A UNIFIED APPROACH TO DISCONTINUOUS FEEDBACK CONTROL OF NON-LINEAR SYSTEMS

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Abstract

In this article a unified approach is proposed for the design of dynamical discontinuous feedback controllers leading to the chattering-free stabilization of nonlinear single input single output systems. The adopted framework is that of a Generalized State representation form of the given nonlinear plant, and use of the associated Generalized Observability Canonical Form of such representation. Unification of discontinuous feedback policies is achieved by meas of the finite time nulling of a suitably specified auxiliary input-dependent output function via simple discontinuous feedback control paradigms of various kinds. The zeroing of such scalar stabilizing function induces asymptotically stable controlled dynamics on the given nonlinear minimum-phase plant. Pulse-Frequency-Modulation, Pulse-Width-Modulation and Sampled Sliding Mode Control strategies are considered from this unified viewpoint. Examples are provided including simulations.

1. INTRODUCTION

Recently, results from the *differential algebraic* approach to control theory, pioneered by Prof. Michel Fliess (1989,1990a), have greatly improved the applicability of discontinuous feedback strategies, specially those of the *sliding mode* (SM) type, leading to asymptotic stabilization, and tracking, in nonlinear systems (see Sira-Ramirez, 1993 and Sira-Ramírez*et al*, 1992) for application examples in mechanical and electro-mechanical systems). Some of the traditional disadvantages of sliding mode control policies are fundamentally related to the chattering of input and state variables response signals (See Utkin, 1978). These difficulties are easily circumvented via *dynamical* sliding mode controllers while retaining the outstanding robustness, and simplicity inherent to this class of feedback control schemes.

In this article, Fliess's Generalized Observability Canonical Form (GOCF) is shown to naturally allow for dynamical feedback controller design based on pulse-frequency-modulation (PFM) strategies, pulse-width-modulation (PWM) policies and sampled sliding modes (SSM). The obtained control input signals are substantially smoothed with respect to their corresponding static alternative and, hence, chattering-free discontinuously controlled responses are generated. The obtained PWM and PFM controller designs do not resort to traditional approximation schemes, based on (infinte frequency) *average* models, of the discontinuously controlled systems (see, Sira-Ramirez, 1989). The smoothing features of dynamical discontinuous feedback policies are particularly important in the regulation of mechanical and chemical systems, in which large and fast input vibrations, or jump discontinuities, cannot be simply allowed on the actuators, while a need still exists for certain degree of robustness (i.e., insensitivity to modeling errors and external perturbations) and quality performance of the proposed regulation scheme.

The synthesis of the several dynamical discontinuous regulators, here proposed, is based on Fliess's *Generalized Observability Canonical Form* (GOCF) for nonlinear single -nput single-output systems (See Fliess, 1989). In Section 2 of this article, we briefly address the dynamical SM control solution to the output stabilization problem and present the basic results of the PFM the PWM and the SSM controller design schemes. In section 3, we present some illustrative examples along with encouraging simulations. The first example deals with the classical robotic manipulator system, for which torque input chattering is effectively eliminated, in spite of the underlying discontinuous feedback control policy. The second example arises from a non-traditional application area for discontinuous feedback control, such as chemical process control. In this example, a discontinuous feedback control regulator is designed for the stabilization of the output concentration of a certain chemical agent, in a double effect evaporator system. In both examples simulations are provided which depict the advantageous features of dynamical discontinuous controls. Concluding remarks are collected at the end of the article.

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2. DYNAMICAL DISCONTINUOUS FEEDBACK CONTROL OF NONLINEAR SYSTEMS

The results of this section may be extended to tracking problems (see Sira-Ramírez, 1993, Sira-Ramírez*et al*, 1992) and to multivariable nonlinear systems.

2.1 Fliess's Generalized Observability Canonical Form.

It has been shown in Fliess 1989 (see also Conte *et al*, 1988) that a nonlinear, single-input single-output n- dimensional system given in *generalized state* representation form:

$$\begin{aligned} \dot{x} &= f(x, u, \dot{u}, \cdots, u^{(\beta)}) \\ y &= h(x, u, \dot{u}, \cdots, u^{(\beta)}) \end{aligned} \tag{2.1}$$

can be locally transformed, via an input-dependent state coordinate transformation of the form :

$$z = \Phi(x, u, \dot{u}, \ldots, u^{(\alpha-1)}) \tag{2.2}$$

into a system of the form:

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ &\cdots \\ \dot{z}_n &= c(z, u, \dot{u}, \ldots, u^{(\alpha)}) \\ y &= z_1 \end{aligned} \tag{2.3}$$

provided the following "observability" matrix of the system (2.1) is full rank:

$$\begin{bmatrix} \dfrac{\partial h(x,u,\dot{u},\cdots,u^{(\beta)})}{\partial x} \\ \dfrac{\partial h^{(1)}(x,u,\dot{u},\cdots,u^{(\beta+1)})}{\partial x} \\ \cdots \\ \dfrac{\partial h^{(n-1)}(x,u,\dot{u},\ldots,u^{(\alpha-1)})}{\partial x} \end{bmatrix} \tag{2.4}$$

In (2.3), α is assumed to be a strictly positive integer. The results, however, can be easily extended to *systems exactly linearizable by static state feedback*, i.e., for systems in which $\alpha = 0$ (see Sira-Ramirez 1992, and the first example presented in Section 3).

