^{EQ}\right]=0 \quad (3.14)$$

i.e.,

$$u^{EQ}(x,U) = -\frac{1}{g\varphi_2}\left\{-c_1\left[x_1-X_1(U)\right](\varphi_1+c_1)+\varphi_2\gamma\right\} \quad (3.15)$$

Remark It is easy to verify, after some tedious but straightforward algebraic manipulations involving (3.12),(3.13), (3.1) and (3.4) that linearization of (3.15) about the equilibrium point $[X_1(U),X_2(U)]$ yields, precisely, equation (3.10) defining the linear equivalent control. However, it is not immediately obvious, in general, how to obtain (3.15) by direct integration of (3.10). ■

The switching strategy for the control function u that accomplishes sliding mode existence for the discontinuously controlled system, provided $g\varphi_2$ is positive, is given by:

$$u = -k\left|u^{EQ}(x,U)\right| \operatorname{sgn} s(x,U) \quad ; \quad k > 1 \quad (3.16)$$

Remark The switching law (3.16) is valid whenever the factor $g\varphi_2$, used in defining (3.6), is positive. Otherwise, the minus sign must be changed in (3.16). (See also example 3.2 below).

3.2 State-Scheduled Sliding Mode Controlled Reorientation Maneuvers for Single-Axis Spacecraft

Consider the kinematic and dynamic model of a single-axis externally controlled spacecraft whose orientation is given in terms of the Cayley-Rodrigues representation of the attitude parameter, denoted by x_1 (See Dwyer and Sira-Ramírez ,1987). The angular velocity is represented by x_2 while J stands for the moment of inertia and u is the applied external torque.

$$\frac{dx_1}{dt} = 0.5\,(1+x_1^2)x_2 \quad ; \quad \frac{dx_2}{dt} = \frac{1}{J}u \quad (3.17)$$

Given arbitrary initial conditions, a slewing maneuver is required which brings the attitude parameter to a final desired value X_1 and the angular velocity to a rest equilibrium. We summarize below the design steps leading to a nonlinear sliding surface where the ideal sliding dynamics is linear and asymptotically stable toward the desired equilibrium point: $x_1 = X_1$, $x_2 = 0$, $u = 0$.

Family of Linearizations Parametrized by Constant Equilibrium Point

$$\begin{bmatrix} \frac{dx_{1\delta}}{dt} \\ \frac{dx_{2\delta}}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 0.5\,(1+X_1^2) \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_{1\delta} \\ x_{2\delta} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{J} \end{bmatrix} u_\delta \quad (3.18)$$

with : $x_{1\delta} = x_1 - X_1$, $x_{2\delta} = x_2 - 0$, $u_\delta = u - 0$.

State Coordinate Transformation to Controllable Canonical Form

$$\begin{bmatrix} z_{1\delta} \\ z_{2\delta} \end{bmatrix} = \begin{bmatrix} \frac{2J}{(1+X_1^2)} & 0 \\ 0 & J \end{bmatrix}\begin{bmatrix} x_{1\delta} \\ x_{2\delta} \end{bmatrix} \quad (3.19)$$

$$\begin{bmatrix} \frac{dz_{1\delta}}{dt} \\ \frac{dz_{2\delta}}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} z_{1\delta} \\ z_{2\delta} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_\delta \quad (3.20)$$

Linear Sliding Surface and Ideal Sliding Dynamics in Transformed and Original Coordinates

In transformed coordinates:

$$\sigma_\delta(z_\delta) = z_{2\delta} + c_1 z_{1\delta} = 0 \; ; \quad c_1 > 0 \quad (3.21)$$

$$\dot{z}_{1\delta} = -c_1 z_{1\delta} \quad (3.22)$$

$$u_\delta^{EQ}(z_{1\delta}) = c_1^2 z_{1\delta} \quad (3.23)$$

In original coordinates:

$$s_\delta(x_\delta) = c_1 x_{1\delta} + 0.5\,(1+X_1^2)x_{2\delta} = 0 \quad (3.24)$$

$$\dot{x}_{1\delta} = -c_1 x_{1\delta} \quad (3.25)$$

$$u_\delta^{EQ}(x_{1\delta}, X_1) = \frac{2J\,c_1^2}{(1+X_1^2)} x_{1\delta} \quad (3.26)$$

