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ON THE DESIGN OF PULSE WIDTH MODULATION CONTROLLERS FOR LINEAR DYNAMICAL SYSTEMS

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Abstract

In this article a method is proposed for the design of Pulse-Width-Modulation (PWM) feedback controllers regulating linear dynamical plants. The design is carried out on the basis of an infinite frequency average model of the actual closed loop controller system which retains all the essential qualitative features of the discontinuous controlled system. The average closed loop system results in the same original plant controlled by a memoryless nonlinearity. This fact allows the use of classical control time domain design techniques in the specification of the PWM controller parameters. An alternative frequency domain approached is proposed for the study of the stability of PWM feedback controllers. Several illustrative examples are presented.

I. Introduction

Pulse-Width-Modulation (PWM) controlled systems constitute a class of discontinuously controlled systems on which the control actions are determined on the basis of periodical error signal sampling. At each sampling instant, the control signal is enabled during a fraction of the sampling period which is proportional to the error magnitude. This practical control technique has been profusely used by control engineers at least for the last 30 years, or so, for the design of practical control systems.

In spite of its practical importance, and the enormous number of application articles appearing in various journals, theoretical developments of PWM controlled systems have remained relatively little explored in the last 18 years. Early contributions are those of Nelson [1], Kadota and Bourne [2], Jury and Nashimura [3] and Tsypkin [4]. Further developments, relating the design of PWM controllers to the input-output functional design techniques, were contributed by Skoog [5] and Skoog and Blankenship [6]. Other works dealing with PWM control are those of Friedland [7], Min et al [8], and more recently that of LaCava et al [9].

The underlying feature in all of the above contributions is the discrete-time considerations (see also Czaki [10, pp. 591]) associated to the analysis and design issues of such controlled systems. Due to the sampling process associated with the modulator operation, discretization techniques have been traditionally seen as natural in the analysis of PWM controlled systems. However, discretization also leads to quite complex calculations that, at some point, necessarily resort to some kind of not easily justifiable approximation scheme. Moreover, the obtained controller design is usually unnecessarily complex and, generally, places an extra on-line computational burden which makes its realization difficult or somewhat expensive.

In recent works [11] [12], a new technique has been proposed to ease the PWM controller design task of nonlinear PWM controlled systems without resorting to discrete-time considerations. This approach is based in considering a smooth, infinite frequency, average model of the PWM controlled system and carrying out the feedback design on the smoothed version of the controlled system actually retains all the basic qualitative features (i.e., stability) of the discontinuous closed loop controlled plant. As a matter of fact, a sliding regime was shown to exist around the average state trajectories or, more precisely, about integral manifolds of the average dynamical system model. The sliding mode trajectories actually converge to the average trajectories as the sampling frequency increases to infinity.

Generally speaking, PWM controlled systems are classed in two types : ON-OFF and ON-OFF-ON modulators. The first type corresponds to systems in which the discontinuous control action takes values on a discrete set, consisting of two elements, with numerical values, say, 0 or 1. Typically, systems controllers by a two-position switch are of this type (DC to DC power converters, local quantizers of the type used in Delta Modulation Circuits for analog signal encoding, switch-capacitor circuits, etc.). The second class of systems typically correspond to control actions that can be made to take values on the discrete set {-1,0,1}. To this type correspond, for instance, symmetric gas reaction control jets regulating reorientation maneuvers in artificial satellites, torque actuators used for the control of joint positions in robotic manipulators, input-relief arrangements of control valves in fluid processes, etc.

In this article we extend the results of [12], restricted to the class of ON-OFF PWM systems, to the case of ON-OFF-ON PWM controllers prescribed for linear time invariant single-input single-output plants. It is shown that a number of classical design techniques such as the circle criterion, the Popov line criterion, the Describing Function Method, and many others, become immediately available for the PWM controller design task. Besides the possibilities of an exact analysis, all the technical difficulties associated to the unbounded character of the PWM operator [6] block are circumvented. The requirement for low pass cascade multipliers [6] is hence dropped from the analysis or design scenario.

