NONLINEAR APPROACHES TO VARIABLE STRUCTURE CONTROL

Hebertt Sira-Ramírez
Departamento Sistemas de Control
Universidad de Los Andes
Mérida-Venezuela

Abstract A differential algebraic apporach for the sliding mode control of nonlinear single-input single-output systems is reviewed in a tutorial fashion. Input-dependent sliding surfaces, possibly including time derivatives of the input signal, are shown to naturally arise from elementary differential algebraic results pertaining Fliess' Generalized Controller Canonical Forms of nonlinear systems. This class of switching surfaces generally lead to chattering-free dynamically synthesized sliding regimes, in which the highest time derivative of the input signal undergoes all the bang-bang type discontinuities. A definite relationship among controllability of a nonlinear system and the possibility of creation of "higher order" sliding regimes is readily established via differential algebra. Examples illustrating the obtained results are also included.

1. INTRODUCTION

Sliding mode control of dynamical systems has a long history of theoretical and practical developments. A rather complete chronological collection of journal articles and conference presentations has been gathered by Professor S.V.Emelyanov (1989), (1990a), who is one of the founding fathers of the technique. Extensive surveys, with an enormous wealth of information, have been presented over the years by Utkin (1977), (1984), (1987). Several books have also been published on the subject : Emelyanov (1967), Itkis (1976), Bühler (1986), Utkin (1978),(1992). Contributed volumes by A. Zinober (1990) and K.K.D.Young (to appear) reveal sliding mode control as an active discipline of research with enough theoretical maturity. A survey of the numerous industrial and laboratory applications of sliding regimes around the world is well beyond the scope of this article. In the following paragraphs we provide a necessarily incomplete overview of some of the contributions in sliding mode control for nonlinear dynamical systems. Many interesting developments in controller robustness, adaptive regulation, and observer design are not mentioned.

In recent years, the outstanding developments for nonlinear control systems based on differential geometric ideas (see the books by Isidori, 1989, Nijmeijer and Van der Schaft , 1990) have found immediate applications, and extensions, to sliding mode control, and closely related areas, such as high-gain, pulse-width-modulation and pulse-frequency modulation. Seminal work on sliding regimes for nonlinear systems is due to Luk'yanov and Utkin (1981). Starting with the contributions by Slotine and Sastry, (1983) especially devoted to the field of robotics automation, the differential geometric approach to nonlinear systems control was exploited and put in perspective, within the sliding mode control area, by the independent work of several authors. An important contribution relating sliding mode systems to high-gain feedback controlled systems from a geometric standpoint was given by Marino (1985), Bartolini and Zolezzi (1986) presented interesting developments of sliding mode control as applied to robust linearization of nonlinear plants. A full case study of sliding mode control design for a nonlinear system was presented by Mathews *et al* (1986). The sliding mode control of nonlinear multivariable systems was addressed by Fernandez and Hedrick (1987). A quite readable tutorial dealing with multivariable nonlinear systems was written by DeCarlo *et al* (1988). Later, in a series of articles, Sira-Ramirez (1987, 1989a, 1989b, 1990) contributed with some formalizations, application examples and generalizations, of sliding regimes in nonlinear systems. More recently, a rather complete picture of the nonlinear multivariable case has been provided by Kwatny and Kim (1990).

Extensions to the problem of (quasi) sliding regimes in discrete time nonlinear systems have also been published in recent years. Work in this area was initiated, within the framework of sampled data systems, by Miloslavjevic (1985), followed by that of Opitz (1986). Sarpturk et al (1987) provided new definitions of sliding regimes for the discrete time case. Zak and Magaña (1987) developed the design ideas from a Lyapunov stability theory standpoint. Utkin and Drakunov (1989), explored the main difficulties in this field and provided rather general results. The contribution of Furuta (1990) in this field is related to the linear case, with emphasis in robustness and self tunning regulation aspects. Recently, Sira-Ramirez (1991a) treated the general nonlinear discrete time case using extensions of the normal canonical forms and the relative degree concept. The works of Spurgeon (1991) and Yu (1992) is centered around the linear systems case.

In the area of distributed sliding regimes for infinite dimensional systems, the first constributions were given by Breger *et al* (1980), Orlov and Utkin (1982) (1987), Utkin (1990) and more recently by Zolezzi (1989), Sira-Ramírez (1989c), Sira-Ramírez and Rivero-Mendoza (1990), and Rebiai and Zinober (1990, 1991).

Recent developments in nonlinear systems include the use of *differential algebra* for the formulation, understanding, and conceptual solution of long standing problems in automatic control. Developments in this area are fundamentally due to Prof. M. Fliess (1986,1988, 1989a, 1989b,1989c,1990a, 1990b). Some other pioneering contributions were also independently presented by Pommaret (1983, 1986). Sliding mode control, and discontinuous feedback control, in general, have also benefited from this new trend. A seminal contribution in the use of differential algebraic results to sliding mode control was given by Fliess and Messager (1990). These results were extended and used in several case studies by Sira-Ramirez *et al* (1992), Sira-Ramirez and Lischinsky- Arenas (1991) and by Sira-Ramirez (1992a-1992d). A most interesting article extending some of the ideas to multivariable linear systems and to the regulation of non-minimum phase linear systems is that of Fliess and Messager (1992). Extensions to pulse-width-modulation control strategies from this viewpoint were also contributed by Sira-Ramirez (1991b, 1992e).

This article is an attempt to present, in a tutorial fashion, some of the developments in sliding mode control theory from the differential algebraic viewpoint. It should be pointed out that some of the results obtained for sliding mode control via the use of differential algebra are closely related to previous ideas presented by Emelyanov (1987, 1990b), from a quite different viewpoint, in his "binary systems" formulation of control problems. Also, in a contribution by Bartolini and Pydynowsky (1991) smoothing of the input signals is achieved through continuous first order estimators. Again, in their work, the basic developments are not drawn from differential algebra.

Section 2 of this article is devoted to present some simple examples which utilize sliding surfaces which not only depend on the state of the system but also on the system's inputs. These examples motivate the need for the more general class of sliding surfaces which directly lead to dynamical sliding mode control. Section 3 presents some fundamental results from differential algebra and their formal implications in sliding mode controller synthesis.

2. SOME MOTIVATING EXAMPLES

In this section we provide simple, yet motivating examples which not only justify the differential algebraic apporach in sliding mode controller design, but they also point to the need and advantages associated to more general classes of sliding surfaces, which include expressions in the input signal and (possibly) some of its time derivatives.

This work was supported by the Consejo de Desarrollo Científico, Humanístico y Tecnológico of the Universidad de Los Andes, under Research Grant I-358-91.

2.1 An example of smoothing the banging of the input signal in discontinuous feedback control

Let us begin by a simple example in which the smoothing properties of dynamical sliding regimes, arising from input-dependent switching surfaces, are clearly portrayed.

Consider the scalar system:

$$\dot{x} = u \qquad y = x - X \tag{2.1}$$

where y represents the state error with respect to a preassigned constant reference value X. The variable u is the input signal, constrained to take values in the discrete set $\{-U, U\}$, where $U > 0$.

It is well known that the following discontinuous feedback policy, given by :

$$u = -U \text{ sign } (y) \tag{2.2}$$

results in a sliding regime on the line $y = 0$. This is easily seen from the fact that the product $\sigma d\sigma/dt := y \, dy/dt = -U| y | \leq 0$. The required sliding surface is then represented as:

$$S = \{ x : \sigma = x - X = 0 \} \tag{2.3}$$

The ideal sliding dynamics is obtained from the condition $d\sigma/dt = 0$, i.e., $dx/dt = 0$, and $\sigma = 0$, i.e $x = X$.

A simulation of the controlled system is shown in figure 1, with $X = 1$ and $U = 1$. The controlled state response, the sliding surface coordinate response and the discontinuous (bang-bang) features of the resulting input signal u are separately portrayed in such a figure.

The effects of the above discontinuous feedback policy are summarized in two important features: 1) The condition $\sigma = y = 0$ is reached in finite time (given by $T = U^{-1} | x(0)|$, 2) After reaching of the desired condition, the same is indefinitely guaranteed to hold. It may be easily proved that this condition can be sustained, in spite of the presence of bounded perturbations affecting the system behavior through the input channel u.

Suppose we would like to trade the finite time reachability of the zero state error condition by a smoother behavior of the input variable u while still, possibly, being constrained to utilize auxiliary input signals (here denoted by υ) taking values in the set $\{-U, U\}$. In order to achieve this purpose, let us propose the following asymptotically stable closed loop behavior of the controlled scalar state:

$$\dot{x} = u = -\lambda(x - X) \tag{2.4}$$

If we take now as the sliding surface one representing a suitable input-dependent switching condition depicting the feedback input signal error :

$$S = \{ (x,u) : \sigma = u + \lambda(x - X) = 0 \} \tag{2.5}$$

one ideally obtains the required closed loop behavior whenever $\sigma = 0$. A sliding regime guaranteeing such a condition can be estabilished by requiring now that $\sigma d\sigma/dt \leq 0$. This may be accomplished by imposing on σ the discontinuous dynamics specified by $d\sigma/dt = -W\text{sign}(\sigma)$, where $W > 0$ is an arbitrary positive real number. Using the new expression for σ, one obtains

$$\dot{u} + \lambda u = -W \text{ sign } [u + \lambda(x - X)] \tag{2.6}$$

which is a differential equation with discontinuous right hand side, whose solution represents the required control input variable. It is easy to see from the above equation (2.6) that the control input signal u is actually the outcome of a first order low pass filter with cut-off frequency represented by λ. Ideed, using the by now popular hybrid notation that merges frequency domain quantities with others in the time domain, one easily obtains:

$$u = -\frac{\lambda}{s+\lambda} \left[\left(\frac{W}{\lambda}\right) \text{sign } [u + \lambda(x - X)] \right] = -\frac{\lambda}{s+\lambda} \upsilon \tag{2.7}$$

Thus, the input u may be synthesized as the output of a low pass filter which accepts as an input a discontinuous (bang-bang) signal υ, of amplitude W/λ. By virtue of the amplitude restriction on the ultimate (auxiliary) input signal υ, mentioned above, this ratio is taken as :

$$\frac{W}{\lambda} = U \iff W = \lambda U \tag{2.8}$$

The diagram in figure 2 depicts the structure of the dynamical discontinuous feedback controller explicitly exhibiting the imbedded low pass filter characteristics which are excited by a bang-bang input signal of amplitude U, as initially required. For a given fixed value of U, relation (2.8) estabilishes a trade-off between the exponential rate of approach of the controlled state x to its desired value (alternatively, the cut-off frequency of the low pass filter , or filter bandwitdth) and the design value of the amplitude W, which indirectly measures the reaching time of the condition $\sigma = 0$, through $T = W^{-1}|\sigma(0)|$. The faster it is desirable to reach $\sigma = 0$, the faster x will approach X, but then, the larger the cut-off frequency of the low pass filter and a larger number of harmonic components of the bang-bang signal υ, and external noise, directly affect the input to the system.