Nonlinear Sliding Surface, Ideal Sliding Dynamics and Nonlinear Sliding Mode Controller

The nonlinear sliding surface coordinate function $s(x,X_1)$ must satisfy the following relations:

$$\left.\frac{\partial s(x,X_1)}{\partial x_1}\right|_{\substack{x_1=X_1\\ x_2=0}} = c_1 \quad ; \quad \left.\frac{\partial s(x,X_1)}{\partial x_2}\right|_{\substack{x_1=X_1\\ x_2=0}} = 0.5\,(1+X_1^2) \quad (3.27)$$

with the condition: $s([X_1,0]^T, X_1) = 0$.

After substitution of X_1 by x_1 in (3.27), and by direct integration one obtains:

$$s(x,X_1) = c_1(x_1-X_1) + 0.5\,(1+x_1^2)x_2 \quad (3.28)$$

From (3.17) one obtains, on $s(x,X_1) = 0$, the ideal sliding dynamics as:

$$\dot{x}_1 = -c_1(x_1 - X_1) \quad (3.29)$$

The equivalent control $u^{EQ}(x,X_1)$, associated to (3.28) , is seen to be given by :

$$u^{EQ}(x_1, X_1) = \frac{2J\,c_1^2\,(x_1-X_1)}{(1+X_1^2)}\,(1-x_1^2+2x_1X_1) \quad (3.30)$$

It can be easily verified that linearization of (3.30) about the equilibrium point $[X_1,0]$ yields back the linear equivalent control (3.26). However, one does not naturally obtain (3.30) by integration of (3.26), after substitution of X_1 by x_1. Indeed, direct integration of (3.26) leads to $u^{EQ}(x,U) = 2\,Jc_1{}^2[\tan^{-1}(x_1) - \tan^{-1}(X_1)]$, but this controller does not leave the nonlinear sliding surface invariant as it may be easily verified. Due to the lack of uniqueness of solutions of the "unlinearization" procedure, only feedback equivalent control laws computed on the basis of the obtained nonlinear sliding surface are actually valid. (See also the remark below).

Finally, according to (2.18), the switching logic is given by:

$$u(x, X_1) = -k\,\frac{2J\,c_1^2}{(1+X_1^2)}\left|(x_1-X_1)(1-x_1^2+2x_1X_1)\right| \operatorname{sgn} s(x,X_1) \; ; \; k > 1 \quad (3.31)$$

Remark Notice that from (3.21) one could have also obtained, instead of (3.24), the following linear sliding manifold:

$$s_\delta(x_\delta, X_1) = \frac{2c_1}{(1+X_1^2)} x_{1\delta} + x_{2\delta} = 0 \quad (3.32)$$

The ideal sliding dynamics, and the equivalent control, corresponding to this manifold are, respectively, still the same as in (3.25),(3.26). However, the nonlinear sliding surface coordinate function $s(x,X_1)$ must now satisfy the following relations:

$$\left.\frac{\partial s(x,X_1)}{\partial x_1}\right|_{\substack{x_1 = X_1 \\ x_2 = 0}} = \frac{2\,c_1}{(1+X_1^2)} \quad ; \quad \left.\frac{\partial s(x,X_1)}{\partial x_2}\right|_{\substack{x_1 = X_1 \\ x_2 = 0}} = 1 \quad (3.33)$$

with the condition: $s([X_1, 0]^T, X_1) = 0$.