This article is organized as follows : Section 2 contains the general theoretical considerations that validate the time domain design approach based on the average model. Section 3 is devoted to the derivation of a frequency domain stability criterion. Section 4 contains several design examples including simulations. Section 5 is devoted to conclusions and some suggestions for further work.

II. Background Results

Consider the linear time-invariant single-input single-output controlled system described by (see figure 1) :

$$\begin{aligned} y(s) &= G(s)\,u(s) \\ e &= y_d - y \\ u &= M\,\mathrm{PWM}\,e(t_k) \end{aligned} \qquad (2.1)$$

where the PWM operator is characterized by a periodic sampling, with period T, of the error signal e(t), at discrete instants t_k (i.e., $t_{k+1} = t_k + T$). M will be a constant control amplitude gain but it could also be, say, a dynamic shaping filter or compensator network (see example 1). Following [6], the control signal u(t) produced by the modulator is characterized by :

$$\mathrm{PWM}(e) = \begin{cases} \mathrm{sign}\,[e(t_k)] & \text{for } t_k < t \le t_k + \tau_\beta[e(t_k)]T \\ 0 & \text{otherwise} \end{cases} \qquad (2.2)$$

with

$$\tau_\beta[e(t_K)] = \begin{cases} 1 & \text{for} \quad |e(t_K)| > 1/\beta \\ |\beta e(t_K)| & \text{for} \quad |e(t_K)| \le 1/\beta \end{cases} \qquad (2.3)$$

τ_β is known as the <u>duty ratio</u> and β will be addressed as the PWM <u>gain</u>. The sampling frequency is simply F = 1/T. G(s) is a strictly proper causal rational transfer function.

Proposition 1

As the sampling frequency tends to infinity, the closed loop PWM controlled system (2.1) represented in figure 2 is described by :

$$\begin{aligned} y(s) &= G(s)\,v(s) \\ e &= y_d - y \\ u &= M\,\mathrm{sat}_\beta e \end{aligned} \qquad (2.4)$$

where

$$sat_\beta e = \begin{cases} \text{sign } e & \text{for} \quad |e| > 1/\beta \\ \beta e & \text{for} \quad |e| \le 1/\beta \end{cases} \tag{2.5}$$

System (2.4)-(2.5) will be addressed as the average controlled system (see figure 2). System (2.1)-(2.3) will be referred to as the actual PWM controlled system.

Proof : Let $\Sigma(A,B,C)$ be any state space realization of G(s) with state vector represented by x(t) at time t, i.e., $G(s) = C(sI-A)^{-1}B$.

At time $t_{k+1} = t_k + T$ the system state may be expressed as :

$$x(t_{k+1}) = x(t_k) + \int_{t_k}^{t_k+T} Ax(\tau)d\tau + \int_{t_k}^{t_k+\tau\beta[e(t_k)]T} BM \text{ sign } e(t_k)d\tau$$

hence :

$$\begin{aligned}\lim_{T\to 0,\, t_k \to t}[x(t_{k+T}) - x(t_k)]/T &= dx/dt = Ax(t) + BM\tau\beta[e(t)] \text{ sign } e(t) \\ &= Ax(t) + BM \text{ sat}_\beta e(t) \\ &= Ax(t) + Bv(t)\end{aligned}$$

i.e.

$$y(s) = C(sI - A)^{-1} Bv(s) = G(s)\, v(s)$$

with

$$v(t) = M \text{ sat}_\beta e(t)$$

The proposition implies that for infinite sampling frequency, the actual closed loop PWM controlled system is equivalent to a system in which the PWM operator is replaced by a saturation block.

In order to use this results as a basis for the PWM controller design (i.e., for the specification of b and, possibly, the sampling frequency F), we must show that the qualitative features of the actual PWM controlled system (2.1)-(2.3) are retained by the average model (2.4)-(2.5). As a matter of fact, we shall prove that, starting both systems from close enough initial conditions, the responses of the actual and the average PWM controlled system remain arbitrarily close to each other for all finite time. This is the topic of the next theorem.