A simulation of the dynamically discontinuosly controlled system (2.1),(2.6) is shown in figure 3 with $X = 1$, $W = 1$ and $\lambda = 1$. The resulting input signal u is shown to be substantially smoothed out with respect to its previous behavior when the static discontinuous contoller was used. Further smoothing of the controlled scalar state x can be equally inferred from such a figure.

2.2 Zeroing of input-dependent output signals via dynamical discontinuous control

A rather general model for nonlinear single-input single-output nonlinear systems is constituted by the following n-dimensional analytic system, in Kalman form :

$$\dot{x} = f(x,u) \qquad y = h(x,u) \tag{2.9}$$

in which the output y is allowed to explicitly depend on the input variable u. (such systems may be properly addressed as systems with *relative degree zero*). Suppose it is desired to zero out the output variable y, possibly in finite time, through a discontinuous feedback control policy. This control task is possible by imposing, again, the following autonomous dynamics on the scalar output signal:

$$\dot{y} = -W \text{ sign}(y) \tag{2.10}$$

and computing the required control signal u. Using (2.9) and (2.10) one obtains:

$$\left[\frac{\partial h}{\partial x}\right] f(x,u) + \left[\frac{\partial h}{\partial u}\right] \frac{du}{dt} = -W \text{ sign}[h(x,u)] \tag{2.11}$$

which may be locally rewritten as a first order, time-varying, ordinary differential equation with discontinuous right hand side :

$$\frac{du}{dt} = -\left[\frac{\partial h}{\partial u}\right]^{-1} \left\{ \left[\frac{\partial h}{\partial x}\right] f(x,u) + W \text{ sign}[h(x,u)] \right\} \tag{2.12}$$

A block diagram depicting the dynamical discontinuous feedback control scheme summarized in (2.12) is shown in Figure 4.

The ideal sliding mode behavior obtained on the input-dependent manifold $y = h(x,u) = 0$ is obtained as follows: Let the feedback law $u = \varphi(x)$ be the (unique) control law satisfying $h(x, \varphi(x)) = 0$. Then $\varphi(x)$ also plays the role of the *equivalent control* and it is, evidently, a particular solution of (2.12), for suitable initial conditions. Indeed, the solutions of (2.12) locally yield $dy/dt = 0$, i.e., they yield constant output responses under ideal sliding mode conditions. If the initial value of the output is zero, then the dynamical controller locally induces the condition $y = 0$ on an open interval of time. This means, by virtue of the assumed uniqueness, that the actual (dynamically generated) applied control input u is taking precisely the same values as $\varphi(x)$.

The ideally controlled dynamics is then obtained as :

$$\dot{x} = f(x, \varphi(x)) \qquad y = 0 \tag{2.13}$$

which, for obvious reasons, is assumed to be locally asymptotically stable to a desired equilibrium state.

Remark An interesting feature of the above class of problems lies in the possibilities of robustly imposing ideally designed feedback control solutions to nonlinear plants. For instance, let $u = -k(x)$ be a desirable scalar feedback control law for the plant $dx/dt = f(x,u)$. Then, adopting as an output function the expression: $y = h(x,u) = u + k(x)$, the dynamical controller obtained from (2.12) imposes, in finite time, the required feedback control law on the given system.

■

2.3 A simple application example in rest-to-rest reorientation maneuvers for single axis spacecraft

Consider the nonlinear second order plant representing the kinematic and dynamic equations of single-axis jet-controlled spacecraft with the attitude variable measured with respect to a skewed axis and specified in terms of the Cayley-Rodrigues parametrization (see Dwyer and Sira-Ramirez, 1988):

$$\begin{aligned} \dot{x} &= 0.5(1+x^2)\omega \\ \dot{\omega} &= \frac{1}{J} u \\ y &= x - X \end{aligned} \tag{2.14}$$

where x represents the Cayley-Rodrigues orientation parameter, ω is the main axis angular velocity and u is the externally applied input torque. J is the moment of inertia of the spacecraft around its principal axes.

It is easy to show that the following nonlinear feedback control law, arising from extended linearization considerations, asymptotically stabilizes the system toward the desired reference attitude value $x = X$, with zero final angular velocity ω (see also: Sira-Ramirez and Lischinsky-Arenas, 1990):

$$u = -2J \left\{ \zeta\omega_n\omega + \omega_n^2 \left[\tan^{-1}(x) - \tan^{-1}(X)\right] \right\} \tag{2.15}$$

where ω_n and ζ are positive design parameters with: $0 < \zeta < 1$.

Figure 5 depicts the simulated responses of the state variables x and ω as well as the required control input signal u, as computed from (2.15) with $\zeta = 0.707$, $\omega_n = 2$ [rad/s], $X = 1.5$ [rad].

One may alternatively take, as remarked above, an auxiliary output function y, for system (2.14), which is constituted by the control input error with respect to the required stablizing feedback function. i.e.,

$$y = u + 2J \left\{ \zeta\omega_n\omega + \omega_n^2 \left[\tan^{-1}(x) - \tan^{-1}(X)\right] \right\}$$

The discontinuous dynamical feedback controller induces a sliding regime on the input- dependent sliding surface S with coordinate function σ given, evidently, by:

$$S = \{ (x, \omega, u) : \sigma = u + 2J \left[\zeta\omega_n\omega + \omega_n^2 \left(\tan^{-1}(x) - \tan^{-1}(X)\right) \right] = 0 \}$$

A dynamical sliding mode controller, which robustly enforces the feedback control law (2.15) by zeroing the above input-dependent (auxiliray) output function y, is given, according to the previously stated results, by:

$$\dot{u} = -(2\zeta\omega_n u + J\omega_n^2\, \omega) - W\, \mathrm{sign}(y) \tag{2.17}$$

Simulations were carried out for the dynamical sliding mode controlled system (2.14),(2.17) with sliding surface given by (2.16). The spacecraft moment of inertia was taken as $J = 70$ N-m/s^2. The desired attitude $X = 1.5$ rad, and the controller design parameters were taken as : $\zeta = 0.707$, $\omega_n = 2$, and $W = 40$.

Figure 6 depicts the dynamically sliding mode controlled state variables responses for x and ω, the sliding surface coordinate σ and the smoothed externally applied input torque u, expressed in N-mt.

In spite of the slower response of the dynamical sliding mode controlled system, the applied input torque is considerably smaller than the one obtained with the continuous feedback control strategy represented by (2.15). This fact has a definite bearing on the stability and performance features of the closed loop system when amplitude control input torque restrictions are enforced. If, for instance, one limits the amplitude of the applied input torque to a reasonable value of, say, 2.5 N-m, the (saturated) continuous feedback controller (2.15) leads to a stable, but quite degraded. response for the attitude parameter x, with exceedingly large overshoot. The dynamical sliding mode controller, on the other hand, still yields a perfectly asymptotically stable response with reasonably small overshoot. This is depicted in figure 7.

3. A DIFFERENTIAL ALGEBRAIC APPROACH TO SLIDING MODE CONTROL OF NONLINEAR SYSTEMS

In the previous section, some of the advantages of using input-dependent sliding surfaces were explored through quite simple illustrative examples. These examples point, essentially, to new possibilities of sliding mode control when input-dependent switching surfaces are used. Such possibilities could have also been arrived at, by using the concept of the *extended system* (Nijmeijer and Van der Schaft, 1990) in combination with traditional static sliding mode controller design. However, input-dependent sliding surfaces may be seen as natural switching surfaces for nonlinear systems. This fact is a direct consequence of the *differential algebraic* approach, proposed by M. Fliess (1986,1988, 1989a, 1989b,1989c,1990a, 1990b), for the study of control systems. In this section we present some simple results of such differential algebraic approach, as related to sliding mode control. The required background may be found in Fliess's numerous articles and outstanding contributions (see references). However, we will try to be as self-contained as possible. The following developments closely follow those found in Fliess (1990a).

3.1 Fliess's Generalized Controller Canonical Forms

Definition 3.1 An *ordinary differential field* K is a commutative field in which a single operation, denoted by "d/dt" or "·" and called *derivation*, is defined, which satisfies the usual rules: $d(ab+c)/dt = (da/dt)b + a(db/dt) + dc/dt$ for any a,b and c in K. If all elements c in K satisfy $dc/dt = 0$, then K is said to be a *field of constants*.

Examples The field R of real numbers, with the operation of time differentiation d/dt, trivially constitutes a differential field, which is a field of constants. The field of rational functions in t with coefficients in R, denoted by R(t), is a differential field with respect to time derivation. R(x) is also a differential field for any differentiable indeterminate x.

Definition 3.2 Given a differential field L which contains K, we say L is a *differential field extension* of K, and denote it by L/K, if the derivation in K is a restriction of that defined in L.

Examples R(t)/R is a differential field extension over the set of real numbers. The differential field R(t)/Q(t) is also a differential field extension over the field Q(t) of all rational functions in t with coefficients in the set of rational numbers Q. Similarly, the field C(t) of rational functions in t with complex coefficients, is a differential field extension of, both, R(t) and Q(t). Evidently, C(t)/Q, and C(t)/C are also a differential field extensions.

In the following developments u is considered as a differential scalar indeterminate and k stands for an ordinary *differential field*, with derivation denoted by d/dt.

Definition 3.3 By k<u>, we denote the *differential field generated by* u *over the ground field* k. i.e., the smallest differential field containing both k and u. This field is clearly the intersection of all differential fields which contain the union of k and u.