After substitution of X_1 by x_1 in (3.33), and by direct integration one obtains:

$$s(x,X_1) = 2\,c_1\left[\tan^{-1}(x_1) - \tan^{-1}(X_1)\right] + x_2 = 0 \quad (3.34)$$

The nonlinear equivalent control, associated to (3.34), is given by:

$$u^{EQ}(x,X_1) = \frac{2\,Jc_1^2\,(x_1 - X_1)}{1+X_1^2} \qquad (3.35)$$

Once again, linearization of (3.35) about the equilibrium point yields back (3.26), but the equivalent control obtained by use of the integration formula is, as it was seen before, $u^{EQ}(x,U) = 2Jc_1^2\,[\tan^{-1}(x_1) - \tan^{-1}(X_1)]$, which does not produce invariance of (3.34) as it can be easily verified. Hence, only (3.35) is a valid equivalent controller. The corresponding ideal sliding dynamics taking place on the sliding surface (3.34) is no longer a linear system, in the original coordinates. However, a suitable nonlinear state coordinate transformation reveals the underlying linear nature of such an ideal sliding dynamics. Indeed, one obtains by virtue of (3.34) and (3.17) that:

$$\dot{x}_1 = -c_1\,(1 + x_1^2)\left[\tan^{-1}(x_1) - \tan^{-1}(X_1)\right] \qquad (3.36)$$

Letting $\xi = \tan^{-1}(x_1)$, and denoting the constant equilibrium point by: $\Xi = \tan^{-1}(X_1)$, one readily obtains:

$$\frac{d\xi}{dt} = -c_1\,(\xi - \Xi) \qquad (3.37)$$

■

Simulations Computer simulations were carried out for the synthesized sliding mode controller (3.28),(3.31) on a spacecraft with moment of inertia : $J = 90$ N-mt-sec^2, with c_1 chosen as 0.11 sec^{-1}, and $k = 1.2$. Figure 1 shows the state trajectories of the sliding mode controlled system when the reference operating point, for the Cayley-Rodrigues attitude orientation parameter x_1, abruptly changes from $X_1 = -0.4$ rad to $X_1 = 0.6$ rad at $t = 60$ sec. The parametrized sliding surfaces, corresponding to both operating points, are also depicted in this figure and labeled as S_1, S_2. Figure 2 shows the corresponding time responses of the state variables x_1 and x_2 under such a large change in the operating equilibrium point.

Remark In the previous second order example, the nonlinear system was already in *regular canonical form*. Hence, in accordance with the results of Example 3.1, the linearizing sliding manifold could have been obtained directly from the systems equations. However, it should be remarked that this is not the case for (single input) higher order systems, nor for systems which are not in regular canonical form (even if they are affine in the control). For the last class of systems, obtaining a linearizing sliding manifold is by no means a trivial task. Moreover, transformation to regular canonical form, of a nonlinear system, involves a quite complicated procedure dealing with the solution of certain associated *Pfaffian systems* (See Luk'yanov and Utkin, 1981). The method of extended linearization, thus, provides with an alternative synthesis approach for such particular cases and, more importantly, for the general case represented by systems of the form (2.1). The next example below deals with a second order control-affine (bilinear) system which is not in regular canonical form. ■

3.3 Input-Current Scheduled Sliding-Mode Control of Angular Velocity in a DC Motor

Consider a DC, field-controlled, motor provided with separate excitation. Let V_a be the constant armature voltage and let u be the field current, acting as a control parameter. The set of bi-linear differential equations describing the dynamics of such a controlled system, acting on a load which exhibits non negligible damping reaction, is given by (See Rugh, 1981, pp. 98-99]) :

$$\begin{aligned} \dot{x}_1 &= -\frac{R_a}{L_a}x_1 - \frac{K}{L_a}x_2 u + \frac{V_a}{L_a} \\ \dot{x}_2 &= -\frac{B}{J}x_2 + \frac{K}{J}x_1 u \end{aligned} \qquad (3.38)$$

where x_1 is the armature current, x_2 is the motor shaft angular velocity, L_a and R_a are, respectively, the inductance and the resistance in the armature circuit, K is the torque constant, while J and B are, respectively, the load's moment of inertia and the associated viscous damping coefficient.

