AN EXTENDED SYSTEM APPROACH TO DYNAMICAL PULSE-FREQUENCY-MODULATION CONTROL OF NONLINEAR SYSTEMS

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Abstract The design of stabilizing Pulse-Frequency- Modulation (PFM) controllers is addressed, in all generality, for the case of nonlinear single-input single-output analytic systems. Both, static and dynamical PFM control strategies are developed, in full detail, on the basis of an elementary scalar system result. Some Illustrative design examples are provided.

1. INTRODUCTION

Pulse-Frequency-Modulation (PFM) feedback control strategies have been relatively little studied in the second half of this century. The main references in this area are constituted by the works of Skoog [1] , Skoog and Blankenship [2], Frank [3] and Frank and Wiechmann [4] where many early references can be found. These works are all centered around the case of linear time-invariant systems. To our knowledge, no further extensions of these works, to the nonlinear case, were pursued later on.

In this article, we present a general design method for synthesizing static and dynamical PFM feedback control laws stabilizing to a constant equilibrium point any *minimum phase* single-input single-output nonlinear dynamical system. A static PFM controller is proposed which asymptotically zeroes a suitably designed auxiliary scalar output function with the property that the restricted PFM controlled dynamics results, in turn, in an asymptotic stabilization of the original system output. The static controller case is dealt via *Normal Canonical Forms* (Isidori [5]). Being a discontinuous feedback regulation policy, static PFM controllers may induce undesirable chattering in the obtained controlled responses. This is due to the high frequency bang-bang character of the synthesized control input signals. In this article, we effectively circumvent this problem by also proposing *dynamical* PFM control strategies. In the dynamical feedback alternative, continuous, instead of bang-bang, feedback control signals are obtained which robustly stabilize to a constant operating point the closed loop system, without chattering effects on the controlled variables. The dynamical PFM controller design is accomplished by first proposing a static PFM controller on the corresponding Normal Canonical Form of a generalized version of the *extended system* (Nijmeijer and Van der Schaft [6]). In contradistinction to the extended system, which only uses one integrator before the input, the *Generalized Extended System* is obtained by adjoining to the original system input a string of integrators of length equal to the the dimension of the *zero dynamics* of the original input-output system.

PFM control, much as its PWM particularization, constitutes a robust feedback control policy due to its insensitivity to external disturbance inputs, certain immunity to model parameter variations, within known bounds, and to the ever present modeling errors (see [7]).

Section 2 presents a fundamental result on the PFM control of an elementary scalar dynamical system. It is shown that an entire nonlinear PFM controller design procedure, for higher order nonlinear plants, may be based on this elementary result. Section 2 also introduces the general version of the extended system and discusses some of its properties. The Normal Canonical Form of the proposed generalized extended system is intimately related to Fliess'*Generalized Observability Canonical Form* (see Fliess [8] see also Conte *et al* [9]). Section 3 is devoted to develop both a static and a dynamical PFM control design scheme for nonlinear minimum-phase systems. Section 4 presents an application example drawn from chemical process control, a rather non-traditional application area for discontinuous control techniques. Customarily, discontinuous feedback control policies are not allowed in the regulation of plants where flow rates are regarded as input variables. This is due to the fact that bang-bang behavior of the main flow rate control valve may severly limit the lifespan of the actuator. A dynamical PFM controller design is then carried out for a a chemical process control as an illustrative example in which smoothed (i.e., implementable) inputs are obtained.The dynamical feedback viewpoint adopted in this article opens up new possibilities for applications of discontinuous feedback control policies. The conclusions of the article are collected in Section 5.

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2. SOME FUNDAMENTAL RESULTS

2.1 PFM Control of a scalar system.

Consider the scalar PFM controlled dynamical system, in which the constants r_1 , r_2 , r_3 and W, are all strictly positive quantities.