Example Consider the field of all possible rational expressions in u, and its time derivatives, with coefficients in R. This differential field is R<u>. A typical element in R<u> may be:

$$\frac{u^{(3)}}{u^2} + \frac{3u^2\dot{u} + \pi(\ddot{u})u^4 - 1.02(\dot{u})^3}{\sqrt{7}u^{(5)} + u} - 5^{0.2}u$$

Let $x_1,\ldots,x_n$ be differential indeterminates. Consider the differential field k<u>. One may then extend k<u> to a differential field K containing all possible rational expressions in the variables $x_1,...,x_n$, and their time derivatives, with coefficients in k<u>. For instance, a typical element in K/R<u> now looks like:

$$\frac{u^2[x_1^{(3)}]^2 - \frac{1}{\log(\pi)}\left(\frac{2\ddot{u} + u^3u^{(7)}}{1+\pi u}\right)(\dot{x}_5)^2 x_6 + x_2}{\sqrt[5]{5}\; x_3x_4(\ddot{x}_1)^3 + u^{(3)} - e^2\,\dot{u}x_2}$$

such a differential field is addressed as a *finitely generated field extension* over k<u>. In general, K does not coincide with k<u,x> and it is somewhat larger.

Definitions 3.4 Any element of a differential field extension, say L/K, only has two possible characterizations. Either it satisfies an algebraic differential equation with coefficients in K, or it doesn't. In the first case, the element is said to be *differentially algebraic* over K, otherwise it is said to be *differentially transcendental* over K. If the property of being differentially algebraic is shared by all elements in L, then L is said to be a *differentially algebraic extension of* K. If, on the contrary, there is, at least one element in L which is differentially transcendent over K, then L is said to be a *differentially transcendent extension of* K.

Example in the previous example, the differential extension k<x,u>/k<u> is algebraic.

Example Consider k<u>, if x is an element which satisfies:

$$\dot{x} - ax - u = 0 \text{ for some } a \in k$$

then x is differentially algebraic over k<u>. However, since no further qualifications have been given, u is differentially transcendent over k.

Definition 3.5 A *differential transcendence basis* of L/K is the largest set of elements in L which do not satisfy any algebraic differential equation with coefficients in K, i.e., they are not differentially K-algebraically dependent. A *non-differential transcendence basis* of L/K is constituted by the largest set of elements in L which do not satisfy any algebraic differential equation with coefficients in K. The number of elements constituting a differential transcendence basis is called the *differential transcendence degree*. The (non-differential)*transcendence degree* refers to the cardinality of a non-differential transcendence basis.

Example in the previous example, the differential field extension k<x,u>/k<u> is algebraic over k<u>, but, on the other hand, k<u>/k is differentially transcendent over k, with u being the differential transcendence basis. Notice that x is non-differentially transcendent over k<u> as it does not satisfy any algebraic equation, but a differential one. Hence, x is a nondifferential transcendence basis of k<x,u>/k<u>. Evidently the differential transcendence degree of k<x,u>/k<u> is zero. The non-differential transcendence degree is just one.

Theorem 3.6 A finitely generated differential extension L/K is differentially algebraic if, and only if its (non-differential) transcendence degree is finite.

Definition 3.7 A *dynamics* is defined as a *finitely generated differentially algebraic extension* K/k‹u› of the differential field k‹u›.

The *input* u is regarded as an *independent* indeterminate. This means that u is a *differentially transcendent* element of K/k, i.e., u does not satisfy any algebraic differential equation with coefficients in k. It is easy to see, that if u is a differential transcendent element of k<u> then it is also a differential transcendence element of K/k<u>.

The following result is quite basic:

Proposition 3.8 Suppose $x = (x_1,x_2,...,x_n)$ is a *nondifferential transcendence basis* of K/k‹u›, then, the derivatives dx_i/dt $(i=1,...,n)$ are k‹u›-algebraically dependent on the components of x.

Proof: immediate.

One of the consequence of all these results, drawn by Fliess (1990a), is that a more general and natural representation of nonlinear systems requires implicit algebraic differential equations. Indeed, from the preceeding proposition, it follows that there exist exactly n polynomial differential equations with coefficients in k, of the form:

$$P_i(\dot{x}_i, x, u, \dot{u}, \cdots, u^{(\alpha)}) = 0\,;\; i=1,\cdots,n \qquad (3.1)$$

implicitly describing the controlled dynamics.

It has been shown by Fliess and Hassler (1990) that such implicit representations are not entirely unusual in physical examples. The more traditional form of the state equations, known as *normal form* is recovered, in a local fashion, under the assumption that such polynomials locally satisfy the following rank condition :

$$\text{rank}\begin{bmatrix} \frac{\partial P_1}{\partial \dot{x}_1} & 0 & 0 \\ \vdots & \ddots & \vdots \\ 0 & 0 & \frac{\partial P_n}{\partial \dot{x}_n} \end{bmatrix} = n$$

The time derivatives of the x_i's may then be, locally, solved for as:

$$\dot{x}_i = p_i(x, u, \dot{u}, \cdots, u^{(\alpha)}) = 0\,;\; i = 1,\cdots,n \qquad (3.2)$$

It should be pointed out that even if (3.1) is in polynomial form, it may happen, in general, that (3.2) is not. The representation (3.2) is now known as the *generalized state representation* of a nonlinear dynamics.

The following theorem constitutes a direct application of the *theorem of the differential primitive element* which may be found in Kolchin (1973). This theorem plays a fundamental role in the study of systems dynamics from the differential algebraic approach (Fliess, 1990a).

Theorem 3.9 Let K/k<u> be a dynamics. Then, there exists an element $\xi \in K$ such that K = k<u,ξ> i.e., such that K is the smallest field generated by the indeterminates u and ξ .

Proof see Fliess (1990a).

The (nondifferential) transcendence degree n of K/k<u> is the smallest integer n such that $\xi^{(n)}$ is k‹u›-algebraically dependent on $\xi, d\xi/dt, ..., d^{(n-1)}\xi/dt^{(n-1)}$. We let $q_1 = \xi$, $q_2 = d\xi/dt$,...., $q_n = d^{(n-1)}\xi/dt^{(n-1)}$. It follows that $q = (q_1,...,q_n)$ also qualifies as a (non-differential) transcendence basis of K/k‹u›. One, hence, obtains a nonlinear generalization of the controller canonical form, known as the Global Generalized Controller Canonical Form (GGCCF) :

$$\begin{aligned} \frac{d}{dt}q_1 &= q_2 \\ \frac{d}{dt}q_2 &= q_3 \\ &\vdots \\ \frac{d}{dt}q_{n-1} &= q_n \\ C(\dot{q}_n, q, u, \dot{u}, \ldots, u^{(\alpha)}) &= 0 \end{aligned} \qquad (3.3)$$

where C is a polynomial with coefficients in k. If one can locally solve for the time derivative of q_n in the last equation, one locally obtains and explicit system of first order differential equations, known as the *Local Generalized Controller Canonical Form* (LGCCF):

$$\begin{aligned} \frac{d}{dt}q_1 &= q_2 \\ \frac{d}{dt}q_2 &= q_3 \\ &\vdots \\ \frac{d}{dt}q_{n-1} &= q_n \\ \frac{d}{dt}q_n &= c(q, u, \dot{u}, \ddot{u}, \ldots, u^{(\alpha)}) \end{aligned} \qquad (3.4)$$

Remark We assume throughout that $\alpha \geq 1$. The case $\alpha = 0$ corresponds to that of exactly linearizable systems under state coordinate transformations and static state feedback. One may still obtain the same smoothing effect of the dynamical sliding mode controllers we derive in this article by considering arbitrary *prolongations* of the input space. This is accomplished by succesively considering the "extended system" (see Nijmeijer and Van der Schaft, 1990) of the original one, and proceeding to use the same differential primitive element yielding the Generalized Controller Canonical Form of the original system. ■

3.2 Dynamical sliding regimes based on Fliess's GCCF.

The preceeding general results on canonical forms for nonlinear systems have an immediate consequence in the definition of sliding surfaces for stabilization and tracking problems in nonlinear systems.

Consider the following *sliding surface coordinate function*, expressed in the generalized phase coordinates q :

$$\sigma = c_1 q_1 + \ldots + c_{n-1} q_{n-1} + q_n \tag{3.5}$$

where the scalar coefficients c_i (i=1,...,n-1) are chosen in such a manner that the following polynomial, p(s), in the complex variable s, is Hurwitz:

$$p(s) = c_1 + c_2 s + \cdots + c_{n-1} s^{n-2} + s^{n-1} \tag{3.6}$$

Imposing on the sliding surface coordinate function σ the discontinuous dynamics:

$$\dot{\sigma} = -W \operatorname{sign}(\sigma) \tag{3.7}$$

then, the trajectories of σ are seen to exhibit, in finite time T given by $T = W^{-1} \mid \sigma(0) \mid$, a sliding regime on $\sigma = 0$. Substituting on (3.7) the expression (3.5) for σ, and using (3.4), one obtains, after some straightforward algebraic manipulations, the following dynamical implicit sliding mode controller :

$$\begin{aligned} & c(q,u,\dot{u},\cdots,u^{(\alpha)}) = \\ & c_1 c_{n-1} q_1 + (c_2 c_{n-1} - c_1) q_2 + \cdots + (c_{n-2} c_{n-1} - c_{n-3}) q_{n-2} + (c_{n-1} c_{n-1} - c_{n-2}) q_{n-1} \\ & -W \operatorname{sign}[c_1 q_1 + \ldots c_{n-1} q_{n-1} + q_n] \end{aligned} \tag{3.8}$$

Evidently, under ideal sliding conditions $\sigma = 0$, the variable q_n no longer qualifies as a state variable for the system since it is expressible as a linear combination of the remanining states and, hence, q_n is no longer a non-differentially transcendental element of the field extension K. The ideal (autonomous) closed loop dynamics may then be expressed in terms of a reduced non-differential transcendence basis K/k which only includes the remaining n-1 phase coordinates associated to the original differential primitive element. This leads to the following *ideal sliding dynamics*:

$$\begin{aligned} \frac{d}{dt} q_1 &= q_2 \\ \frac{d}{dt} q_2 &= q_3 \\ &\vdots \\ \frac{d}{dt} q_{n-1} &= -c_1 q_1 - \cdots - c_{n-1} q_{n-1} \end{aligned} \tag{3.9}$$

The characteristic polynomial of (3.9) is evidently given by (3.6) and, hence, the (reduced) autonomous closed loop dynamics is asymptotically stable to zero. Notice that by virtue of (3.5), the condition $\sigma = 0$, and the asymptotic stability of (3.9), that q_n also tends in an asymptotically stable fashion to zero.