Theorem 1

Let x(t) and z(t) denote, respectively, the states of the actual and the average PWM controlled systems and denote by $e^*(t) = x(t) - z(t)$, the vector discrepancy among such stated at time t. Then, given an arbitrarily small positive quantity ε, there exists a sampling frequency F_0 and a small positive quantity $\delta(\varepsilon)$ such that for any initial states discrepancy satisfying $\|e^*(0)\| < \varepsilon$ and $F > F_0$, $\|e^*(t)\| \le \delta(\varepsilon)$ for all finite time t.

Proof : Consider first the case in which $|e| > 1/\beta$. Then, under the hypothesis of the theorem, both the average and the PWM controllers are saturated to the same extreme control values and both systems descriptions entirely coincide. By virtue of the continuity of solutions with respect to initial states, the theorem holds obviously true. The case of interest is then represented by $|e| < 1/\beta$. Consider then :

$$x(t_{k+T}) = x(t_k) + \int_{t_k}^{t_k+T} Ax(\tau)\,d\tau + \int_{t_k}^{t_k+\tau_\beta[e(t_k)]T} BM \text{ sign } e_1(t_k)\,d\tau \tag{2.6}$$

$$z(t_{k+T}) = z(t_k) + \int_{t_k}^{t_k+T} Az(\tau)\,d\tau + \int_{t_k}^{t_k+T} BM \text{ sat}_\beta e_2(t_k)\,d\tau \tag{2.7}$$

with $e_1 = y_d - y = y_d - Cx$ and $e_2 = y_d - Cz$

Subtracting (2.7) from (2.6) and taking norms, one finds :

$$\|e^*(t_{k+T})\| \le \|e^*(t_k)\| + \int_{t_k}^{t_k+T} \|Ae^*(\tau)\|d\tau$$

$$+ \left\|\int_{t_k}^{t_k+t_\beta[e(t_k)]T} BM \text{ sign } e_1(t_k)d\tau\right\| + \left\|\int_{t_k}^{t_k+T} BM \text{ sat}_\beta e_2(t_k)d\tau\right\|$$

$$\le \|e^*(t_k)\| + a\int_{t_k}^{t_k+T} \|e^*(\tau)\|d\tau + \int_{t_k}^{t_k+t_\beta[e(t_k)]T} \|BM \text{ sign } e_1(t_k)\|\,d\tau$$

$$+ \int_{t_k}^{t_k+T} \|BM \text{ sat}_\beta e_2(t_k)\|\,d\tau$$

$$\le \|e^*(t_k)\| + a\int_{t_k}^{t_k+T} \|e^*(\tau)\|d\tau + \int_{t_k}^{t_k+T} \|BM \text{ sat}_\beta e_2(t_k)\|d\tau$$

$$\le \|e^*(t_k)\| + a\int_{t_k}^{t_k+T} \|e^*(\tau)\|d\tau + 2MbT$$

where a is the induced norm of A, $b = \|B\|$ and use was made of the fact that for any vectors p and q, $\|p-q\| \le \|p\| + \|q\|$. Using the Gronwall-Bellamn lemma(Beckenbach and Bellman [13, pp. 134]) one obtains :

$$\|e^*(t_{k+T})\| \le (\|e^*(t_k)\| + 2MbT)\, e^{aT} \tag{2.8}$$

Using (2.8) recursively, from $t_k = 0$, the following crude estimate the norm of the vector discrepancy at time NT is readily established :

$$\|e^*(NT)\| \le (\|e^*(0)\| + 2MN\, bT)\, e^{aNT}$$

i.e.

$$\|e^*(NT)\|e^{-aNT} \le (\|e^*(0)\| + 2MN\, bT) \tag{2.9}$$

Let now $\|e^*(0)\| \le \varepsilon$ and suppose one wishes to impose the bound : $\|e^*(NT)\| \le (1+\Delta)\varepsilon$, on the discrepancy vector at time NT. With Δ being an arbitrarily small, but positive, quantity. Then, it is easy to see that the following transcendental equation, obtained directly from (2.9), has a unique solution for some $T_0 > 0$:

$$(1 + \Delta)\varepsilon\, e^{-aNT} = (\varepsilon + 2MN\, bT) \tag{2.10}$$

Hence, given an initial discrepancy bound, $\|e^*(0)\| \le \varepsilon$, and an arbitrarily small positive constant Δ, a sampling frequency $F_0 = 1/T_0$ exists for which a preassigned error response bound δ, of the form $\delta = (1 + \Delta)\varepsilon$, can be obtained for $\|e^*(NT)\|$, at any later finite time NT. The result follows.