$$\dot{s} = -W v$$
$$v = PFM_{\tau, T}(s) = \begin{cases} \operatorname{sgn} s(t_k) & \text{for } t_k \le t < t_k + \tau[s(t_k)]T[s(t_k)] \\ 0 & \text{for } t_k + \tau[s(t_k)]T[s(t_k)] \le t < t_k + T[s(t_k)] \end{cases} \quad (2.1)$$
$$\tau[s(t)] = \begin{cases} 1 & \text{for } |s(t)| > \frac{1}{r_1} \\ r_1 |s(t)| & \text{for } |s(t)| \le \frac{1}{r_1} \end{cases}$$
$$T[s(t)] = \begin{cases} T_{max} & \text{for } |s(t)| \ge \frac{1}{r_2} \\ T_{min} + \frac{r_2 r_3}{r_3 - r_2}[T_{max} - T_{min}]\left(s(t) - \frac{1}{r_3}\right) & \text{for } \frac{1}{r_3} < |s(t)| < \frac{1}{r_2} \\ T_{min} & \text{for } |s(t)| \le \frac{1}{r_3} \end{cases}$$
$$k = 0,1,2,\dots \quad ; \quad t_{k+1} = t_k + T[s(t_k)].$$

where it is assumed that $r_2 < r_1 < r_3$. The t_k's represent irregularly spaced sampling instants, determined by the sampled values of the *duty cycle function*, denoted here by $T[s(t_k)]$. The duty cycle function, $T[s(t)]$, takes values on the closed interval $[T_{min}, T_{max}]$ and it varies linearly with respect to s(t) in the region $|s| < 1/r_2$. The duty cycle, or sampling period, saturates to T_{max} for large values of s, and remains fixed at the constant lower bound T_{min} for small values of s. At each sampling instant, t_k, the value of the width of the sign-modulated, unit amplitude, control pulse is determined by the sampled value of the *duty ratio function*, represented by $\tau[s(t_k)]$. In general, the duty cycle and the duty ratio functions may be quite independent of each other. The function "sgn" stands for the *signum* function:

$$\text{sgn}(s) = \begin{cases} +1 & \text{if } s > 0 \\ 0 & \text{if } s = 0 \\ -1 & \text{if } s < 0 \end{cases}$$

The condition $r_2 < r_1 < r_3$ indicates that the pulse width, τ, is saturated to the value of the duty cycle, T, as long as the value of the duty cycle is itself saturated to T_{max} (see figure 1). When the state, s, of the scalar system is decreased, in absolute value, below the boundary value $1/r_2$. the duty cycle, T, starts also decreasing, in a linear fashion with respect to s, while the pulse width temporarily continues to be saturated to the same values adopted by T. If the state s further decreases and reaches the interval $[-1/r_1, 1/r_1]$ (notice that $1/r_1$ is intermediate between $1/r_3$ and $1/r_2$), the pulse width also starts decreasing linearly with respect to s. When the state s is finally confined to the band $[-1/r_3, 1/r_3]$, the duty cycle (sampling period) reaches its minimum value T_{min}. In this region, the duty ratio still continues to linearly decrease towards zero, even if the duty cycle is already saturated to its minimum value T_{min}.

The following proposition establishes a sufficient condition for the asymptotic stability to zero of the PFM controlled system (2.1).

Proposition 2.1

The PFM controlled system (2.1) is asymptotically stable to s = 0, if

$$0 < r_3 W T_{max} < 2 \tag{2.2}$$

Proof (see [10]).

2.2 Normal Canonical Forms and the Generalized Extended System

2.2.1 The Normal Canonical Form [5]

Consider the analytical n-dimensional state variable representation of a single-input single-output system :

$$\begin{aligned} \dot{x} &= F(x,u) \\ y &= h(x) \end{aligned} \tag{2.3}$$

which is assumed to have *strong relative degree* r ([5]). The integer r is roughly defined as the minimum number of times that the scalar output function y = h(x) must be differentiated, with respect to time, so that the control input u appears explicitly in the derivative expression. It is assumed that (2.3) exhibits a constant equilibrium point of interest characterized by F(X(U),U) = 0, for which h(X(U)) = 0. We refer to this point as (U,X(U),0).

Associated to the relative degree of the system one defines its *Normal Canonical Form* as given by ([5]):

$$\begin{aligned} \dot{\xi}_i &= \xi_{i+1} \quad ; \; i = 1,2,\ldots,r-1 \\ \dot{\xi}_r &= f(\xi,\eta,u) \\ \dot{\eta} &= q(\xi,\eta,u) \\ y &= \xi_1 \end{aligned} \tag{2.4}$$

where $\partial f/\partial u \neq 0$, $\xi = (\xi_1,\ldots,\xi_r)$ and $\eta = (\eta_1,\ldots,\eta_{n-r})$. It is assumed that n-r is a positive integer. The case in which n = r corresponds to the class of *exactly input-output linearizable systems* by means of static state feedback. The design of a static stabilizing PFM controller for this case requires no special consideration. However, the dynamic PFM controller for such a case requires use of the extended system as defined in [6] .