The *equivalent control*, denoted by u_{EQ} is a virtual feedback control action achieving ideally smooth evolution of the system on the constraining sliding surface $\sigma = 0$, provided initial conditions are precisely set on such a switching surface. The equivalent control is formally obtained from the condition $d\sigma/dt = 0$. i.e.:

$$\begin{aligned} & c(q,u_{EQ},\dot{u}_{EQ},\cdots,u_{EQ}^{(\alpha)}) = \\ & c_1 c_{n-1} q_1 + (c_2 c_{n-1} - c_1) q_2 + \cdots + (c_{n-2} c_{n-1} - c_{n-3}) q_{n-2} + (c_{n-1} c_{n-1} - c_{n-2}) q_{n-1} \end{aligned} \tag{3.10}$$

Since q asymptotically converges to zero, the solutions of the above time-varying implicit differential equation, describing the evolution of the equivalent control, asymptotically approach the solutions of the following autonomous implicit differential equation :

$$c(0,u,\dot{u},\cdots,u^{(\alpha)}) = 0 \tag{3.11}$$

Equation (3.11) constitutes the *zero dynamics* (See Fliess, 1990b) associated to the problem of zeroing the differential primitive element, considered now as an (auxiliary) output of the system. Notice that (3.10) may also be regarded as the *zero dynamics* associated with zeroing of the sliding surface coordinate function σ. If (3.11) locally asymptotically approaches a constant equilibrium point u = U, then the system is said to be locally *minimum phase* around such an equilibrium point, otherwise the system is said to be *non-minimum phase*. The equivalent control is, thus, locally asymptotically stable to U, whenever the underlying input-output system is minimum phase.

One may be tempted to postulate, for the sake of physical realizability of the sliding mode controller, that a sliding surface σ is properly defined whenever the associated zero dynamics is constituted by an asymptotically stable motion towards equilibrium. In other words, that the input-sliding surface system is minimum phase. It should be pointed out, however, that non-minimum phase systems might make perfect physical sense and that, in some instances, instability of a certain state variable, or input, does not necessarily means desastrous effects on the controlled system. The following example illustrates this fact.

Example 3.10 (control of a non-minimum phase system).

Consider the problem of maneuvering a motor-driven unicycle which advances with constant (ground) speed V on a plane equipped with cartesian coordinates, given by the ordered pairs (x,y), describing the position of the contact point. The control input is represented by the heading angle u, measured with respect to the x axis. The objective is to maneuver the unicycle to follow a circle of radius R, drawn on the plane, and centered at the origin O of coordinates (see figure 8). For simplicity, we assume that u takes values in the interval $(-\infty, \pi/2)$ and, hence, only counter- clockwise solutions are considered.

It is easy to see that the the motions may be described by the following set of *analytic* differential equations :

$$\begin{aligned} \dot{x} &= V\cos(u) \\ \dot{y} &= V\sin(u) \end{aligned} \tag{3.12}$$

or, in polar coordinates ρ,φ by :

$$\begin{aligned} \dot{\rho} &= V\cos(u-\varphi) \\ \dot{\varphi} &= \frac{V}{\rho}\sin(u-\varphi) \end{aligned} \tag{3.13}$$

In spite of the analyticity of the expressions in the differential equations, the system may be reduced, by straightforward elimination, to an algebraic implicit differential equation:

$$\left\{ \ddot{\rho} - \frac{1}{\rho}\left[V^2 - (\dot{\rho})^2\right] \right\}^2 - (\dot{u})^2 \left[V^2 - (\dot{\rho})^2\right] = 0 \tag{3.14}$$

The condition:

$$(\dot{\rho})^2 \neq V^2$$

must be enforced, so that the radial position coordinate does not become uncontrollable. The uncontrollable motions correspond with uniformly sustained purely radial motions from (or towards) the origin of coordinates. Moreover, notice that, unless u is allowed to become constant (i.e., unless du/dt = 0), the implicit differential equation (3.14) does not have any real solutions if the following strict inequality :

$$(\dot{\rho})^2 < V^2 \tag{3.15}$$

is violated.

Remark The phenomenon of obtaining implicit differential equations and inequalities as the input-output description of a system, arising from a state elimination procedure, has been demonstrated to hold in full generality by Diop (1989). ■

We consider the following position error: $\zeta = \rho - R$, with respect to the circle line.

The control task consists in stabilizing the value of ζ to zero and, thus, obtain a perfectly circular motion of radius R for the unicycle. Notice that under perfect tracking of the circle, $d\rho/dt = 0$ and the inequality (3.15) is always satisfied.

It is easy to see that $q_1 = \zeta = \rho - R$ qualifies as a differential primitive element. The GCCF for the system is, evidently, given by :

$$\dot q_1 = q_2 \qquad (3.16)$$
$$\left\{ \dot q_2 - \frac{1}{q_1+R}\left[V^2 - (q_2)^2\right] \right\}^2 - (\dot u)^2\left[V^2 - (q_2)^2\right] = 0$$

The sliding surface candidate σ is constituted, in this case, by an appropriate stabilizing linear combiatinon of the generalized state components :

$$\sigma = q_2 + c_1 q_1 \quad ; c_1 > 0 \qquad (3.17)$$

Notice that in original coordinates, σ is an input dependent switching surface.

Under ideal sliding conditions $\sigma = 0$, the unicycle asymptotically approaches the circle of radius R. The dynamical sliding mode controller is obtained by imposing the discontinuous dynamics (3.7) on σ. Such dynamical discontinuous controller is, implicitely, given, in terms of the transformed coordinates q_1, q_2, by:

$$\left\{ c_1 q_2 + W\,\mathrm{sign}(q_2 + c_1 q_1) - \frac{1}{q_1+R}\left[V^2 - (q_2)^2\right] \right\}^2 - (\dot u)^2\left[V^2 - (q_2)^2\right] = 0 \qquad (3.18)$$

The zero dynamics associated to the stabilized (closed loop) system is immediately obtained, according to (3.11), from (3.16) by letting q_1, q_2 and dq_2/dt be zero:

$$|\dot u|^2 = \frac{V^2}{R^2} \qquad (3.19)$$

The imposed restrictions on the heading angle u dictate that the physically meaningful solution to the zero dynamics implicit equation is given by: $du/dt = -V/R$, which is, evidently, unstable.

Remark The physical meaning of such an unstable zero dynamics is quite clear: in order to mantain the motion of the unicycle on the prescribed circle, one must turn the heading of the unicycle at a fixed rate, which precisely coincides with the constant angular velocity $-V/R$ of the contact point moving, counter-clockwise along the circle, with fixed tangential velocity V. The fact that the heading angle is constantly decreasing, without bound, can hardly be considered to represent a physically harmful behavior for the system or for the associated control task. ∎

An explicit representation of the system, which is necessarily local, may be obtained by solving with respect to dq_2/dt from the second equation in (3.16):

$$\dot q_1 = q_2 \qquad (3.20)$$
$$\dot q_2 = \frac{1}{q_1+R}\left[V^2 - (q_2)^2\right] \pm (\dot u)\sqrt{V^2 - (q_2)^2}$$

It is easy to see, from equilibrium considerations, that the two possible solutions for dq_2/dt represent the possibility of clockwise and counter-clockwise motions along the circle, in inverse correspondence with the sign adopted for the (unstable) zero dynamics above. We take the positive sign as the solution for dq_2/dt in (3.20), since we have explicitly assumed that only counter-clockwise motions are allowed.

The explicit dynamical sliding mode controller is then readily obtained as:

$$\dot u = -\frac{1}{\sqrt{V^2 - (q_2)^2}}\left[c_1 q_2 + \frac{1}{q_1+R}\left[V^2 - (q_2)^2\right] + W\,\mathrm{sign}(q_2 + c_1 q_1) \right] \qquad (3.21)$$

or, by carefully taking into account the right angular relation, in original polar coordinates, as :

$$\dot u = \frac{\left\{ c_1\cos(u-\varphi) + \frac{V}{\rho}\sin^2(u-\varphi) + \frac{W}{V}\,\mathrm{sign}\left[V\cos(u-\varphi) + c_1(\rho - R)\right] \right\}}{\sin(u-\varphi)}$$

Simulations of the dynamically sliding mode controlled unicycle were performed with the following parameters: $V = 5m/s$, $R = 5m$, $W = 10$, $c_1 = 2\ s^{-1}$. The results are shown in figure 9.

The smooth trajectory on the plane is portrayed showing asymptotic approach to the target circle. The sliding surface coordinate evolution is also shown in that figure and it is easily seen to comply with the impossed discontinuous dynamics. The angular position of the contact point of the unicycle on the plane exhibits an (unstable) ever decreasing behavior as pointed out above. The heading angle response, acting as an external control input, is also shown to grow without bound, asymptotically to a linear growth, as demanded by the nature of the equivalent control dynamics and its limiting behavior represented by the zero dynamics.

3.3 Some Formalizations of Sliding Mode Control through the Differential Algebraic Approach

Consider a (nonlinear) dynamics K/k<u>. Let, furthermore, $\zeta = (\zeta_1, \ldots, \zeta_n)$ be a *non-differential transcendence baisis* for K, i.e., the *transcendence degree* of K/k<u> is, then, assumed to be n.

Definition 3.11 A *sliding surface candidate* is any element σ of K/k<u> such that its time derivative $d\sigma/dt$ is k<u>-algebraically dependent on ζ. That is, there exists a polynomial S over k such that :

$$S(\dot\sigma, \zeta, u, \dot u, \ldots, u^{(\alpha)}) = 0 \qquad (3.23)$$

A more traditional definition of sliding surface coordinate function is related to the fact that no input signals, nor any of its time derivatives, were customarily allowed to be part of the expression defining a sliding surface candidate. In this unnecessarily restricted sense, the sliding surface candidates were only allowed to be an (algebraic) function of the state components. One recovers this definition, and its inherent limitations, using differential algebra.

Proposition 3.12 The element σ in K/k<u> is a sliding surface candidate if it is k-algebraically dependent on all the elements of a transcendence basis ζ.

Proof the time derivative of σ is k-algebraically dependent on the derivatives of every element in the transcendence basis ζ. Therefore, $d\sigma/dt$ is k<u>-algebraically dependent on ζ.

The condition in the proposition is clearly not necessary as σ may well be k-algebraically dependent only on some elements of the transcendence basis ζ, and still have $d\sigma/dt$ being k<u>-algebraically dependent on ζ.