The preceding result indirectly establishes that one can always find a sufficiently large sampling frequency in the PWM controller so that the closed loop output responses of the average and actual PWM controlled systems evolve arbitrarily close to each other. The required sampling frequency is always computable in terms of the systems parameters and the desired precision of the actual response with respect to that of an average design.

We point out that obtaining the required sampling frequency is largely a matter of taste and it is highly dependent upon the particular need at hand. If precise simulations cans be afforded, the task of finding an appropriate sampling frequency boils down to several educated trials. For this reason, we do not make further considerations about this issue and refer to (2.9) or (2.10), if a precise calculation needs to be carried out. Instead, we concentrate on the problem of specifying a stabilizing parameter β for the actual PWM controller on the basis of the corresponding stability features of the average PWM closed loop system.

It is obviously clear that for the average design problem, in which a suitable stabilizing gain β is sought for the saturation block in the classical configuration of figure 2, a vast number of design techniques become immediate available. Among these we find : the Small Gain Theorem, the Circle Criterion, the Popov Line Criterion, Tsypkin's Criterion, the Describing Function Method and many others. These methods are well documented and readily found in the literature (see McFarlane [14] for a complete review). We shall illustrate the use of some of these methods through the examples provided in section IV.

III. The Frequency Domain Approach

In the following, we will show that a PWM wave can be expressed as a frequency dependant sector [15]. To show that, we develop a general formula for the spectrum of an ON-OFF PWM wave. We assume such a wave G(w) is built around a periodic clock signal of amplitude M, average duration τ_0 and frequency $F = 1/T = \Omega/2\pi$ (Figure 3). We will also assume that the duration $\tau(t)$ of the PWM wave vary according to the modulating excitation signal $m(t) = \sin\omega t$, so that $\tau(t) = \tau_0 + Km(t)$ in which $K < \tau_0$ is a constant. The Fourier series of such a wave will be :

$$g(t) = \frac{M}{T}\tau(t) + \frac{2M}{T}\tau(t)\sum_{n=1}^{\infty} sa\left(\frac{n\Omega}{2}\tau(t)\right)\cos n\Omega t$$

in which the sampling function sa(·) represent $\frac{\sin(\cdot)}{(\cdot)}$. We write :

$$\frac{2M}{T}\tau(t)\, sa\left(\frac{n\Omega}{2}\tau(t)\right) = \frac{4M}{Tn\Omega} I_m\left\{e^{j\frac{n\Omega}{2}\tau(t)}\right\}$$
$$= \frac{2\Delta}{n\pi} I_m\left[e^{j\alpha}e^{jn\gamma m(t)}\right]$$

where $\alpha = \frac{\Omega\tau_0}{2}$, and $\gamma = \frac{\Omega K}{2}$. Using the properties of Bessel functions [16], we get for $m(t) = \sin\omega t$

$$e^{jn\gamma\sin\omega t} = \sum_{k=0}^{\infty} J_k(n\gamma)\, e^{jn\omega t}$$

where $J_k(\cdot)$ represents the Bessel function of order k. Therefore,

$$G(\omega) = \frac{M\tau_0}{T} + \frac{MK\sin\omega t}{T} + \frac{2M}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sum_{k=0}^{\infty} J_k(n\gamma)\sin(n\omega t + n\alpha)\cos n\Omega t$$