The state coordinates transformation which yields the normal canonical form (2.4) for the system (2.3) is given by:

$$(\xi,\eta) = \Phi(x) = [\, h(x), \dot{h}(x), \ldots, h^{(r-1)}(x), \phi_{r+1}(x), \cdots, \phi_n(x) \,] \tag{2.5}$$

where $\eta_j = \phi_{r+j}$ (j=1,...,n-r) is an arbitrary set of coordinate functions which are functionally independent among themselves, and also independent of the first r coordinates ξ, which are obtained by repeated differentiation of the output function y = h(x).

Let there exist a (possibly discontinuous) feedback control law $u(\xi,\eta) = u(\Phi(x))$ which locally asymptotically stabilizes the system trajectories to the manifold $\xi = 0$. The system (2.3) is said to be locally *minimum phase* at a given equilibrium point $F(X(U)) = (0,\eta^0)$ if the resulting autonomous set of nonlinear ordinary differential equations represented by:

$$\dot{\eta} = q(0,\eta,u(0,\eta)) =: q_0(\eta) \tag{2.6}$$

is locally asymptotically stable towards η^0, otherwise the system is said to be *nonminimum phase* (see [5]). The marginally stable case is usually treated via *Center Manifold theory* (see also [5], Appendix B). We assume, henceforth, that the given system is locally minimum phase around the equilibrium point of interest.

2.2.2 The Generalized Extended System

Consider the nonlinear system (2.3). One defines a *Generalized Extended System* of (2.3), as the 2n-r dimensional system obtained by placing a chain of n-r integrators before the original system input u, and feeding the resulting system by an external auxiliary input signal v, i.e.,

$$\begin{aligned} \dot{x} &= F(x,x_{n+1}) \\ \dot{x}_{n+j} &= x_{n+j+1} \quad ; \; j = 1,2,\ldots,n+r-1 \\ \dot{x}_{2n-r} &= v \\ y &= h(x) \end{aligned} \tag{2.7}$$

The *extended system*, as defined in [6], only requires placing a single integrator before the input u, regardless of the value of the relative degree of the system.

The following state coordinate transformation takes the 2n-r dimensional system (2.7) into normal canonical form :

$$(\hat{\xi},\eta) = \hat{\Phi}(x,x_{n+1},x_{n+2},\cdots,x_{2n-r}) =: \tag{2.8}$$

$$\hat{\Phi}(\hat{x}) = \begin{bmatrix} h(x) \\ \dot{h}(x) \\ \cdots \\ h^{(r)}(x,x_{n+1}) \\ \cdots \\ h^{(n-1)}(x,x_{n+1},x_{n+2},\cdots,x_{2n-r}) \\ \phi_{n+1}(x) \\ \cdots \\ \phi_{2n-r}(x) \end{bmatrix}$$

where:

$$\hat{\xi} := (\xi_1,\cdots,\xi_r,\xi_{r+1},\cdots,\xi_n) = (\xi,\xi_{r+1},\cdots,\xi_n)$$

$$\eta := (\phi_{n+1},\cdots,\phi_{2n-r})$$

Notice that the state coordinate functions: ϕ_j (j = n+1,...,2n-r) may be chosen in precisely the same way as the η's were chosen, in (2.5), when the normal canonical form of the original system (2.3) was obtained. The components of η are, then, set independently of the first r coordinate components comprising the vector ξ. It should be obvious that if the η's, in (2.8), are chosen in this manner, they are also independent of the new set of extra state coordinates $\xi_{r+1},\ldots,\xi_n$, and the transformation (2.8) is full rank.

The normal canonical form of system (2.7) is, therefore, given by:

$$\begin{aligned} \dot{\xi}_i &= \xi_{i+1} \ ; \ i = 1,2,\ldots,n-1 \\ \dot{\xi}_n &= \hat{c}(\xi,\eta,v) \\ \dot{\eta} &= q(\xi_1,\cdots\xi_r,\eta) \\ y &= \xi_1 \end{aligned} \tag{2.9}$$

Notice that the dimension, and the form of the equations, of the zero dynamics, associated to the generalized extended system, are not altered with respect to those of the original system. This reveals the invariance of the zero dynamics, which is a well known fact [5].