Remark in the traditional definition of sliding surface candidate for systems in "Kalman form" with state ξ, the time derivative of the sliding surface was only required to be algebraically dependent on ξ. Hence, all the resulting sliding mode controllers were necessarily static. The differential algebraic approach naturally points to the possibilities of dynamical sliding mode controllers, specially in nonlinear systems where elimination of input derivatives may not be possible at all (see Fliess *et al* 1991, for a physical example of this nature). ∎

Imposing on σ a discontinuous sliding dynamics of the form:

$$\dot\sigma = -W\,\mathrm{sign}(\sigma) \qquad (3.24)$$

one obtains, from (3.23), an *implicit dynamical sliding mode controller* given by :

$$S(-W\,\mathrm{sign}(\sigma), \xi, u, \dot u, \ldots, u^{(\alpha)}) = 0 \qquad (3.25)$$

which is to be viewed as an implicit, time-varying, discontinous ordinary differential equation for the control input u.

The two "structures" associated to the underlying variable structure control system are represented by the pair of implicit dynamical controllers:

$$
\begin{aligned}
&S(-W,\xi,u,\dot u,\dots,u^{(\alpha)})=0\ ;\ \text{for}\quad \sigma>0\\
&S(W,\xi,u,\dot u,\dots,u^{(\alpha)})=0\ ;\ \text{for}\quad \sigma<0
\end{aligned}
\tag{3.26}
$$

each one valid, respectively, on one of the "regions" : $\sigma>0$ and $\sigma<0$. Precisely on the condition $\sigma=0$ neither one of the control structures is valid. One then, ideally, characterizes the motions by formally assuming $\sigma=0$ and $d\sigma/dt=0$ in (3.23).

We formally define the *equivalent control dynamics* as the dynamical state feedback control law obtained by letting $d\sigma/dt$ become zero in (3.23), and considering the resulting implicit differential equation for u :

$$
S(0,\xi,u_{EQ},\dot u_{EQ},\dots,u_{EQ}^{(\alpha)})=0 \tag{3.27}
$$

Equation (3.23) is implicit with respect to σ. Whenever $\partial S/\partial(d\sigma/dt)\neq 0$, then one locally obtains:

$$
\dot\sigma = s(\xi,u,\dot u,\dots,u^{(\alpha)}) \tag{3.28}
$$

and the corresponding dynamic sliding mode controller, complying with (3.24), is given by:

$$
s(\xi,u,\dot u,\dots,u^{(\alpha)}) = -W\,\mathrm{sign}(\sigma) \tag{3.29}
$$

If, furthermore, $\partial s/\partial u^{(\alpha)}$ is non zero, one locally obtains a time-varying state space representation for the dynamical sliding mode controller (3.29), of the form:

$$
\begin{aligned}
&\dot u_1 = u_2\\
&\dot u_2 = u_3\\
&\ \vdots\\
&\dot u_\alpha = \theta(u_1,\cdots,u_\alpha,\xi,W\mathrm{sign}(\sigma))\\
&u = u_1
\end{aligned}
\tag{3.30}
$$

All discontinuities arising from the bang-bang control policy are seen to be confined to the highest derivative of the control input through the nonlinear function θ. The output u of the dynamical controller is clearly the outcome of α integrations performed on such discontinuous time derivative and, for this reason, u is sufficiently smoothed.

3.4 An alternative definition of the equivalent control dynamics

One may generate a differential algebraic extension of $k\langle u\rangle$ by adjoining σ to it and consider $k\langle u,\sigma\rangle$. The differential field extension $k\langle u,\sigma\rangle/k\langle u\rangle$ is an input-output system, or, more properly, an input-sliding surface system. The element σ is then a non-differential transcendence element of $k\langle u,\sigma\rangle$ over $k\langle u\rangle$ and it, thus, satisfies an algebraic differential equation with coefficients in $k\langle u\rangle$. This means that there exists a polynomial with coefficients in k such that:

$$
P(\sigma,\dot\sigma,\dots,\sigma^{(\rho)},u,u,\dots,u^{(\gamma)})=0 \tag{3.31}
$$

where we have implicitly assumed that ρ is the smallest integer such that $d^\rho\sigma/dt^\rho$ is dependent on σ, $d\sigma/dt$,..., u, du/dt,...

This, useful, characterization of sliding surface coordinate functions has not been clearly established in the sliding mode control literature. Obtaining a differential equation for the sliding surface coordinate σ, which is indpendent of the system state, has direct implications in the area of "higher order" sliding motions (see Chang, 1991, for a second order sliding motion example) and some recent developments in "binary control systems" (Emelyanov, 1990b). We explore these relations in section 3.5, below.

A state-independent, implicit, definition of the "equivalent control dynamics" can then be immediately obtained from (3.31) by setting σ, and its time derivatives, to zero and, hence, obtain:

$$
P(0,0,\dots,0,u,\dot u,\dots,u^{(\gamma)})=0 \tag{3.32}
$$

In the language of differential algebra we may relate the definition of the equivalent control dynamics with that of a *differential specialization*. This allows to formally define the above procedure.

Indeed, using closely related ideas, appearing in Fliess (1990b), associated to the *zero dynamics* of smoothly controlled systems, the *equivalent control dynamics*, corresponding to the sliding surface candidate σ, may be obtained as follows: Take the input-sliding surface system $k\langle u,\sigma\rangle/k\langle u\rangle$. Find the largest differential subfield J of $k\langle u,\sigma\rangle$ which contains $k\langle\sigma\rangle$ and such that the extension $J/k\langle\sigma\rangle$ is differentially algebraic. Give now the value of zero to σ and extend the corresponding *differential specialization* to J, and obtain a differential field Jeq. Then, the extension Jeq/k constitutes the *equivalent control dynamics*. This extension is, evidently, differentially algebraic.

The above procedure amounts to elimination of the transcendence basis ξ adopted as a state of the given system, and obtaining an (implicit) differential algebraic expression relating u, and its time derivatives, to σ and a finite number of its time derivatives (see Diop, 1989). On the resulting differential equation, we would then set σ to be (ideally) zero. The obtained autonomous differential equation for u is the *equivalent control dynamics*. This definition of the equivalent control dyamics has the advantage of being state-free and, therefore, independent of the particular representation of the system.

3.5 Higher order sliding regimes

In recent times some efforts have been devoted to smoothing of sliding regimes through the so called "higher order" sliding regimes. The ideas behind "binary control systems" as applied to variable structure control are also geard towards obtaining asymptotic convergence towards the sliding surface, in a manner that avoids control input chattering through integration. These two develpments are also closely related to the differential algebraic approach. In the following paragraphs we explain how the same ideas may be formally derived from differential algebra, in all generality.

Consider (3.31), with σ as an output. We may rewrite such an implicit dynamics as the following Global Generalized Observability Canonical Form (GGOCF) (see Fliess, 1988):

$$
\begin{aligned}
&\dot\sigma_1 = \sigma_2\\
&\dot\sigma_2 = \sigma_3\\
&\ \vdots\\
&P(\sigma_1,\cdots,\sigma_\rho,\dot\sigma_\rho,\ ,u,\dot u,\cdots,u^{(\gamma)})=0\\
&\sigma=\sigma_1
\end{aligned}
\tag{3.33}
$$

As before, an explicit LGOCF can be obtained for the element σ whenever $\partial P/\partial(d\sigma_\rho/dt)\neq 0$:

$$
\begin{aligned}
&\dot\sigma_1 = \sigma_2\\
&\dot\sigma_2 = \sigma_3\\
&\ \vdots\\
&\dot\sigma_\rho = p(\sigma_1,\cdots,\sigma_\rho,\ ,u,\dot u,\cdots,u^{(\gamma)})\\
&\sigma=\sigma_1
\end{aligned}
\tag{3.34}
$$

Definition 3.13 An element σ of the dynamics $K/k\langle u\rangle$ admits a *ρ-h order sliding* regime if the GOCF (3.34), associated to σ, is ρ-th order.

One defines a *ρ-th order sliding surface candidate* as any arbitrary (algebraic) function of σ and its time derivatives, up to ρ-1-st order. For obvious reasons, the most convenient type of function is represented by a stabilizing linear combination of σ and its time derivatives.

$$
s = m_1\sigma_1 + m_2\sigma_2 + \cdots + m_{\rho-1}\sigma_{\rho-1} + \sigma_\rho \tag{3.35}
$$

A first-order sliding motion is then imposed on such a linear combination of generalized phase variables by means of the discontinuous sliding mode dynamics:

$$
\dot s = -M\,\mathrm{sign}(s)\ \ ;\ \ M>0 \tag{3.36}
$$

This results in the implicit dynamical higher order sliding mode controller:

$$
\begin{aligned}
&p(\sigma_1, \ldots, \sigma_\rho, u, \dot{u}, \cdots, u^{(\gamma)}) = \\
&m_1 m_{\rho-1}\sigma_1 + (m_2 m_{\rho-1} - m_1)\sigma_2 + \cdots \\
&+(m_{\rho-2} m_{\rho-1} - m_{\rho-3})\sigma_{\rho-2} + (m_{\rho-1} m_{\rho-1} - m_{\rho-2})\sigma_{\rho-1} \\
&-M \,\mathrm{sign}[m_1\sigma_1 + \ldots m_{\rho-1}\sigma_{\rho-1} + \sigma_\rho]
\end{aligned}
\qquad (3.37)
$$

As previously discussed, s goes to zero in finite time and, provided the coefficients in (3.35) are properly chosen, an ideally asymptotically stable motion can be then obtained for σ, as governed by the following autonomous linear dynamics:

$$
\begin{aligned}
&\dot{\sigma}_1 = \sigma_2 \\
&\dot{\sigma}_2 = \sigma_3 \\
&\ \vdots \\
&\dot{\sigma}_{\rho-1} = -m_1\sigma_1 - m_2\sigma_2 - \cdots - m_{\rho-1}\sigma_{\rho-1} \\
&\sigma = \sigma_1
\end{aligned}
\qquad (3.38)
$$

Example in example 3.10, the first order sliding regime obtained for σ is actually a second order sliding regime for the radial position error: $q_1 = \zeta = \rho - R$. As it is easily seen from (3.13), such an error quantity does qualify as a sliding surface candidate and, hence, a non-smoothed first order sliding regime could have also been created on it.