It must be remarked, however, that, in general, (2.3) is not, necessarily, n-dimensional.

The input-dependent state coordinate transformation (2.2) is given by the following local diffeomorphism:

$$z = \Phi(x, u, u, \ldots, u^{(\alpha-1)}) = \begin{bmatrix} h(x,u,\dot{u},\cdots,u^{(\beta)}) \\ h^{(1)}(x,u,\dot{u},\cdots,u^{(\beta+1)}) \\ \cdots \\ h^{(n-1)}(x,u,\dot{u},\ldots,u^{(\alpha-1)}) \end{bmatrix} \tag{2.5}$$

Suppose $u = U$, $x = X(U)$ describe a constant equilibrium point for the original system (2.1), such that $h(X(U),U,0,...,0)$ is zero, then $z = 0$ is an equilibrium point of (2.3). The autonomous dynamics described by:

$$c(0,u,\dot{u},\dots,u^{(\alpha)}) = 0 \tag{2.6}$$

is known as the *zero dynamics* (see Fliess, 1990b). The stability nature of any constant equilibium point $u = U$ of (2.6) determines the *minimum* or *non-minimum phase* character of the system about the corresponding equilibrium point. We denote such constant equilibrium point for system (2.1) as $(X(U),U,0)$.

2.2 A GOCF Approach to Dynamical Discontinuous Feedback Controller Design for Nonlinear Systems.

Consider the following auxiliary output function $\sigma : R^n \to R$, defined in terms of the transformed variables z,

$$\sigma(z) = \left(\sum_{i=1}^{n-1} \gamma_i z_i\right) + z_n \tag{2.7}$$

such that the following corresponding polynomial in the complex variable λ is Hurwitz:

$$\sum_{i=1}^{n-1} \gamma_i \lambda^{i-1} + \lambda^{n-1} \tag{2.8}$$

Suppose the system is locally minimum phase around $(X(U),U,0)$. It is easy to see that if (2.7) is forcefully constrained to zero (whether in finite time, or in an asymptotically stable fashion) by means of appropriate control actions (possibly of discontinuous nature), the resulting controlled dynamics locally evolves in accordance with:

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ &\cdots \\ \dot{z}_{n-1} &= -\sum_{i=1}^{n-1} \gamma_i z_i \\ y &= z_1 \end{aligned} \tag{2.9}$$

which is asymptotically stable to zero.

Two of the dynamical discontinuous feedback controller design schemes, here proposed, rely on inducing an asymptotically stable linear time invariant controlled dynamics such as (2.9), with eigenvalues placeable at will. This is done by driving the proposed auxiliary output function $\sigma(z)$ to zero. SM controllers can always accomplish such a task in finite time. PFM and PWM controllers, on the other hand, can only accomplish this task in an asymptotially stable fashion, while SSM control can only do it approximately.

Dynamical Sliding Mode Control of Nonlinear Systems

Proposition 2.1 Let W be a strictly positive quantity and let "sgn" stand for the *signum* function. The one dimensional discontinuous system :

$$\dot{\sigma} = -W \operatorname{sgn} \sigma \tag{2.10}$$

globally exhibits a sliding regime on $\sigma = 0$. Furthermore, any trajectory starting on the value $\sigma = \sigma(0)$, at time 0, reaches the condition $\sigma = 0$ in finite time T, given by : $T = W^{-1}|\sigma(0)|$.

Proof Immediate upon checking that globally: $\sigma\, d\sigma/dt < 0$ for $\sigma \neq 0$, which is a well known condition for sliding mode existence (Utkin, 1978). The second part follows trivially from the fact that $|\sigma(t)| = -Wt + |\sigma(0)|$ ■

Proposition 2.2 A minimum phase nonlinear system of the form (2.1) is locally asymptotically stabilizable to the equilibrium point $(U,X(U),0)$ if the control action u is specified as a dynamical SM control policy given by the solution of the following implicit, time-varying, nonlinear discontinuous differential equation :

$$c(z,u,\dot{u},\cdots,u^{(\alpha)}) = -\sum_{i=1}^{n} \gamma_{i-1} z_i - W \operatorname{sgn}\left[\sum_{i=1}^{n-1} \gamma_i z_i + z_n\right] \tag{2.11}$$

where $\gamma_0 = 0$.