In order to be able to solve, even if in a local sense, for the auxiliary control input v on any relation involving the function $c(\xi,\eta,v)$, it is assumed that $\partial c/\partial v \neq 0$ in the region of interest. This is equivalent to avoiding the regions where *impasse* points may exist for the traditional definition of the state of a dynamical controller, derived on the basis of (2.9) (see Fliess and Hasler [11] for related details).

3. STATIC AND DYNAMICAL PFM CONTROL OF NONLINEAR SYSTEMS

3.1. Static PFM Control of Nonlinear Systems.

Let p(l) be an (r-1)th order Hurwitz polynomial with constant coefficients:

$$p(\lambda) = \lambda^{r-1} + a_{r-1}\lambda^{r-2} + \cdots + a_2\lambda + a_1 \tag{3.1}$$

Associated to this polynomial, consider the following auxiliary output function of the system (2.3):

$$s = \xi_r + a_{r-1}\xi_{r-1} + \cdots + a_2\xi_2 + a_1\xi_1 \tag{3.2}$$

It should be evident that if the condition s = 0 is achieved by means of appropriate control actions, the restricted motions of the minimum phase system (2.3) satisfy the following asymptotically stable time-invariant linear dynamics:

$$\begin{aligned} \dot{\xi}_i &= \xi_{i+1} \ ; \ i = 1,2,\ldots,r-2 \\ \dot{\xi}_{r-1} &= -a_{r-1}\xi_{r-1} - \cdots - a_2\xi_2 - a_1\xi_1 \end{aligned} \tag{3.3}$$

The following proposition is one of the main results of this article and a direct consequence of the above considerations and of Proposition 2.1.

<u>Proposition 3.1</u> A minimum phase nonlinear system of the form (2.6) is locally asymptotically stable to the equilibrium point (U,X(U),0) if the control action u is specified as a static PFM control policy given by the solution of the following implicit algebraic equation :

$$f(\xi,\eta,u) = -\sum_{i=1}^{r} a_{i-1}\xi_i - W\,PFM_{t,T}[\sum_{i=1}^{r} a_i\xi_i] \tag{3.4}$$

where $a_0 = 0$, and $a_r = 1$.

<u>Proof</u>

Immediate upon imposing on the auxiliary output function $s(\xi)$ in (3.2) the asymptotically stable discontinuous PFM dynamics defined by (2.1). ■

In original coordinates, the implicit static PFM controller, given by (3.4), is expressed as:

$$f(\Phi(x),u) = -\sum_{i=1}^{r} a_{i-1}h^{(i-1)}(x) - W\,PFM_{t,T}\left[\sum_{i=1}^{r} a_i h^{(i-1)}(x)\right] \tag{3.5}$$

Notice that by virtue of the strong relative degree assumption on (2.3), one globally has : $\partial f/\partial u \neq 0$. By invoking the Implicit Function theorem, one concludes that there always exist a locally unique solution of (3.4), or of (3.5), for the control input function u.

3.2. Dynamical PFM Control of Nonlinear Systems.

Let $p(\lambda)$ be an (n-1)th order Hurwitz polynomial with constant coefficients:

$$p(\lambda) = \lambda^{n-1} + a_{n-1}\lambda^{n-2} + \cdots + a_2\lambda + a_1 \tag{3.6}$$

Consider now the following auxiliary output function of the system (2.3):

$$\hat{s}(\xi) = \xi_n + a_{n-1}\xi_{n-1} + \cdots + a_2\xi_2 + a_1\xi_1 \tag{3.7}$$

As before, if the condition s = 0 is achieved by means of suitable controls, the restricted motions of the generalized extended system (2.7) satisfy the following asymptotically stable linear time-invariant dynamics:

$$\begin{aligned} \dot{\xi}_i &= \xi_{i+1} \ ; \ i = 1,2,\ldots,n-2 \\ \dot{\xi}_{n-1} &= -a_{n-1}\xi_{n-1} - \cdots - a_2\xi_2 - a_1\xi_1 \end{aligned} \tag{3.8}$$

The following proposition is a direct consequence of the preceding considerations and of Proposition 2.1.