It follows that the spectrum $G(\omega)$ includes an average D.C. value $\frac{M\tau_0}{T}$, a fundamental frequency of amplitude $\frac{MK}{T}$, and a cluster of frequency components around $n\Omega$, at frequencies $n\Omega \pm k\omega$ and of amplitude $\frac{2M}{\pi}J_k(n\gamma)$. Figure 4 illustrates a typical spectrum which assume for simplicity that $\Omega >> \omega$. As $J_k(\gamma) < J_0(0) = 1$ for any k, γ, we deduce that the amplitude of the spectrum of the PWM wave is always bounded by $\max(M\tau_0/T,\ 2M/\pi) = \hat{\Delta}$. From which we deduce that $G(\omega)$ belongs to the real sector $(0, \Delta)$ and has an L_2 induced norm of :

$$\Delta = M\sqrt{\frac{K^2 + 2\tau_0^2 + 2T^2}{T^2}} \text{ (see appendix)}$$

Remarks

We have developed the frequency response of the ON-OFF PWM signal. If we set $\tau_0 = 0$, $K = \beta$, and $g'(t) = g(t)$. sgn(t), and limit our interest to one period of the modulating signal, we can derive a corresponding frequency response for the ON-OFF-ON PWM signal.

We can also take into account a saturating effect whenever the amplitude k of the input is higher than a given threshold β in the following manner. Let p(t) be the unit pulse defined between $-\alpha/2$ and $\alpha/2$, with $\alpha = 1/\omega \arcsin(1/\beta K)$. The input sine wave is than truncated by $P(t) + P(-t)$ for $P(t) = P(t-\alpha/2) + P(t-\pi/\omega + \alpha/2)$. The remaining part has a saturation value M whenever $P(t) + P(-t) \neq 1$ within the interval $[-\pi/\omega, \pi/\omega]$. The corresponding frequency response is a convolution product, and an upper bound on the induced norm can be easily derived. This could be used for the application of the small gain theorem.

IV. Stability Results

The following stability results are based on the small gain theorem and the positivity theorem [17]. Consider the system in Figure 3 in which G is some linear operator represented by its frequency response $G(\omega)$ and H is a PWM modulator whose sector is $(0,\Delta)$.

Theorem 2

The system is stable provided the H_∞ norm of $\Delta G(\omega)$ is smaller than unity.

This is a direct application of the small gain theorem, which can be interpreted graphically by the Circle Criterion.

Theorem 3

Assuming $1 - \Delta\, \|G\|_\infty > 0$, the system is stable.

Indeed, the positivity of $(1 - \Delta\, \|G\|_\infty)$ ensures that $(I + GH)$ is invertible as $\|e\| \leq \| r + GHe\| \leq \|r\| + \|G\| \, . \, \|H\|\, \|e\|$ leads to $\|e\| \leq (I - \|G\|\cdot\|H\|)^{-1}\, \|r\|$. Such a criterion is less conservative than the circle criterion.

V. Sensitivity Results

Moreover, we can extrapolate a result regarding the sensitivity bound of the system represented in Figure 5. Considering the error signal $e = r - Hy$, $y = He$, we can deduce the following inequality which shows the effect of the sector Δ on the overall sensitivity bound.

$$\|e\| \leq [1 - \|G\| \cdot \|H\|]^{-1}\, \|r\|$$
$$= [1 - \Delta\, \|G\|]^{-1}\, \|r\|$$

VI. The Multivariable Case

We now consider that, in Figure 3, G is a matrix of strictly proper frequency responses and H might be a multivariable matrix of PWM modulators ; typically H would be a diagonal matrix. We assume that all inputs are of finite energy, i.e. they belong to L_2.

Each of the PWM modulators has belongs to a sector $(0,\Delta_{ij})$, the L_2 induced norm of which is Δ_{ij}. The induced norm of the matrix H is defined to be [18,19] the maximal singular value norm of the matrix $\hat{H}$ whose entries are Δ_{ij}. Similarly, the L_2 induced norm of the $n \times n$ matrix $G(\omega)$ is the supremum over all frequencies of its maximum singular values which is the multivariable H_∞ norm, while the $n \times n$ matrix $\hat{G}$ represents the matrix where entries are the H_∞ norm of the entries of $G(\omega)$.

We define E to be the vector whose entries are $\|e_i\|$, $i = 1, \ldots n$ and similarly, $R = (\|r_1\|, \|r_2\|, \ldots \|r_n\|)^T$, as in [6], so that :

$$E \leq R + \|G\| \cdot \|H\|\, E$$

Theorem 4

If $\|G\| \cdot \|H\| < 1$, the system is stable.