Proof Immediate upon imposing on the auxiliary output function $\sigma(z)$ defined in (2.7) the dynamics defined by (2.10). ■

We assume that in (2.11) the quantity $\partial c/\partial u^{(\alpha)}$ is locally nonzero and, hence, no singularities need to be locally considered.

Controller (2.11) is represented in terms of the original state space coordinates x by using the input dependent state coordinate transformation (2.5).

Dynamical PFM Control of Nonlinear Systems

Consider the scalar PFM controlled dynamical system, in which the constants r_1, r_2, r_3 and W, are all strictly positive quantities.

$$\dot{\sigma} = -W v$$

$$v = PFM_{\tau,T}(\sigma) = \begin{cases} \operatorname{sgn} \sigma(t_k) & \text{for } t_k \le t < t_k + \tau[\sigma(t_k)]T[\sigma(t_k)] \\ 0 & \text{for } t_k + \tau[\sigma(t_k)]T[\sigma(t_k)] \le t < t_k + T[\sigma(t_k)] \end{cases}$$

$$\tau[\sigma(t)] = \begin{cases} 1 & \text{for } |\sigma(t)| > \frac{1}{r_1} \\ r_1|\sigma(t)| & \text{for } |s(t)| \le \frac{1}{r_1} \end{cases} \tag{2.12}$$

$$T[\sigma(t)] = \begin{cases} T_{max} & \text{for } |\sigma(t)| \ge \frac{1}{r_2} \\ T_{min} + \frac{r_2 r_3}{r_3 - r_2}[T_{max} - T_{min}]\left(\sigma(t) - \frac{1}{r_3}\right) & \text{for } \frac{1}{r_3} < |\sigma(t)| < \frac{1}{r_2} \\ T_{min} & \text{for } |\sigma(t)| \le \frac{1}{r_3} \end{cases}$$

$$k = 0,1,2,\ldots \quad ; \quad t_{k+1} = t_k + T[\sigma(t_k)].$$

where it is assumed that $r_2 < r_1 < r_3$. The t_k's represent irregularly spaced sampling instants, determined by the sampled values of the *duty cycle function*, denoted here by $T[\sigma(t_k)]$. The duty cycle function, $T[\sigma(t)]$, takes values on the closed interval $[T_{min}, T_{max}]$ and it varies linearly with respect to $\sigma(t)$ in the region $|\sigma| < 1/r_2$. The duty cycle, or sampling period, saturates to T_{max} for large values of σ, and remains fixed at the constant lower bound T_{min} for small values of σ. At each sampling instant, t_k, the value of the width of the sign-modulated, unit amplitude, control pulse is determined by the sampled value of the *duty ratio function*, represented by $\tau[\sigma(t_k)]$. In general, the duty cycle and the duty ratio functions may be quite independent of each other. The function "sgn" stands for the *signum* function.

The following proposition establishes a sufficient condition for the asymptotic stability to zero of the PFM controlled system (2.12).

Proposition 2.3 The PFM controlled system (2.12) is asymptotically stable to $\sigma = 0$, if

$$0 < r_3 W T_{max} < 2 \tag{2.13}$$

Proof Due to the piecewise constant nature of the control input and the linearity of the continuous system, it suffices to study the stability of the discretized version of (2.12) at the sampling instants. An exact discretization of the PFM controlled system (2.12) yields :

$$\sigma(t_k + T) = \sigma(t_k) - W \operatorname{sgn}[\sigma(t_k)]\, \tau[\sigma(t_k)]\, T[\sigma(t_k)] \tag{2.14}$$

The stability of (2.14) follows easily using Lyapunov type of arguments. For a proof of this proposition the reader is refered to Sira-Ramírez and Llanes-Santiago (1992).

Proposition 2.4 A minimum phase nonlinear system of the form (2.1) is locally asymptotically stabilizable to the equilibrium point $(U,X(U),0)$ if the control action u is specified as a dynamical PFM control policy given by the solution of the following implicit, time-varying, nonlinear discontinuous differential equation :

$$c(z,u,\dot{u},\cdots,u^{(\alpha)}) = -\sum_{i=1}^{n} \gamma_{i-1} z_i - W\, PFM_{\tau,T}\left[\sum_{i=1}^{n-1} \gamma_i z_i + z_n\right] \tag{2.16}$$

where $\gamma_0 = 0$.