<u>Proposition 3.2</u> A minimum phase nonlinear system of the form (2.3) is locally asymptotically stable to the equilibrium point (U,X(U),0) if the control action u is specified as a dynamical PFM control policy given, with slight abuse of notation, by the solution of the following implicit, time-varying, nonlinear discontinuous differential equation :

$$\begin{aligned} c(\Phi(x),u,\dot{u},\cdots,u^{(n-r)}) = \\ -\sum_{i=1}^{r} a_{i-1}h^{(i-1)}(x) - \sum_{i=r+1}^{n} a_{i-1}h^{(i-1)}(x,u,\dot{u},\cdots,u^{(i-r-1)}) \\ - W\,PFM_{t,T}\left[\sum_{i=1}^{r} a_i h^{(i-1)}(x) + \sum_{i=r+1}^{n} a_i h^{(i-1)}(x,u,\dot{u},\cdots,u^{(i-r-1)})\right] \end{aligned} \tag{3.9}$$

where $a_0 = 0$, and $a_n = 1$.

<u>Proof</u>

Imposing on the auxiliary output function $s(\xi)$, given in (3.7), the asymptotically stable discontinuous PFM controlled dynamics defined by (2.1), one immediately obtains an implicit PFM static controller for v, in terms of the transformed state variables. In original state and input coordinates the controller adopts the form (3.9). ■

Notice that one cannot, in general, assume that a global state variable representation exists for the dynamics of the implicit controller given by (3.10). As it is now known from the *differential algebraic* approach to system analysis, state variable representations are only locally possible, in general (see the outstanding work of Fliess [12], and the references therein).

4. AN APPLICATION EXAMPLE

Example (A Dynamical PFM Control Approach for Concentration Control in an Exothermic Continuously Stirred Tank Reactor).

Consider the following nonlinear dynamical controlled model of an exothermic reaction occuring inside a CSTR (see Parrish and Brosilov [13]), where the control objective is to regulate the outlet concentration through manipulation of the water jacket temperature :

$$\begin{aligned} \dot{x}_1 &= \frac{F}{V}(c_0 - x_1) - a x_1 e^{-b/x_2} \\ \dot{x}_2 &= \frac{F}{V}(T_0 - x_2) + \frac{aL}{c_p} x_1 e^{-b/x_2} - \frac{h}{Vc_p}(x_2 - u) \\ y &= x_2 - T \end{aligned} \tag{4.1}$$

Where x_1 represents the product concentration. The state variable x_2 represents the reactor temperature. The control variable u is the water jacket temperature. F is the reactor throughput in lb/hr, c_0 is the inlet flow concentration in lb/lb, T_0 is the inlet flow temperature measured in deg.R, c_p is the material heat capacity in BTU/lb.R while V and L are, respectively, the reactor holdup (in lb.) and the heat of the reaction (in BTU/lb.) . The constant h is the heat transfer parameter (in BTU/hr.R) , b is the activation constant (in deg R) and a is the pre-exponential factor in hr^{-1} . A constant temperature T is to be stably maintained to indirectly control the product concentration x_1 to its constant equilibrium value X_1.

A stable constant equilibrium point for this system is then given by :

$$\begin{aligned} x_2 &= T \; ; \; x_1 = X_1(T) = \frac{c_0}{1 + V/_F\, a\, e^{-b/T}} \; ; \\ u &= U(T) = T - \frac{c_p F}{h}(T_0 - T) - \frac{aLV}{h} \frac{c_0 e^{-b/T}}{1 + V/_F\, a\, e^{-b/T}} \end{aligned} \tag{4.2}$$

We next summarize the design procedure leading to a dynamical stabilizing PFM controller for system (4.1), based on the extended model. As it is easily verified the relative degree of the system (4.1) is equal to one and, hence, the dimension of the zero dynamics is also one.