Theorem 5

If $(1 - \hat{H}\hat{G})$ is an M matrix, the system is stable.

Theorems 4 and 5 are a direct application of [18,19]. Moreover, the sensitivity bound can be easily got. An adequate choice of parameters of H could indeed reduce the sensitivity upperbound.

VII. Elastic Joint Manipulator

Albert and Spong [21] proposed the following linearized version of a single link elastic manipulator controlled by a PD scheme based on motor position feedback (see Figure 6).

$$y(s) = K_p(1 + T_D s)\, e(s)$$

$$e(s) = y_d - \theta_m(s) \quad (3.1)$$

$$\theta_m(s) = \frac{(J_1 s^2 + B_1 s + k)}{(J_m s^2 + B_m s + k)(J_1 s^2 + B_1 s + k) - k^2}$$

where B_1 and B_m are respectively the viscous friction coefficients associated to the link and the motor, k is the elastic torsion coefficient modeling the non-rigid coupling among the motor and the link, θ_m is the motor angular position, y_d is a desired final angular position, and K_p and T_d are the PD controller parameters.

A PWM controller block is to be inserted between the compensated error output y(s) and the input u(s). We let G(s) denote the open loop transfer function relating y(s) to u(s) in (3.1) with $y_d = -0$. An application of the circle criterion [14, pp. 188-195] readily yields the necessary gain β of the saturation block in the average PWM model of the closed loop system. For the particular saturation non linearity, the critical disk in the complex plane degenerates into an unbounded plane to the left of the vertical line $Re(s) < -1/\beta$. A sufficient condition for asymptotic stability is then given by :

$$Re\, G(jw) + 1/\beta > 0 \qquad (3.2)$$

The Popov stability criterion (see [14, pp. 163-181]), on the other hand, establishes that if a linear stable plant with transfer function G(s) is being fedback by a memoryless nonlinearity, bounded by a sector with slopes 0 and $\beta > 0$ (notice that such is the case of the saturation nonlinearity representing the average PWM operator), then the closed loop system is asymptotically stable for all $0 < \beta < k_m$ if there exists a positive constant a, such that the Nyquist plot of the modified transfer function $G^*(jw) = ReG(jw) + jwImG(jw) = X + jY$ (known as the Popov plot) lies entirely to the right of the (Popov) line : $X - \alpha Y + 1/k_m = 0$, in the complex plane XY.

Figure 4 shows the real part of G(jw), as a function of w, for the following manipulator coefficients [15] : $B_1 = 0$ N.m.s/rad, $B_m = 0.015$ N.m.s/rad, $J_1 = J_m = 0.0004$ N.m.s^2/rad, $k = 0.8$ N.m/rad, $K_p = 0.105$ and $T_D = 0.005$. The region of admissible values for $-1/\beta$ is also depicted in Figure 7 $-\beta^{-1} < -0.3383$, i.e. $\beta < 2.9557$ is sufficient to guarantee an asymptotically stable design). Figure 8 shows simulated state trajectory responses for the average PWM controlled plant using several values of the PWM gain parameter β($\beta = 1$, $\beta = 5$, $\beta = 10$). Notice that as β increases the system remains stable but elastic modes excitation leads to a poor damped, or highly oscillatory, link position response.

It can be shown, using the Popov line criterion [14] that an infinite gain is possible (see Figure 6 where the Popov plot is shown for the open loop transfer function G(s) of the motor-link system and a Popov line candidate). It is easy to see that for a high gain β, the saturation block nears the description of an ideal switch and a Variable Structure Controlled system is obtained. The average system exhibits then a sliding regime around $y = 0$ with asymptotically stable motor position dynamics governed by the eigenvalue $-1/T_d$. (i.e., from $e = 0$ in (3.1) the asymptotically stable ideal sliding dynamics, $d\theta_m/dt = -(1/T_d)(\theta_m - y_d)$, is readily obtained). In such a case, the link position trajectory becomes an "almost" pure harmonic oscillation.