Proof Immediate upon imposing on the auxiliary output function $\sigma(z)$ in (2.7) the asymptotically stable discontinuous dynamics defined by (2.12). ∎

Dynamical PWM Control of Nonlinear Systems

Consider the scalar PWM controlled system, in which $r > 0$ and $W > 0$:

$$\dot{\sigma} = -W v$$
$$v = PWM_\tau(\sigma) = \begin{cases} \text{sgn}\,\sigma(t_k) & \text{for } t_k \le t < t_k + \tau[\sigma(t_k)]T \\ 0 & \text{for } t_k + \tau[\sigma(t_k)]T \le t < t_k + T \end{cases}$$
$$\tau[\sigma(t)] = \begin{cases} 1 & \text{for } |\sigma(t)| > \frac{1}{r} \\ r|\sigma(t)| & \text{for } |\sigma(t)| \le \frac{1}{r} \end{cases} \tag{2.17}$$
$$k = 0,1,2,\ldots \quad ; \quad t_{k+1} = t_k + T.$$

where the t_k's represent regularly spaced sampling instants and "sgn" stands for the *signum* function.

It is easy to see that (2.17) is just a particular case of the PFM controlled system (2.12) in which the duty cycle function $T(\sigma(t_k))$ is now taken as a constant of value T. The following results follow immediately from this fact.

Proposition 2.5 The PWM controlled system (2.17) is asymptotically stable to $\sigma = 0$ if and only if:

$$0 < rWT < 2 \tag{2.18}$$

Proof Sufficiency is clear from the preceeding proposition. Necesity follows from the fact that (2.18) is necessary to have $\sigma(t_k)$ lie in the region $|\sigma(t_k)| \le 1/r$, for some k, independently of the initial condition. In this region, the PWM controlled dynamics adopts the form $\sigma(t_{k+1}) = (1-rWT)\,\sigma(t_k)$. The result follows. ∎

Proposition 2.6 A minimum phase nonlinear system of the form (2.1) is locally asymptotically stabilizable to the equilibrium point (U,X(U),0) if the control action u is specified as a dynamical PWM control policy given by the solution of the following implicit, time-varying, nonlinear discontinuous differential equation :

$$c(z,u,\dot{u},\cdots,u^{(\alpha)}) = -\sum_{i=1}^{n} \gamma_{i-1} z_i - W\, PWM_\tau[\sum_{i=1}^{n-1} \gamma_i z_i + z_n] \tag{2.19}$$

where $\gamma_0 = 0$.

Proof Immediate upon imposing on the auxiliary output function $\sigma(z)$ in (2.7) the asymptotically stable discontinuous dynamics defined by (2.17). ∎

Dynamical Sampled Sliding Mode Control of Nonlinear Systems

Proposition 2.7 Consider the following one-dimensional Sampled Sliding Mode controlled system:

$$\dot{\sigma} = -W v$$
$$v = SSM[\sigma(t)] = \text{sign}[\sigma(t_k)] \quad \text{for } t_k < t < t_k + T \tag{2.20}$$
$$k = 0,1,\ldots \; ; \quad t_{k+1} = t_k + T$$

Then, given an $\varepsilon > 0$, there exist a sampling interval $T(\varepsilon) = \varepsilon / W$ for which the trajectories of (2.20) satisfy the condition $|\sigma(t)| \le 2\varepsilon$ for all $t > T(\varepsilon) / |\sigma(0)|$.

Proof The proof is immediate from the exact discretization of (2.20):

$$\sigma(t_k+T) = \sigma(t_k) - W\,T\,\text{sign}[\sigma(t_k)]$$

hence,

$$|\sigma(t_k+T) - \sigma(t_k)| = W\,T$$

The first part follows by letting $WT = \varepsilon$. The second part is immediate from the linearity of the system and the fact that for all $t \ge 0$, $|\,d\sigma/dt\,| = W$ ∎

Chattering of σ, around the value $\sigma = 0$, can be made of arbitrarily small amplitude, according to the width of the sampling interval $T(\varepsilon)$. As $T \to 0$, the response of σ to a SSM strategy asymptotically converges to the response of a SM policy.

Proposition 2.8 A minimum phase nonlinear system of the form (2.1) is locally stabilizable around the equilibrium point (U,X(U),0), modulo some small chattering, if the control action u is specified as a dynamical SSM control policy given by the solution of the following implicit, time-varying, nonlinear discontinuous differential equation :

$$c(z,u,\dot{u},\cdots,u^{(\alpha)}) = -\sum_{i=1}^{n} \gamma_{i-1} z_i - W\, SSM[\sum_{i=1}^{n-1} \gamma_i z_i + z_n] \tag{2.21}$$

where $\gamma_0 = 0$.

Proof Immediate upon imposing on the auxiliary output function $\sigma(z)$ in (2.7) the discontinuous dynamics defined by (2.20). ∎

A sampled sliding mode control policy may also be viewed as a particular case of a PWM control policy in which the pulse width $\tau[\sigma(t_k)]T$ is saturated to the value of the sampling interval T (i.e., the duty ratio, $\tau[\sigma(t_k)]$, is equal to 1 for all k).