<u>Extended System Model of CSTR</u>

$$\begin{aligned} \dot{x}_1 &= \frac{F}{V}(c_0 - x_1) - a x_1 e^{-b/x_2} \\ \dot{x}_2 &= \frac{F}{V}(T_0 - x_1) + \frac{aL}{c_p} x_1 e^{-b/x_2} - \frac{h}{Vc_p}(x_2 - x_3) \\ \dot{x}_3 &= v \\ y &= x_2 - T \end{aligned} \tag{4.3}$$

<u>State Coordinate Transformation to Normal Canonical Form for the Extended System.</u>

$$\begin{aligned} \xi_1 &= x_2 - T \\ \xi_2 &= \frac{F}{V}(T_0 - x_2) + \frac{aL}{c_p} x_1 e^{-b/x_2} - \frac{h}{Vc_p}(x_2 - x_3) \\ \eta &= x_3 \end{aligned} \tag{4.4}$$

$$\begin{aligned} x_1 &= \frac{c_p}{aL} e^{b/(\xi_1+T)} \left\{ \xi_2 - \frac{F}{V}(T_0 - T - \xi_1) + \frac{h}{Vc_p}(\xi_1 + T - h) \right\} \\ x_2 &= \xi_1 + T \\ x_3 &= \eta \end{aligned} \tag{4.5}$$

<u>Normal Canonical Form of the Extended System</u>

$$\begin{aligned} \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= \frac{aL}{c_p} e^{-b/(\xi_1+T)} \\ & \left[\frac{F}{V} c_0 - \frac{c_p}{aL} e^{b/(\xi_1+T)} \frac{F}{V} \left(1 + \frac{V}{F} a e^{-b/(\xi_1+T)} - \frac{bV}{F} \frac{\xi_2}{(\xi_1+T)^2} \right) \right. \\ & \qquad \left. \left(\xi_2 - \frac{F}{V}(T_0 - T - \xi_1) + \frac{h}{Vc_p}(\xi_1 + T - h) \right) \right] \\ & - \left(\frac{F}{V} + \frac{h}{Vc_p} \right) \xi_2 + \frac{h}{Vc_p} v \\ \dot{\eta} &= v \\ y &= \xi_1 \end{aligned} \tag{4.6}$$

<u>Auxiliary Output Function</u>

$$s = \xi_2 + a_1 \xi_1 \quad ; \quad a_1 > 0 \tag{4.7}$$

<u>Restricted Asymptotically Stable Motions of the Controlled Dynamics</u>

$$\dot{\xi}_1 = -a_1 \xi_1 \tag{4.8}$$

<u>Static PFM Controller for the Extended System</u>

$$\begin{aligned} v = \frac{Vc_p}{h} \Bigg\{ & - \frac{aL}{c_p} e^{-b/(\xi_1+T)} \left[\frac{F}{V} c_0 \right. \\ & - \frac{c_p}{aL} e^{b/(\xi_1+T)} \frac{F}{V} \left(1 + \frac{V}{F} a e^{-b/(\xi_1+T)} - \frac{bV}{F} \frac{\xi_2}{(\xi_1+T)^2} \right) \\ & \left. \left(\xi_2 - \frac{F}{V}(T_0 - T - \xi_1) + \frac{h}{Vc_p}(\xi_1 + T - \eta) \right) \right] \\ & + \left(\frac{F}{V} + \frac{h}{Vc_p} - a_1 \right) \xi_2 - W\, PFM_{\tau,T}[s(\xi)] \Bigg\} \end{aligned} \tag{4.9}$$

<u>Asymptotically Stable Zero Dynamics</u>

$$\begin{aligned} \dot{\eta} = & - \frac{F}{V}\left(1 + \frac{V}{F} a e^{-b/T}\right) \\ & \left[\eta - T + \frac{c_p F}{h}(T_0 - T) + \frac{aLV}{h} \frac{c_0 e^{-b/T}}{1 + V/_F\, a\, e^{-b/T}} \right] \end{aligned} \tag{4.10}$$

<u>Dynamical PFM Controller in Original State and Input Coordinates</u>

$$\begin{aligned} \dot{u} = \frac{Vc_p}{h} \Bigg\{ & \left[\frac{F}{V}(T_0 - x_2) + \frac{aL}{c_p} x_1 e^{-b/x_2} - \frac{h}{Vc_p}(x_2 - u) \right] \\ & \left(\frac{F}{V} + \frac{h}{Vc_p} - \frac{abL}{c_p} e^{-b/x_2} \frac{x_1}{x_2^2} - a_1 \right) \\ & - \frac{aL}{c_p} e^{-b/x_2} \left(\frac{F}{V}(c_0 - x_1) - a x_1 e^{-b/x_2} \right) - W\, PFM_{\tau,T}[\, s(x,u) \,] \Bigg\} \\ s(x,u) = & \frac{F}{V} T_0 - a_1 T - \left(\frac{F}{V} + \frac{h}{Vc_p} - a_1 \right) x_2 + \frac{aL}{c_p} x_1 e^{-b/x_2} + \frac{h}{Vc_p} u \end{aligned} \tag{4.11}$$