Example. (continuously stirred tank biological reactor)

The following differential equations describe a simplified model of methanol growth in a continuously stirred tank biological reactor which utilizes*methylomonas* organisms (see Hoo and Kantor, 1986, and Sira-Ramírez, 1992e). Let x_1 represent the density of methylomonas cells and let x_2 represent the methanol concentration :

$$
\begin{aligned}
\dot{x}_1 &= A_\mu \varphi(x_2)x_1 - u x_1 \\
\dot{x}_2 &= -A_\sigma \varphi(x_2)\, x_1 + u(A_f - x_2) \\
y &= x_2
\end{aligned}
\qquad (3.39)
$$

where

$$\varphi(x_2) = \frac{x_2}{B + x_2} \qquad (3.40)$$

The control input u represents the dilution rate of the substrate and A_f is the feed substrate concentration, assumed to be constant. A_μ and A_σ are known constants.

For constant values u = U, of the dilution rate, the system exhibits two constant equilibrium points. One of the equilibrium points is located at $(0, A_f)$, which is of no physical interest, and the second one is given by :

$$X_1(U) = \frac{A_f A_\mu - (A_f + B)U}{A_\sigma} A_\mu \quad ; \quad X_2(U) = \frac{BU}{A_\mu - U} \quad (3.41)$$

The equilibrium value U, for the dilution rate, must necessarily satisfy the following relation:

$$U < \frac{A_f A_\mu}{(A_f + B)}$$

in order to have physically meaningful (i.e., positive and finite) equilibrium values for x_1 and x_2.

Suppose it is desired to regulate the methanol concentration x_2 to its equilibrium point $X_2(U)$ for a given U.

The methanol concentration error, $\sigma = x_2 - X_2(U)$, evidently qualifies as a sliding surface candidate, since its time derivative is dependent on the control input u. The resulting static "first order" sliding mode controller does not seem to have much practical sense, since a discontinuous dilution rate u, i.e., one including arbitrarily large frequency switchings, is difficult, if not impossible, to achieve:

$$u = \frac{1}{(A_f - x_2)}\left[A_\sigma \varphi(x) - W \,\mathrm{sign}\,(x_2 - X_2(U))\right] \qquad (3.42)$$

A simulation of the static sliding mode controlled biological tank reactor is shown in figure 10. The system, and design, parameter values used for the simulation were chosen as :

$$A_f = 1.8\,,\ A_\mu = 0.504\ ,\ A_\sigma = 1.32\ ,\ B = 8.49 \times 10^{-4}$$
$$U = 0.4\,,\ X_2(U) = 3.3 \times 10^{-3}\ ,\ W = 10$$

The state trajectory response for x_1 is sufficiently smooth and it is seen to slowly converge to its equilibrium value $X_1(U) = 0.6849$, while the trajectory of x_2 exhibits significant chattering around its preassigned equilibrium point. The feedback control input also exhibits a chattering response, thus making the feasibility of the controller quite questionable from practical grounds.

The concentration error σ is seen to satisfy a second order algebraic differential equation of the form:

$$
\begin{aligned}
\ddot{\sigma} = &-A_\sigma\Big[\varphi'(\sigma + X_2(U))\,\dot{\sigma} + \varphi(\sigma + X_2(U))\big(A_\mu\varphi(\sigma + X_2(U)) - u\big] \\
&\left[\frac{u(A_f - \sigma - X_2(U))}{A_\sigma\varphi(\sigma + X_2(U))} - \dot{\sigma}\right) + \dot{u}\,(A_f - \sigma - X_2(U)) - u\dot{\sigma}
\end{aligned}
\qquad (3.43)
$$

where $\varphi'(.)$ stands for $d\varphi(.)/d(.)$.

The LGOCF, which in this case is also a GGOCF, associated to the concentration error σ is then given by:

$$
\begin{aligned}
&\dot{\sigma}_1 = \sigma_2 \\
&\dot{\sigma}_2 = A_\sigma\Big[\varphi'(\sigma_1 + X_2(U))\,\sigma_2 + \varphi(\sigma_1 + X_2(U))\big(A_\mu\varphi(\sigma_1 + X_2(U)) - u\big)\Big] \\
&\left(\frac{u(A_f - \sigma_1 - X_2(U))}{A_\sigma\varphi(\sigma_1 + X_2(U))} - \sigma_2\right) \\
&+ \dot{u}\,(A_f - \sigma_1 - X_2(U)) - u\,\sigma_2 \\
&\sigma = \sigma_1
\end{aligned}
\qquad (3.44)
$$

A second order sliding regime may now be created for σ using the sliding surface:

$$s = \sigma_2 + m_1\sigma_1 \qquad (3.45)$$

Notice that, expressed in terms of the state variables, such a sliding surface is actually an input -dependent switching function. Indeed, one obtains the following alternative expression for s:

$$s = -A_\sigma\varphi(x_2)\,x_1 + u(A_f - x_2) + m_1\,(x_2 - X_2(U)) \qquad (3.46)$$

Imposing the discontinuous dynamics ds/dt = -M sign(s), on the second order sliding surface candidate s, yields the following dynamical sliding mode controller:

$$
\begin{aligned}
&\dot{u}\,(A_f - \sigma_1 - X_2(U)) = \\
&-A_\sigma\Big[\varphi'(\sigma_1 + X_2(U))\,\sigma_2 + \varphi(\sigma_1 + X_2(U))\big(A_\mu\varphi(\sigma_1 + X_2(U)) - u\big)\Big] \\
&\left(\frac{u(A_f - \sigma_1 - X_2(U))}{A_\sigma\varphi(\sigma_1 + X_2(U))} - \sigma_2\right) \\
&+ (u - m_1)\,\sigma_2 - M\,\mathrm{sign}\,(\sigma_2 + m_1\,\sigma_1)
\end{aligned}
\qquad (3.47)
$$

which expressed now in terms of the state variables of the system reads :

$$
\begin{aligned}
\dot{u} = \frac{1}{A_f - x_2}\Bigg\{&\Big[-A_\sigma\varphi(x_2)\,x_1 + u\,(A_f - x_2)\Big]\big(A_\sigma\varphi'(x_2)x_1 + u - m_1\big) \\
&+ A_\sigma\varphi(x_2)x_1\big(A_\mu\varphi(x_2) - u\big) \\
&- M\,\mathrm{sign}\Big[-A_\sigma\varphi(x_2)x_1 + u\,(A_f - x_2) + m_1(x_2 - X_2(U))\Big]\Bigg\}
\end{aligned}
\qquad (3.48)
$$

The dynamical controller (3.48) exhibits a singularity (*impasse point*) at $x_2 = A_f$. The desired value $X_2(U)$ must then be chosen far away from A_f. If, however, trajectories must necessarily cross through this singularity, then suitable discontinuities must be appropriately devised on the control input prescription (see Abu el Ata-Doss *et al* , 1992 for details).

The simulations shown in figure 11 depict the higher order sliding mode controlled state responses x_1 and x_2 converging towards their equilibrium points, the smoothed nature of the dilution rate u, acting now as a dynamically generated feedback input, and, finally, the asymptotic convergence of the concentration error σ, to zero.

3.6 Sliding regimes in controllable nonlinear systems

The *differentially algebraic closure* of the ground field k in the dynamics K is defined as the differential field κ, where $K \supseteq \kappa \supseteq k$, consisting of the elements of K which are differentially algebraic over k. The field k is *differentially algebraically closed* if, and only if, $\kappa = k$.

The following definition is taken from Fliess (1991) (see also Pommaret, 1988):

Definition 3.14 The dynamics K/k<u> is said to be *controllable* if, and only if, the groud field k is differentially algebraically closed in K.

Controllability implies, then, that all elements of K are necessarily influenced by the input u, since they satisfy a differential equation which is not independent of u and of, possibly, some of its time derivatives.

Proposition 3.15 A higher order sliding regime can be created for any element σ of the dynamics K/k<u> if, and only if, K/k<u> is controllable.

Proof sufficiency is obvious from the fact that σ satisfies a differential equation with coefficients in k<u>. For the necessity of the condition, suppose, contrary to what is asserted, that K/k<u>is not controllable and yet a higher order sliding regime can be created on any element of the differential field extension K/k<u>. Since k is not differentially algebraically closed, then, there are elements in K, which belong to a differential field κ containing k, which satisfy differential equations with coefficients in k. Clearly, these elements are not related to the control input u through differential equations. It follows that a higher order sliding regime cannot be created on such elements. A contradiction is established.

Examples consider the unicycle example, which is easily seen to be controllable. The radial error coordinate ρ-R qualifies as a sliding surface candidate since its first time derivative is already (analytically) dependent on u and , hence, a first order sliding motion can be created on it. This very same element exhibits a second order GOCF and, consequently, a second order sliding motion can also be created on such a sliding surface candidate.

In the single-axis satellite example, the system is clearly controllable. Notice, however, that the attitude error is not a sliding surface candidate and, therefore, a first order sliding motion can not be created on it. However, a second order (although static) sliding regime clearly exists for this element.

In this more relaxed notion of a higher order sliding regime, one may say that a sliding regime can be created on any element of the dynamics of the system, if, and only if, the system is controllable. This characterization of sliding mode existence through controlability is believed to be new, and a direct consequence of the differential algebraic approach.

4. CONCLUSIONS AND SUGGESTIONS FOR FURTHER RESEARCH

The differential algebraic approach to system dynamics provides, both, theoretical, and practical grounds, for the development of sliding mode control of nonlinear dynamical systems. More general classes of sliding surfaces, which include the presence of inputs and, possibly, their time derivatives, were shown to naturally allow for chattering-free sliding mode controllers of dynamical nature. Although equivalent smoothing effects can be similarly obtained by simply resorting to appropriate systems extensions, or prolongations of the input space, the theoretical simplicity, and conceptual advantages, stemming from the differential algebraic approach, bestow new possibilities to the broader area of discontinous feedback control. For instance, the same smoothing effects, and theoretical richness, can be used for the apropriate formulation and the attack of many potential application areas based on pulse-width-modulated control strategies (see Sira-Ramirez, 1992e). The less explored pulse-frequency-modulated control techniques have also been shown to benefit from this new approach (Sira-Ramirez, 1992f).