3. SOME APPLICATION EXAMPLES

3.1 Dynamical PFM, PWM and SSM control of a single link rigid robotic manipulator.

Our first example is concerned with the chattering-free discontinuous feedback control of an exactly linearizable system constituted by a single link rigid robotic manipulator. The various dynamical discontinuous feedback controllers are obtained by resorting to the GOCF of the "extended" system model of the original given nonlinear plant.

Consider the following nonlinear dynamical model of a single link robotic manipulator (Khalil 1992):

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{g}{L}\sin x_1 - \frac{k}{M} x_2 + \frac{1}{ML^2} u \tag{3.1}$$

where x_1 is the link angular position, x_2 is the angular velocity and u represents the applied torque. The mass M is assumed to be concentrated at the tip of the manipulator of length L. The constant k is the viscous damping coefficient, while g is the gravity constant.

It is desired to synthesize, dynamical PFM, PWM and SSM feedback control policies which drive the angular position of the system to a constant desired angular position x_{1d}. We, therefore, consider the position error $x_1 - x_{1d}$ as the output y of the system.

Consider now the extended system of (3.1) (see Nijmeijer and Van der Schaft , 1990).

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{g}{L}\sin x_1 - \frac{k}{M} x_2 + \frac{1}{ML^2} u$$
$$\dot{u} = \upsilon \tag{3.2}$$
$$y = x_1 - x_{1d}$$

where the control input derivative, denoted by υ, plays the role of an *auxiliariy input function*. It is easy to see that the rank condition (2.4) is globally satisfied. Indeed:

$$\text{rank}\begin{bmatrix} \frac{\partial y}{\partial(x,u)} \\ \frac{\partial \dot{y}}{\partial(x,u)} \\ \frac{\partial \ddot{y}}{\partial(x,u)} \end{bmatrix} = \text{rank}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{g}{L}\cos x_1 & -\frac{k}{M} & \frac{1}{ML^2} \end{bmatrix} = 3$$

The resulting GOCF of the extended system is now obtained as :

$$\dot{z}_1 = z_2$$
$$\dot{z}_2 = z_3$$
$$\dot{z}_3 = -\frac{g}{L} z_2 \cos(z_1 + x_{1d}) - \frac{k}{M} z_3 + \frac{1}{ML^2}\upsilon \tag{3.3}$$
$$y = z_1$$

with :

$$z_1 = x_1 - x_{1d} \;;\; z_2 = x_2 \;;\; z_3 = -\frac{g}{L}\sin x_1 - \frac{k}{M}x_2 + \frac{1}{ML^2}u$$

Let the auxiliary output function $\sigma(z)$ be defined as:

$$\sigma(z) = z_3 + \gamma_2 z_2 + \gamma_1 z_1 \tag{3.4}$$

with γ_2 and γ_1 positive constants, chosen in the standard second order system form i.e., with damping factor ζ and natural frequency ω_n: $\gamma_2 = 2\zeta\omega_n$ and $\gamma_1 = \omega_n^2$. Notice that if $\sigma(z)$ is stabilized to zero, possibly in finite time, by means of a discontinuous feedback control policy, the constrained dynamics evolves in accordance to the asymptotically stable second order dynamics : $dz_1/dt = z_2$; $dz_2/dt = -2\zeta\omega_n z_2 - \omega_n^2 z_1$, thus achieving the desired stabilizaton task: $z_2 = x_2 \to 0$ and $z_1 = x_1 - x_{1d} \to 0$.Notice that, in original coordinates, the auxiliary output function σ is a control-input dependent function:

$$\sigma(x) = -\frac{g}{L}\,\text{sen}\, x_1 + \gamma_1(x_1 - x_{1d}) - \left(\frac{k}{M} - \gamma_2\right)x_2 + \frac{1}{ML^2}u \tag{3.5}$$

3.1.1 Dynamical PFM controller design for the manipulator system

Imposing on the auxiliary output function $\sigma(z)$, the asymptotically stable dynamics of the PFM controlled system (2.12) one obtains the following static PWM controller for the extended system, written in terms of the auxiliary input function υ:

$$\begin{aligned}\upsilon = ML^2\Big\{&\left(\frac{k}{M} - 2\zeta\omega_n\right)z_3 - \omega_n^2 z_2 + \\ &\frac{g}{L}z_2\cos(z_1 + x_{1d}) - W\,PFM_{\tau,t}(z_3 + 2\zeta\omega_n z_2 + \omega_n^2 z_1)\Big\}\end{aligned} \tag{3.6}$$

which, in original state and input coordinates, is rewritten as a dynamical PFM controller given by the solution of the following time-varying ordinary differential equation for the original control input torque u with discontinuous (PFM) right hand side :

$$\begin{aligned}\dot{u} = ML^2\Big\{&\left(\frac{k}{M} - 2\zeta\omega_n\right)\left(-\frac{g}{L}\sin x_1 - \frac{k}{M}x_2 + \frac{1}{ML^2}u\right) - \omega_n^2 x_2 + \frac{g}{L}x_2\cos(x_1) \\ &- W\,PFM_{\tau,T}\left[-\frac{g}{L}\sin x_1 + \omega_n^2(x_1 - x_{1d}) - \left(\frac{k}{M} - 2\zeta\omega_n\right)x_2 + \frac{1}{ML^2}u\right]\Big\}\end{aligned} \tag{3.7}$$

Notice that, from (3.6) and the definition in (2.6), the zero dynamics is given, in this case by : $\upsilon = du/dt = 0$.