It is required to maintain a fixed nominal angular velocity W by suitable discontinuous control actions generated by the field circuit input current u. We summarize next the nonlinear sliding mode controller synthesis.

Family of Constant Equilibrium Points

$$u = U\,,\ x_1 = X_1(U) = \frac{BV_a}{R_aB + K^2U^2}\,,\ x_2 = X_2(U) = W(U) = \frac{V_aKU}{R_aB + K^2U^2} \qquad (3.39)$$

Family of Linearizations Parametrized by Constant Equilibrium Point

$$\begin{bmatrix} \dot{x}_{1\delta} \\ \dot{x}_{2\delta} \end{bmatrix} = \begin{bmatrix} -\frac{R_a}{L_a} & -\frac{KU}{L_a} \\ \frac{KU}{J} & -\frac{B}{J} \end{bmatrix} \begin{bmatrix} x_{1\delta} \\ x_{2\delta} \end{bmatrix} + \begin{bmatrix} -\frac{K^2V_aU}{L_a(R_aB + K^2U^2)} \\ \frac{KBV_a}{J(R_aB + K^2U^2)} \end{bmatrix} u_\delta \qquad (3.40)$$

State Coordinate Transformation to Controllable Canonical Form

$$\begin{bmatrix} z_{1\delta} \\ z_{2\delta} \end{bmatrix} = \begin{bmatrix} \frac{B}{J\,\eta(U)} & \frac{KU}{L_a\,\eta(U)} \\ \frac{K^2U^2 - BR_a}{L_a\,J\,\eta(U)} & -\frac{2BKU}{L_aJ\,\eta(U)} \end{bmatrix} \begin{bmatrix} x_{1\delta} \\ x_{2\delta} \end{bmatrix} \quad (3.41)$$

with:

$$\eta(U) = \frac{K^2V_a\,U\left(-2B^2L_a + R_aJB - K^2U^2J\right)}{L_a^2J^2\left[R_aB + K^2U^2\right]} \qquad (3.42)$$

Parametrized Linear Sliding Surface and Ideal Sliding Dynamics in Transformed and Original Coordinates

In transformed coordinates:

$$\sigma_\delta(z_\delta) = z_{2\delta} + c_1 z_{1\delta} = 0; \quad c_1 > 0 \qquad (3.43)$$

$$\dot{z}_{1\delta} = -c_1 z_{1\delta} \qquad (3.44)$$

$$u_\delta^{EQ}(z_{1\delta},U) = \left[\frac{R_aB + K^2U^2}{L_aJ} - c_1\left(\frac{JR_a + BL_a}{L_aJ} - c_1\right)\right] z_{1\delta} \quad (3.45)$$

In original coordinates:

$$s_\delta(x_\delta,U) = -(c_1BL_a - R_aB + K^2U^2)\,x_{1\delta} + (2B - Jc_1)KU\,x_{2\delta} = 0 \quad (3.46)$$

$$\dot{x}_{1\delta} = -c_1 x_{1\delta} \quad ; \quad \dot{x}_{2\delta} = -c_1 x_{2\delta} \qquad (3.47)$$

$$u_\delta^{EQ}(x_{1\delta}) = \frac{1}{\eta(U)}\left[\frac{2K^2U^2BL_a + \left[R_aB - K^2U^2\right]\left[c_1L_aJ - R_aJ\right]}{L_a^2J^2} - \frac{B(c_1BL_a - R_aB + K^2U^2)}{L_aJ^2} - \frac{\left(c_1BL_a - R_aB + K^2U^2\right)^2}{L_a^2J(2B - Jc_1)}\right]x_{1\delta} \quad (3.48)$$