From the simulations in Figure 5, b was chosen as b = 1. An appropriate PWM sampling frequency was found to be that of 100 samples per second. Figure 10a shows the simulated step responses for the motor angular position and the link angular position in the actual PWM controlled system. The desired angular position reference was set to 1 rad. Figure 10b shows superimposed trajectories of the actual and the average PWM controlled plants in the link phase space. The discrepancy among these curves is negligible. Figure 10c shows superimposed trajectories of the actual and the average PWM controlled plants in the motor phase space. The existence of a sliding regime about the average trajectories is clearly protracted in this figure. The reason for which such a sliding motion (chattering) is not observed in the link phase space of Figure 10b is due to smoothing of the discontinuous control action through the motor dynamics.

IX. Conclusions and Suggestions for Further Research

An infinite frequency average model of linear PWM controlled systems was shown to capture the essential (stability) properties of the actual PWM controlled plants while allowing a more exact analysis of the closed loop characteristics of the controlled system. The result directly leads to making available a vast amount of classical design techniques for the specification of an important parameter defining the PWM controller. An implicit estimate was also furnished of the required PWM sampling frequency which guarantees a pre-specified small discrepancy among the actual and the PWM controlled responses. This estimate, however, requires knowledge of any state space realization of the controlled plant and the solution of a simple scalar transcendental equation.

The results here presented hold valid in the case of nonlinear controlled plants (see Sira-Ramirez [23]) even if these include finite time delays (Sira-Ramirez [24]). One may wish to extend the results here presented to the multivariable case (Bensoussan [18-19]), and to explore further connections with H_∞ control theory [25].

X. Appendix

Using $J_k(n\gamma)\sin(\omega t + n\alpha)\cos n\Omega t = J_k(n\gamma)/2\,[\cos(n\Omega + n\omega t + n\alpha) + \cos(n\Omega - n\omega - n\alpha)$, and the two identities :

$$\sum_{\kappa=0}^{\infty} J_k^2(n\gamma) = 1 \text{ and, } \sum_{0}^{\infty} \frac{1}{n^2} = \frac{4\pi^2}{8}$$

we deduce the L_2, norm of the output to be :

$$\sqrt{\left(\frac{M\tau_0}{T}\right)^2 + \frac{1}{2}\left(\frac{MK}{T}\right)^2 + M^2}$$

for a corresponding L_2 norm of the input of $(2^{-1/2})$ from which we get the induced norm :

$$\Delta = M\sqrt{\frac{K^2 + 2\tau_0^2 + 2T^2}{T^2}}$$

In practice, the higher harmonics of the PWM wave would have to be neglected as we restrict ourselves to a finite operational bandwidth and the induced norm will be viewed as an upper bound rather than an exact value. However, in the case of two sine inputs of frequencies f_1 and f_2, and corresponding constants γ_1 and γ_2, the spectrum is enriched by intermodulation products, as the output spectrum includes the harmonics at $F + k f_1, F + l f_2$, $k, l = 0, \ldots \infty$ as well as intermodulation frequencies $F + kf_1 + l f_2$ with amplitudes $\frac{2M}{n\pi} J_k(n\gamma_1) J_l(n\gamma_2)$.

XI. References

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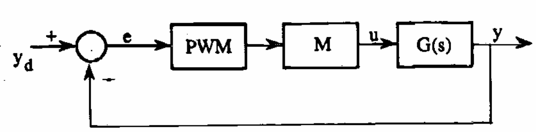