Simulations were run for the dynamically PFM controlled manipulator (3.1) , (3.7) with the following physical parameter values : $M = 1$ [Kg] , $L = 1$[m], $k = 0$, $x_{1d} = 4$ [rad], $g = 9.8$ [m/s^2], with $\zeta = 0.8$ and $\omega_n = 2.8$ [rad/s]. The PFM controller parameters were chosen as: $W = 24$ [rad/s], $T_{max} = 0.2$ [s], $T_{min} = 0.05$ [s], $r_1 = 0.1$, $r_2 = 0.05$, $r_3 = 0.4$. In this case the sufficient condition of Proposition 2.3 is verified as $r_3 W\, T_{max} = 1.92 < 2$ Figure 1 depicts the state trajectories of the controlled system clearly showing asymptotic convergence of x_1 toward the desired angular position $x_{1d} = 4.0$ [rad], while steady state zero angular velociy is achieved with no chattering being exhibited. Figure 2 shows the PFM signal and the substantially smoothed out (chattering-free) applied torque input signal u, as generated by the dynamical PFM controller (3.7). The effect of adding an integrator to the original input u of the exactly linearizable system results, therefore, in a low pass filtering effect on the input u of the original system.

3.1.2 Dynamical PWM controller design for the manipulator system

Imposing on the auxiliary output function $\sigma(z)$, the asymptotically stable dynamics of the PWM controlled system (2.17) one obtains, after some manipulations, the following dynamical PWM controller for the manipulator system:

$$\begin{aligned}\dot{u} = ML^2\Big\{&\left(\frac{k}{M} - 2\zeta\omega_n\right)\left(-\frac{g}{L}\sin x_1 - \frac{k}{M}x_2 + \frac{1}{ML^2}u\right) - \omega_n^2 x_2 + \frac{g}{L}x_2\cos(x_1) \\ &- W\,PFM_{\tau}\left[-\frac{g}{L}\sin x_1 + \omega_n^2(x_1 - x_{1d}) - \left(\frac{k}{M} - 2\zeta\omega_n\right)x_2 + \frac{1}{ML^2}u\right]\Big\}\end{aligned} \tag{3.8}$$

Simulations were run for the dynamically PWM controlled manipulator (3.1),(3.8) with the same physical parameters, and the same (constrained) second order system parameter values as in the previous example. The PWM controller parameters were chosen as: $W = 24$, $T = 0.2$ [s], $r = 0.1$ [s/rad]. In this case the necessary and sufficient condition of Proposition 2.5 is verified as $rWT = 0.48 < 2$. Figure 3 depicts the state trajectories of the dynamically PWM controlled system clearly showing convergence of x_1 to the desired angular position and convergence to zero of the corresponding angular veloctiy. Figure 4 shows the PWM signal and the chattering-free applied torque input signal u, as generated by the dynamical PWM controller (3.8).

3.1.3 Dynamical SSM controller design for the manipulator system

Imposing now on the auxiliary output function $\sigma(z)$, the dynamics in (2.20) one easily obtains the following dynamical SSM controller:

$$\begin{aligned}\dot{u} = ML^2\Big\{&\left(\frac{k}{M} - 2\zeta\omega_n\right)\left(-\frac{g}{L}\sin x_1 - \frac{k}{M}x_2 + \frac{1}{ML^2}u\right) - \omega_n^2 x_2 + \frac{g}{L}x_2\cos(x_1) \\ &- W\,SSM\left[-\frac{g}{L}\sin x_1 + \omega_n^2(x_1 - x_{1d}) - \left(\frac{k}{M} - 2\zeta\omega_n\right)x_2 + \frac{1}{ML^2}u\right]\Big\}\end{aligned} \tag{3.9}$$

Simulations were run for the dynamically SSM controlled manipulator (3.1),(3.9) with the same physical parameters, and the same (constrained) second order system parameter values as in the previous examples. The SSM controller parameter were chosen as: $W = 24$ [rad/s], $T = 0.2$ [s]. Figure 5 depicts the state trajectories of the dynamically PWM controlled system clearly showing convergence to the desired angular position and convergence to zero of the corresponding angular veloctiy, with no chattering being exhibited. Figure 6 shows the evolution of the auxiliary output function signal and the applied torque input signal u, exhibiting a small chattering, as generated by the dynamical SSM controller (3.9).