Nonlinear Sliding Surface, Ideal Sliding Dynamics and Nonlinear Sliding Mode Controller

$$s(x,U) = -(c_1BL_a - 2R_aB)\left(x_1 - \frac{BV_a}{R_aB + K^2U^2}\right) - BV_a\ln\left(\frac{R_aB + K^2U^2}{BV_a}x_1\right) \quad (3.49)$$

$$+\frac{(2B-Jc_1)B}{2x_1}\left[(x_2)^2-\left(\frac{V_aKU}{V_aB+K^2U^2}\right)^2\right] = 0$$

$$\dot{x}_2 = -c_1\left[x_2-\frac{V_aKU}{R_aB+K^2U^2}\right] \qquad (3.50)$$

$$u^{EQ}(x,U) = \frac{J\left[2(c_1L_a-2R_a)x_1^2+2V_ax_1+(2B-Jc_1)\left(x_2^2-x_2^2(U)\right)\right](-R_ax_1+V_a)+2BL_a(2B-Jc_1)x_2^2x_1}{J\left[2(c_1L_a-2R_a)x_1^2+2V_ax_1+(2B-Jc_1)\left(x_2^2-x_2^2(U)\right)\right]Kx_2+2L_a(2B-Jc_1)Kx_1^2x_2} \qquad (3.51)$$

$$u(x,U) = -k\left|u^{EQ}(x,U)\right| \operatorname{sgn} s(x,u) \qquad (3.52)$$

Simulations Computer simulations were carried out for the synthesized sliding mode controller (3.49),(3.52) on a loaded DC-motor with moment of inertia: $J = 1.06 \times 10^{-6}$ N-m-sec^2/rad, $B = 6.04 \times 10^{-6}$N-m-s/rad and $L_a = 120$ mH, $K = 1.41 \times 10^{-2}$N-m/A, $R_a = 7\Omega$, $V_a = 5$ Volts. c_1 was chosen as 5.0 sec^{-1}, and $k = 1.5$. Figure 3 shows the time responses of the state variables x_1 and x_2, of the sliding mode controlled system, when the reference operating point for the motor shaft angular velocity abruptly changes from $W_1 = 159.25$ rad/sec to $W_2 = 280.69$ rad/sec at $t = 1.0$ sec. The corresponding change in the input current nominal value is from $U = 0.1$ Amp to $U = 0.2$ amp.

4. CONCLUSIONS

A general systematic approach has been proposed for the synthesis of sliding mode control regulators for a rather wide class of nonlinear systems, possessing no particular control input structure and exhibiting a continuous family of constant operating points. The method entitles use of the extended linearization technique for the specification of the nonlinear switching manifold, the associated equivalent control and the required switching strategy. As demonstrated by a general second order example, in which the required linearizing sliding surface is readily apparent, the method appears to be a natural one, as it yields the intuitively obvious solution. The self-scheduling properties of the proposed controller were demonstrated in two physically motivated simulation examples.

The fundamental advantages of the proposed design scheme lie in the facts that 1) The approach benefits from an extensive list of well known theoretical contributions for design of linear sliding modes, including efficient computer packages already developed for such design tasks. 2) The possibilities of nontrivial applications can be greatly enhanced, and carried out, by means of existing algebraic manipulation systems. 3) The method naturally enjoys rather useful self-scheduling properties when nominal operating conditions are abruptly changed. This is particularly important in the field of control of mechanical manipulators, aerospace systems and other practical nonlinear control application areas. 4) The method developed in this article also constitutes an alternative approach, for approximate linearization of nonlinear systems, to the method developed by Bartolini and Zolezzi (1988).

5. REFERENCES

Bartolini, G. and Zolezzi, T. (1988). Asymptotic Linearization of Uncertain Systems by Variable Structure Control, Systems and Control Letters, 10, 111-117.