Discontinuous feedback controller design will undoubtedly be enrriched by the differential algebraic approach. For instance, it has been shown, in a most elegant manner, by Fliess and Messager (1991), that non-minimum phase linear systems can be asymptotically stabilized using dynamical precompensators and sliding mode controllers. Such results could be extended to the nonlinear systems case with, possibly, some significant additional efforts. This topic, as well as possible extensions of the theory to nonlinear multivariable systems and to infinite dimensional systems, . deserve some attention in the forseeable future.

REFERENCES

Abu el Ata-Doss, S., Coïc, A., and Fliess, M.,1992, "Nonlinear Predictive Control by Inversion : Discontinuities for Critical Behaviors," *Int. J. of Contr.* (to appear).

Bartolini, G.,and Zolezzi, T., 1986, Control of variable structure systems, *J. of Math. Anal. and Appl.*, **118**, 42-62.

Bartolini,G., and Pydynowsky, P., 1991, Approximate linearization of uncertain nonlinear systems by means of continuous control, *30th IEEE Conf. on Decision and Contr.*, Vol. 3, pp. 2165-2167, Brighton, England, December 11-13.

Breger, A.M., Butkovsky, A.G., Kubyshin, V.A., and Utkin, V.I., 1980, Sliding modes for control of distributed parameter entities systems subjected to a movable multicycle signal, *Autom.and Remote Contr.* **41**, 346-355.

Bühler, H., 1986, *Réglage par mode de glissement*, (Presse Polytechnique Romande: Lausanne).

DeCarlo, R.A., Zak, S.H., and Mathews, G., 1988, Variable structure control of nonlinear multivariable systems : A tutorial, *Proc. of the IEEE*, **76**, 212-232.

Chang, L.W ., 1991, A versatile sliding control with a second-order sliding condition, *Proc. American Control Conference* Vol. 1, pp. 54-55, Boston, MA, June 26-28.

Diop, S., 1989, A state elimination procedure for nonlinear systems," in *New Trends in Nonlinear Control Theory* , J. Descusse, M. Fliess, A.Isidori and D. Leborgne, Eds. Lect. Notes Control Inform. Sci., Vol. 122, (Springer-Verlag: New York).

Drakunov, S.V., Utkin, V.I., 1989, On discrete-time sliding modes," *Proc. of IFAC Symposium on Nonlinear Control Systems* (NOLCOS) pp.484-489, Capri-Italy, June.

Dwyer, T.A.W. III, and Sira-Ramirez, H., 1988, Variable structure control of spacecraft attitude maneuvers, *J. of Guid. Dyn. and Contr.* **11**, 262-270.

Emelyanov, S.V., 1967, *Variable Structure Control Systems*, (Nauka: Moscow); 1987, *Binary Control Systems*, (MIR: Moscow); 1989,*Titles in Theory of Variable Structure Control Systems* (International Research Institute for Mangement Sciences) (Irimis : Moscow); 1990a, *Titles in new types of feedback, variable structure systems and binary control*, (International Research Institute for Mangement Sciences) (Irimis : Moscow) ; 1990b.The principle of duality, new types of feedback, variable structure and binary control, *Proc. of the Int. Workshop on Variable Structure Systems and their Applications*, Sarajevo, March 19-21.

Fernandez, B., and Hedrick, J.K., Control of multivariable nonlinear systems by the sliding mode method, 1987, *Int. J. of Control*, . **46**, 1019-1040.

Fliess, M., 1986, A note on the invertibility of nonlinear input-output differential systems, *Syst. and Contr. Lett.*, **8**, 147-151; 1988, Nonlinear Control Theory and Differential Algebra, in *Modelling and Adaptive Contrrol*, Ch.I. Byrnes and A. Kurzhanski (Eds.), Lect. Notes in Contr. and Inform. Sci., Vol. 105. pp. 134-145, (Springer-Verlag : New York); 1989a, Automatique et corps différentieles, *Forum Math.*, 1, 227-238;

1989b, Generalized linear systems with lumped or distributed parameters and differential vector spaces, *Int. J. of Contr.*, **49**, 1989-1999; 1989c, Généralisation non linéaire de la forme canonique de commande et linéarisation par bouclage, *C.R. Acad. Sci. Paris*, **I-308**, 377-379; 1990a, Generalized controller canonical forms for linear and nonlinear dynamics, *IEEE Trans. on Auto. Contr.*, **35**, 994-1001; 1990b, What the Kalman state variable representaion is good for, *29th IEEE Conf. on Decision and Contr.* Vol. 3, pp. 1282-1287, Honolulu, Hawaii, December 5-7; 1991, Controllability Revisited, in *Mathematical System Theory : The Influence of R.E. Kalman*, pp. 463-474 (New York: Springer-Verlag).

Fliess, M., and Hassler, M., 1990, Questioning the classical state-space description via circuit examples, in *Mathematical Theory of Networks and Systems*. M.A. Kaashoek, A.C.M. Ram and J.H. van Schuppen (Eds), Progress in Systems andControl Theory, (Birkhauser : Boston, MA).

Fliess, M., Lévine, J., and Rouchon, P., 1991, A simplified approach of crane control via a generalized state-space model, *30th IEEE Conf. on Decision and Contr.* Vol. 1, pp. 736-741, Brighton, England, December 11-13.

Fliess, M., and Messager, F., 1990., Vers une stabilisation non linéaire discontinue, in *Anal. Optimiz. Syst.*, A. Bensoussan and J.L. Lions, Eds., Lect. Notes Contr. Inform. Sci., Vol. 144, pp. 778-787, (Springer-Verlag : New York); 1991, Sur la commande en régime glissant, 1991, *C. R. Acad. Sci. Paris*, **313** Serie I, 951-956.

Furuta, K., 1990, Sliding mode control of a discrete system, *Syst. and Contr. Letters*, **14**, 145.

Hoo, K.A., and Kantor, J.C., 1986, Linear feedback equivalence and control of an unstable biological reactor, *Chem. Engrg. Comm.* **46**, 385-399.

Isidori, A., 1989, *Nonlinear Control Systems* (Springer- Verlag, New York).

Itkis, U., 1976, *Control Systems of Variable Structure.* (Wiley: New York).

Kolchin, E.R., 1973, *Differential Algbebra and Algebraic Groups*. (Academic Press: New York).

Kwatny, H., and Kim, H., 1990., *Syst. and Contr. Letters*, **15**, 67.

Luk'yanov, A.G., and Utkin, V.I., 1981, Ways of reducing equations of dynamic systems to a regular form, *Automation and Remote Control*, **42**, 5-13.

Magaña, M. E., Zak, S.H., 1987, The Control of Discrete-Time Uncertain Systems, Technical Report TR-EE 87-32, School of Electrical Engineering, Purdue University. West Lafayette, Indiana 47907.

Marino, R. , 1985, High gain feedback in nonlinear control systems, *Int. J. of Control*, **42**, 1369-1385.

Mathews, G., DeCarlo, R.A., Hawley, P., and Lefebvre, S., 1986, Toward a feasible variable structure control design for a synchronous machine connected to an infinite bus, *IEEE Trans. on Auto. Contr.* **31**, 1159-1163.

Miloslavjevic, C., 1985, General conditions for the existence of a quasisliding mode on the switching hyperplane in discrete variable structure systems, *Autom. and Rem. Control*, **46**, 307-314.

Nijmeijer, H., and Van der Schaft, A., 1990, *Nonlinear Dynamical Control Systems*, (Springer-Verlag: New York).

Opitz, H.P., 1986, Robustness properties of discrete-variable structure controllers, *Int'l J. of Control*, **43**,1003-1014.

Orlov, Y.U., and Utkin, V.I., 1982., Application of sliding modes to control of distributed systems, *Autom.and Remote Contr.* **43**, 1127-1135; 1987, Sliding mode control in indefinite dimensional systems, *Automatica*, **23**, 753-757.

Pommaret, J.F., 1983, *Differential Galois Theory* (Gordon and Breach, New York); 1986, Géométrie différentielle algébrique et théorie du contrôle, *C.R. Acad. Sci. Paris*, **I-302**, 547-550; 1988, *Lie Groups and Mechanics* (Gordon and Breech: New York).

Rebiai,S., and Zinober, A.S.I., 1990, Stabilization of distributed parameter systems, *Proc. of the Int. Workshop on Variable Structure Systems and their Applications*, pp. 17-25, Sarajevo, March 19-21; 1991., Stabilizing Boundary Control of Uncertain Heat Equations, *Proc. 30th IEEE Conf. on Decision and Control.*, Vol. 3, pp. 2142-2147, Brighton, December 11-13.

Sarpturk, S.Z., Istefanopulos, Y., Kaynak, O., 1987, On the stability of discrete-time sliding mode control systems, *IEEE Trans. on Auto. Contr.* **32**, 930-932.

Sira-Ramírez, H., 1987, Variable structure control of nonlinear systems *Int. J. of Syst. Sci.* **18**, 1673-1689; 1988, Differential geometric methods in variable structure control, *Int. J. of Control*, **48**, 1359-1391; 1989a, A geometric approach to pulse-width-modulted control in nonlinear dynamical systems, *IEEE Trans. on Auto. Contr.* **34**, 184-187; 1989b, Sliding regimes in general nonlinear systems in sliding mode: The general case, *Int. J. of Control*, **50**, 1487-1506; 1989c, Distributed sliding mode control in systems described by quasilinear partial differential equations, *Syst. and Contr. Letters*, **13**, 177-181; 1990, Structure at infinity, zero dynamics and normal forms of systems undergoing sliding motions, *Int. J. of Syst. Sci.* **21**, 665-674 ; 1991a, Nonlinear discrete variable structure systems in quasi-sliding mode, *Int. J. of Control*, **54**, 1171-1187; 1991b, Nonlinear dynamically feedback controlled descent on a non atmosphere-free planet : A differential algebraic approach, *Control -Theory and Adv. Technol.*, **7**, 301-320 ; 1992a, Dynamical Variable Structure Control Strategies in Asymptotic Output Tracking Problems, *IEEE Trans.on Auto. Contr.* (accepted for publication, to appear); 1992b, Asymptotic Output Stabilization for Nonlinear Systems via Dynamical Variable Structure Control, *Dyn. and Contr.* **2**, 45-58 ; 1992c., The Differential Algebraic Approach in Nonlinear Dynamical Feedback Controlled Landing Maneuvers, *IEEE Trans. on Auto. Contr.* **37** (to appear); 1992d, Dynamical Sliding Mode Control Strategies in the Regulation of Nonlinear Chemical Processes, *Int. J. of Control.* (accepted for publication, to appear); 1992e, Dynamical Pulse Width Modulation Control of Nonlinear Systems, *Syst. and Contr. Lett.*, (accepted for publication, to appear).