3.2 Dynamical PFM, PWM and SSM Control of a double effect evaporator.

3.2.1 The double effect evaporator model

The following doble effect evaporator model is taken from Montano and Silva (1991).

$$\begin{aligned}\dot{x}_1 &= \delta_1 F_0(c_0 - x_1) + \delta_2 x_1 u \\ \dot{x}_2 &= \delta_3 F_0(x_1 - x_2) + \left[\delta_4 x_1 + \delta_5 x_2\right]u\end{aligned} \tag{3.10}$$

where x_1 represents the product concentration in the first stage of the evaporator, while x_2 stands for the output concentration of the product at the second stage. The control input u is a positive quantity representing the steam flow comming from a boiler. We take as the output of the system the output concentration error : $y = x_2 - x_{2d}$ with x_{2d} being the required constant value of the output product concentration. The rest of the parameters in (3.10) are assumed to be known positive constants, except for δ_4 which is negative. We summarize below the necessary steps to obtain a dynamical PFM regulator for the given system.

3.2.2 Design of a dynamical PFM regulator for concentration control in a double effect evaporator

Rank condition on the output

$$\det\begin{bmatrix}\frac{\partial y}{\partial x} \\ \frac{\partial \dot{y}}{\partial x}\end{bmatrix} = \det\begin{bmatrix}0 & 1 \\ \delta_3 F_0 + \delta_4 u & -\delta_3 F_0 + \delta_5 u\end{bmatrix} = -\left(\delta_3 F_0 + \delta_4 u\right)$$

The rank condition can only be violated by a constant value of the control input u given by : $u = -\delta_3 F_0/\delta_4$.

Input-dependent state coordinate transformation to obtain GOCF

$$z_1 = x_2 - x_{2d} \;;\; z_2 = \delta_3 F_0(x_1 - x_2) + \left[\delta_4 x_1 + \delta_5 x_2\right]u \tag{3.11}$$

$$x_1 = \frac{z_2 - \left(\delta_5 u - \delta_3 F_0\right)\left(z_1 + x_{2d}\right)}{\delta_3 F_0 + \delta_4 u} \;;\; x_2 = z_1 + x_{2d}$$

Generalized observability canonical form of the plant

$$\dot{z}_1 = z_2$$
$$\dot{z}_2 = \left(\delta_5 u - \delta_3 F_0\right) z_2 + \frac{\delta_3 F_0(\delta_4 + \delta_5)(z_1 + x_{2d}) + \delta_4 z_2}{\delta_3 F_0 + \delta_4 u}\dot{u}$$
$$+ \left(\delta_2 u - \delta_1 F_0\right)\left[\left(\delta_3 F_0 - \delta_5 u\right)(z_1 + x_{2d}) + z_2\right] + c_0 \delta_1 F_0\left(\delta_3 F_0 + \delta_4 u\right)$$
$$y = z_1 \tag{3.12}$$

Zero dynamics

$$\frac{\delta_3 F_0(\delta_4 + \delta_5)\ x_{2d}}{\delta_3 F_0 + \delta_4 u}\dot{u} + \left(\delta_2 u - \delta_1 F_0\right)\left[\left(\delta_3 F_0 - \delta_5 u\right) x_{2d}\right]$$
$$+ c_0 \delta_1 F_0\left(\delta_3 F_0 + \delta_4 u\right) = 0 \tag{3.13}$$

The equilibrium points of the zero dynamics are given by the real solutions of the following quadratic algebraic equation:

$$\left(\delta_2 u - \delta_1 F_0\right)\left[\left(\delta_3 F_0 - \delta_5 u\right) x_{2d}\right] + c_0 \delta_1 F_0\left(\delta_3 F_0 + \delta_4 u\right) = 0$$

The smallest root of the above polynomial equation corresponds to the stable equilibrium point for the zero dynamics.

Auxiliary output function in transformed and original coordinates

$$\sigma(z) = z_2 + \gamma_1 z_1$$
$$\sigma(x) = \delta_3 F_0(x_1 - x_2) + \left[\delta_4 x_1 + \delta_5 x_2\right] u + \gamma_1(x_2 - x_{2d}) \tag{3.14}$$

PFM controller in original coordinates

$$\dot{u} = \frac{1}{\delta_4 x_1 + \delta_5 x_2}\Big[-\left(\delta_3 F_0 + \delta_4 u\right)\left(\delta_1 F_0 (c_0 - x_1) + \delta_2 x_1 u\right)$$
$$- \left(\delta_5 u - \delta_3 F_0\right)\left(\delta_3 F_0 (x_1 - x_2) + \left[\delta_4 x_1 + \delta_5 x_2\right] u\right)$$
$$- \gamma_1\left(\delta_3 F_0 (x_1 - x_2) + \left[\delta_4 x_1 + \delta_5 x_2\right] u\right)$$
$$- W\, PFM_{\tau,T}\left(\delta_3 F_0(x_1 - x_2) + \left[\delta_4 x_1 + \delta_5 x_2\right] u + \gamma_1(x_2 - x_{2d})\right)\Big] \tag{3.15}$$

Impasse points for the dynamical controller occur on the line : $\delta_4 x_1 + \delta_5 x_2 = 0$, which may represent a physically meaningful condition, due to negativity of δ_4 and the possitivity of the concentrations. Results are valid far from this singularity condition.