Baumann, W.T. and Rugh, W.J. (1986). Feedback Control of Nonlinear Systems by Extended Linearization. IEEE Transactions on Automatic Control, AC-31, 40-46.

Dwyer, T.A.W. and Sira-Ramirez, H. (1987). Variable Structure Control of Spacecraft Attitude Maneuvers Journal of Guidance, Dynamics and Control, 11 262-270.

Itkis, U. (1976). Control Systems of Variable Structure, Kater Publishing House: Jerusalem.

Luk'yanov, A.G. and Utkin, V.I. (1981). Ways of Reducing Equations of Dynamical Systems to a Regular Form," Avtomatika i Telemekhanika, 42, 5-13.

Rugh,W.J. (1981). Nonlinear System Theory -The Volterra/Wiener Approach, The Johns Hopkins University Press: Baltimore.

Rugh, W.J. (1986). The Extended Linearization Approach for Nonlinear Systems Problems, in Algebraic and Geometric Methods in Nonlinear ControlTheory. pp. 285-309. M. Fliess and M. Hazewinkel (Eds.) D. Reidel Publishing Company: Dordrecht.

Rugh, W.J. (1987). Design of Nonlinear PID Controllers," AIChE Journal, 33, 1738-1742.

Sira-Ramirez, H. (1988). Differential Geometric Methods in Variable Structure Control, International Journal of Control, 48, 1359-1390.

Sira-Ramirez, H. (1989a). Sliding Regimes in General Nonlinear Systems : A Relative Degree Approach, International Journal of Control, 50, 1487-1506.

Sira-Ramirez, H. (1989b). Nonlinear Variable Structure Systems in Sliding Mode : The General Case, IEEE Transactions on Automatic Control, AC-34, 1186-1188.

Utkin, V.I. (1978). Sliding Modes and Their Applications in Variable Structure Systems, Mir Publishers: Moscow.

FIGURES

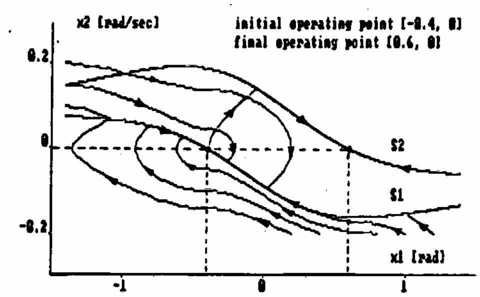

Figure 1. State Trajectories of Sliding Mode Controlled Spacecraft under a Sudden Change of the Orientation Parameter Operating Point.

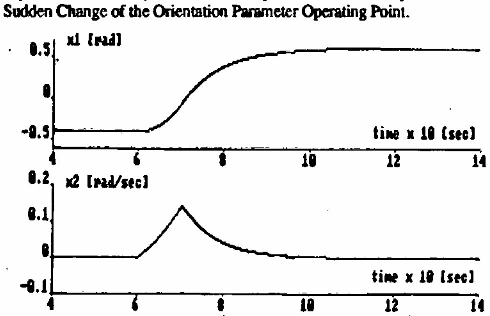

Figure 2. Typical State Responses of Sliding Mode Controlled Spacecraft under a Sudden Change of the Orientation Parameter Operating Point.

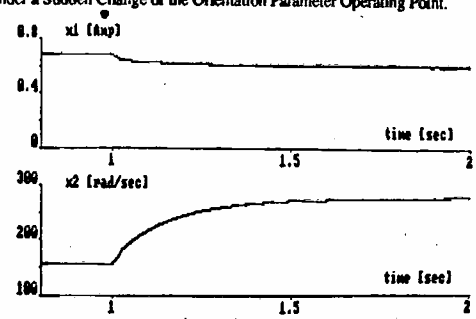

Figure 3. Typical State Responses of Sliding Mode Controlled DC-Motor under a Sudden Change of the Shaft's Angular Velocity Operating Point