Figure 1. PWM Controlled Linear System

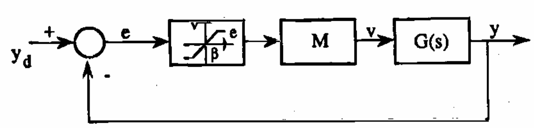

Figure 2. Average PWM Controlled Linear System

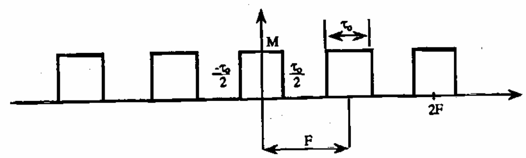

Figure 3

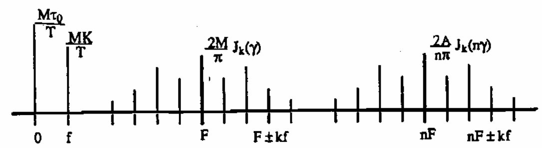

Figure 4

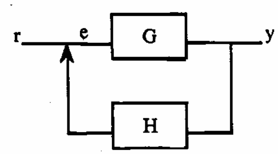

Figure 5

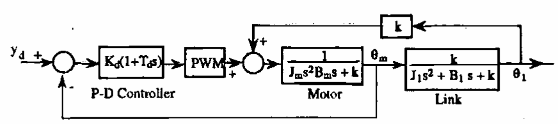

Figure 6. Single Link Elastic Joint Manipulator Model

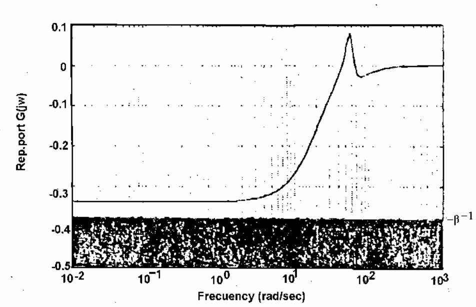

Figure 7. Application of the Circle Criterion to Example

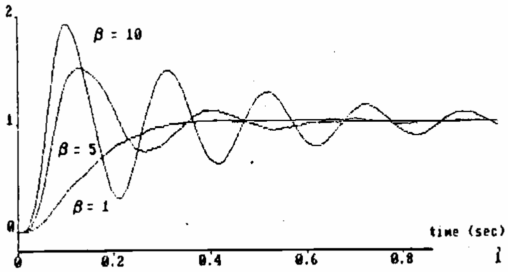

Figure 8. Simulated Responses of Link Angular Position for Several Averages

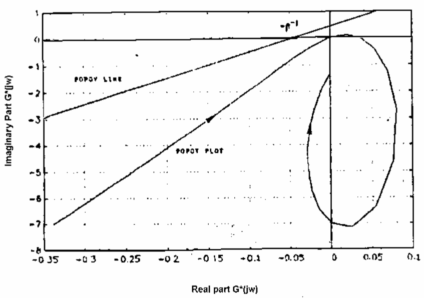

Figure 9. Application of Popov Stability Criterion to Example

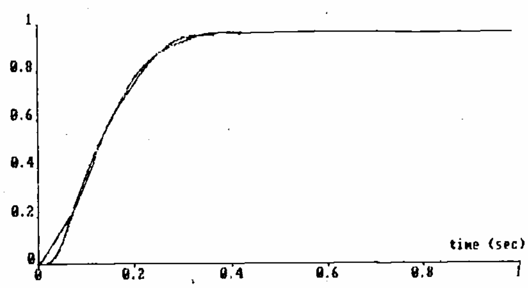

Figure 10a. Actual PWM Response of Motor and Link Angular Positions

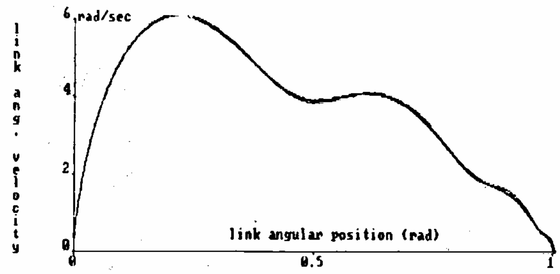

Figure 10b. Actual and Average PWM Controlled Link Phase Space

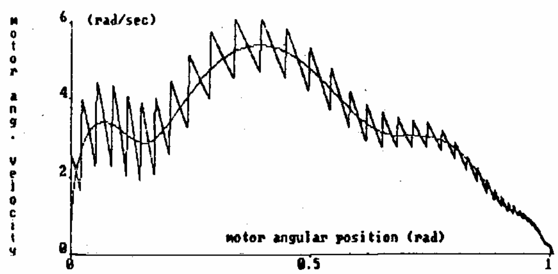

Figure 10c. Actual and Average PWM Controlled Motor Phase Space