Sira-Ramirez, H., Ahmad, S., and Zribi, M., 1992, Dynamical Feedback Control of Robotic Manipulators with Joint Flexibility, *IEEE Trans. Syst. Man and Cyb.* (accepted for publication, to appear).

Sira-Ramirez, H., and Lischinsky-Arenas, P., 1990, Dynamical discontinuous feedback control of nonlinear systems, *IEEE Trans. on Auto. Contr.* **35**, 1373-1378 ; 1991., The differential algebraic approach in nonlinear dynamical compensator design for dc-to-dc power converters, *Int. J. of Control*, **54**, 111-134.

Sira-Ramirez, H., and Rivero-Mendoza, J., 1991 , Sliding mode control of nonlinear first order distributed parameter systems, *Pure Math. and Appl.* Series B, **1**, 123-141.

Slotine, J.J.E., 1985, The robust control of robot manipulators, *Int. J. of Robotics Research*, **4** , 49-64.

Slotine, J.J.E., and Sastry, S., 1983, Tracking control of nonlinear systems using sliding surfaces with applications to robot manipulators, *Int. J. of Control*, **38** , 465-492.

Spurgeon, S., 1991, Sliding mode control of uncertain discrete-time systems, *Proc. 30th IEEE Conf. on Decision and Control*, Vol. 3, pp. 2136-2141, Brighton, England, December 11-13.

Utkin, V.I. , 1977, Variable structure systems with sliding modes, *IEEE Trans. on Auto. Contr.* **22**, 212-222; 1983, Variable structure systems: Present and future, *Autom. and Remote Contr.* **44**, 1105-1120; 1987, Discontinuous Control Systems: State of the Art in the Theory and Applications, *World Triennial Congress IFAC*, pp. 75-94, Munich ; 1978, *Sliding Modes and their Applications in Variable Structure Systems* (MIR: Moscow); 1990. Sliding mode control in infinite dimensional plants, *Proc. of the Int. Workshop on Variable Structure Systems and their Applications*, pp. 1-16, Sarajevo March 19-21; 1992, *Sliding Modes in Optimization and Control Problems* (Springer-Verlag: New York);

Young, K.K.D., (ed) (to appear), *Variable Structure Control for Robotics and Aerospace Applications* (Elsevier Science : Amsterdam).

Yu, X., 1992., Computer controlled variable structure systems, *J. of Australian Math. Soc.*, Series B (to appear).

Zinober, A.S.I., (ed), 1990, *Determnistic Control of Uncertain Systems*, (Peter Peregrinus: London).

Zolezzi, T., 1989, Variable structure control of semilinear evolution equations," in *Partial Differential Equations and the Calculus of Variations*, Vol. II, pp. 997-1018 (Birkhauser : Boston).

FIGURES

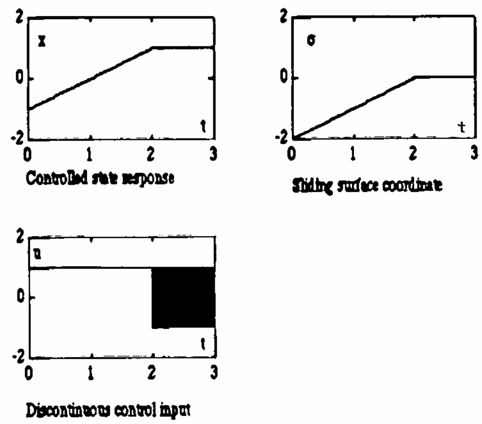

Figure 1. Simulation of (statically) sliding mode controlled responses of single integrator plant.

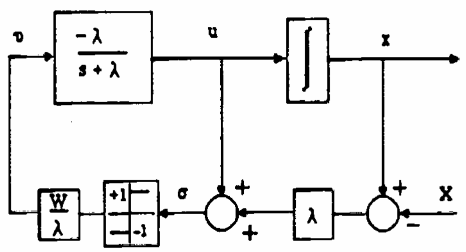

Figure 2. Filtering Effect of Dynamical Sliding Mode Control of a Single Integrator Plant.

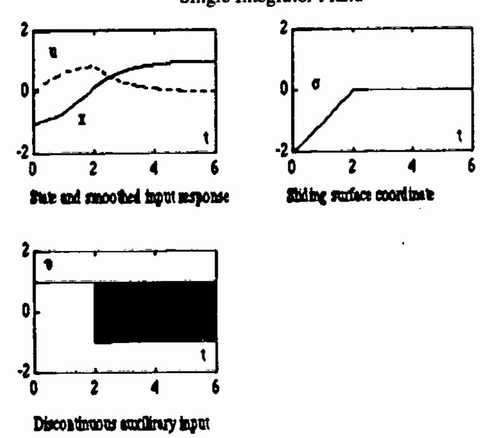

Figure 3. Simulation of dynamically sliding mode controlled responses of single integrator plant.

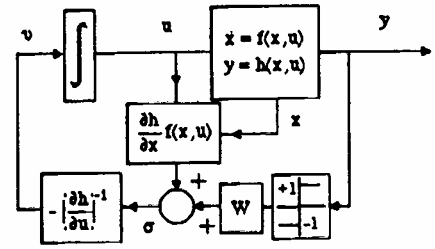

Figure 4. Dynamical sliding mode control scheme for zeroing of input-dependent outputs.

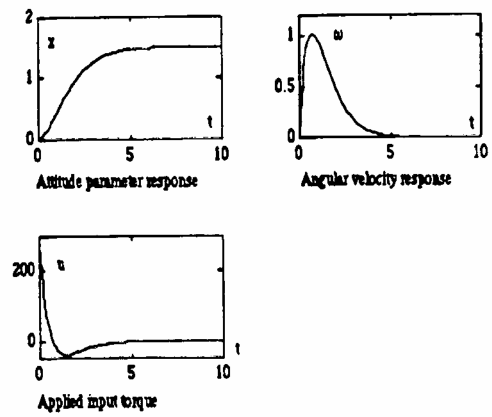

Figure 5. State variables responses and applied input torque for continuous feedback controlled single-axis spacacraft.

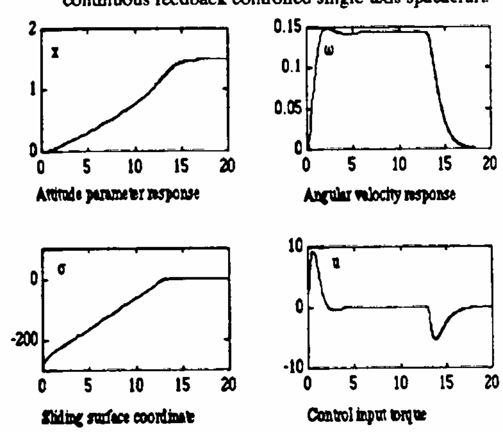

Figure 6. State variables responses, sliding surface coordinate evolution and applied input torque for dynamically sliding- mode controlled spacacraft.

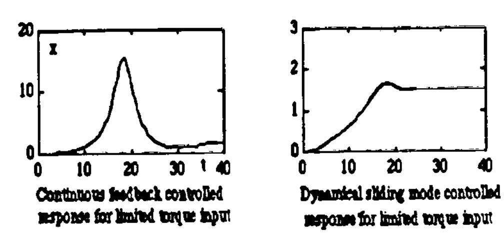

Figure 7. Continuous and dynamical sliding mode feedback controlled responses of attitude parameter subject to saturation of control input torque (torque saturation limits:| u |≤ 2.5 N-m).

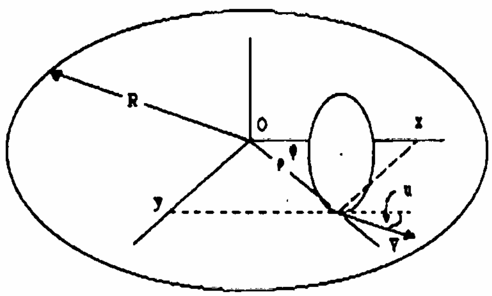

Figure 8. Geometry of the unicycle control problem of following a prescribed circular trajectory with constant velocity.

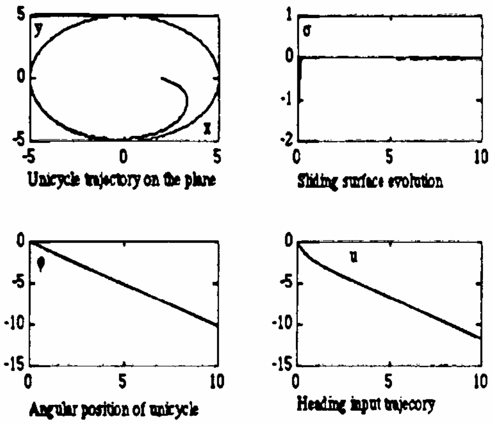

Figure 9. Simulations of dynamically sliding mode controlled unicycle.

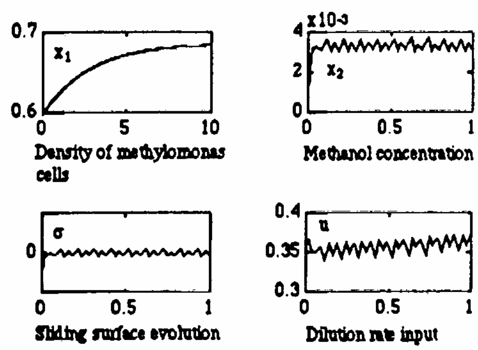

Figure 10. First order sliding mode controlled continuously stirred tank biological reactor.

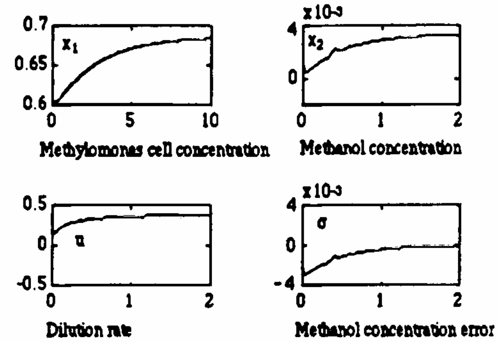

Figure 11. Second order sliding mode controlled continuously stirred tank biological reactor.