Simulations were performed for a dynamical PFM controller CSTR characterized by the following parameters [13]:

F = 2000lb/hr ; c_0 = 0.50 lb/lb ; V = 2400 lb. ; a = 7.08 × 10^{10} hr^{-1} ; b = 15080 deg R ; T_0 = 5320 deg.R ; L = 600 BTU/lb. ; cp = 0.75 BTU/lb.R ; h = 15000 BTU/hr.R ;

For such values of the parameters, the equilibrium point (4.2) of the system results in :

x_2 = T = 600 deg R ; u = U(T) = 107.679 deg. R ; $X_1(T)$ = 0.246 lb/lb.

The PFM controller parameters were chosen as: $a_1 = 8$, W = 50, , $T_{max} = 8 \times 10^{-4}$hr, $T_{min} = 2 \times 10^{-4}$hr , $r_1 = 15$, $r_2 = 10$, $r_3 = 40$. Figure 2 portrays the time response of the dynamical PFM controlled state variables x_1 and x_2, the chattering-free (smoothed) continuous control input trajectory u(t) and the evolution of the auxiliary output function s(x,u).

5. CONCLUSIONS

A general stabilizing design procedure, based on static or dynamical PFM feedback control policies, has been presented for minimum phase nonlinear single-input single-output systems. A stabilizing discontinuous static controller of the PFM type is proposed for an elementary scalar system. Based on this result, a stable PFM controller design is immediately obtained from zeroing an auxiliary output function - defined in terms of the *normal canonical variables* - of any minimum phase nonlinear system. The obtained static controller generates bang-bang control inputs to the system thus producing typical chattering state and output responses. As an alternative to the static controller design, which eliminates this inconvenience, we have also proposed a dynamical discontinuous feedback controller of the PWM type. Such a dynamical controller is also obtained on the basis of zeroing an auxiliary output function, defined now in terms of the *normal canonical variables* of the *Generalized Extended* system. The static and dynamical results are based on elementary considerations concerning the asymptotic stabilization of such a scalar auxiliary output function by means of a simple PFM feedback controller of the ON-OFF-ON type. Zeroing of the auxiliary output function, in both cases, induces an asymptotically stable motion of the constrained dynamics, characterized by a linear time-invariant system with eigenvalues placeable at will. In the dynamical controller case, the discontinuities, generated by the PFM generator, take place in the state space of the dynamical controller, and not in the state space of the system. The resulting integrated control actions are, thus, continuous with substantially reduced (smoothed out) chattering. Aside from the chosen sampling frequency, the chattering reduction effect on the control input was seen to directly depend on the relative degree of the given system. This effect being more important for systems of small relative degree and non-existent for exactly linearizable systems.

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FIGURES

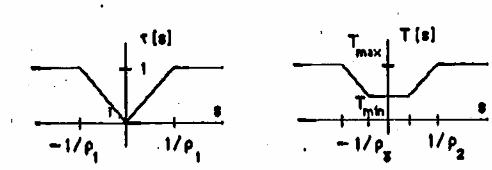

Figure 1. Duty Ratio and Duty Cycle functions for PFM actuator.

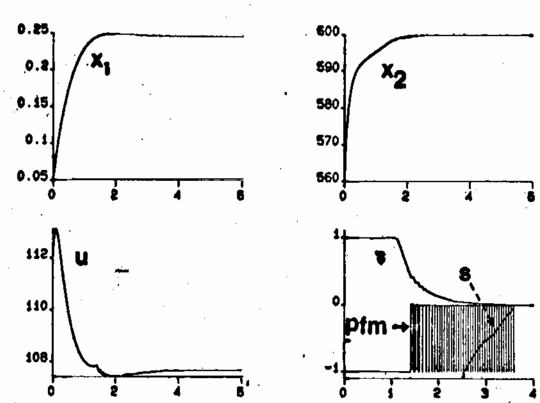

Figure 2. Dynamical PFM Controlled State Variables, Chattering-Free Control Input Trajectory and Auxiliary Output function for CSTR Example.