3.2.3 Simulation results

The following parameter values were used in the simulations of the dynamical PFM controlled system (3.10), (3.15).

$$F_0 = 2.525 \text{ [Kg/min]},\ c_0 = 0.04,$$
$$\delta_1 = 0.0105,\ \delta_2 = 8.509 \times 10^{-3}$$
$$\delta_3 = 9.523 \times 10^{-3},\ \delta_4 = -7.699 \times 10^{-3}$$
$$\delta_5 = 10.304 \times 10^{-3}$$

The PFM controller parameters were set to be:

$$W = 8 \times 10^{-4}\ ;\ r_1 = 250\ ;\ r_2 = 300\ ;\ r_3 = 400$$
$$T_{max} = 2 \text{ [min]}\ ;\ T_{min} = 1 \text{[min]}\ ;\ \gamma_1 = 0.1$$

Figure 7 shows the state response of the dynamical PFM controlled system asymptotically converging toward the desired equilibrium point given by $x_{2d} = 0.0939$ while the concentration x_1 converges to its equilibrium value 0.07. Figure 8 depicts the smoothed control input trajectory. Rather similar responses were obtained for the PWM and the SSM control strategies based on the same auxiliary output function.

4. CONCLUSIONS

The feasibility of effective chattering-free discontinuous feedback controllers of the PFM, PWM and SSM types for robust stabilizaton of nonlinear systems has been demonstrated via use of dynamical feedback control strategies. These strategies are based on stabilization of suitably specified auxiliary output functions defined on the basis of generalized phase variables of Fliess's GOCF. Stabilizing sliding mode controllers, sampled or not, pulse-frequency-modulation, and pulse-width-modulation controller design procedures, for nonlinear dynamical plants, are unified via this technique, which is based on elementary results derived from the differential algebraic approach to system dynamics. The fundamental stability features and quality of the controlled responses obtained by using the various studied discontinuous feedback alternatives are basically the same. Their main differences being perhaps located on the transient responses. The results here presented can be extended to decouplable multivariable nonlnear plants and to other classes of dynamical systems.

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FIGURES

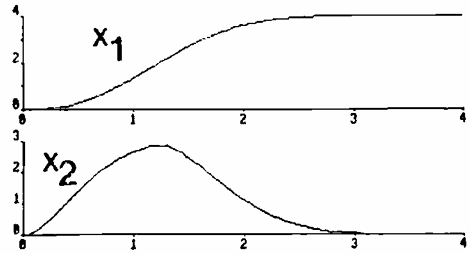

Figure 1. Dynamical PFM controlled response of robotic maniplator

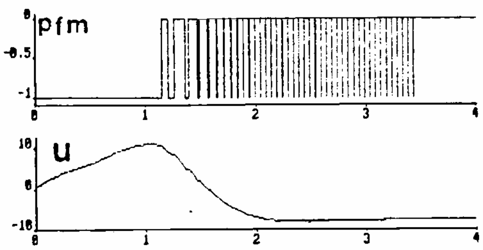

Figure 2. PFM input signal and applied input torque.

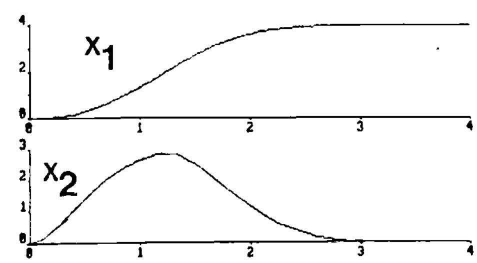

Figure 3.Dynamical PWM controlled response of robotic maniplator

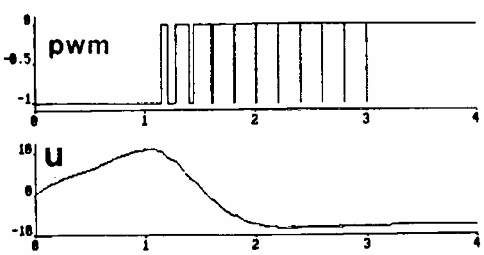

Figure 4. PWM input signal and applied input torque.

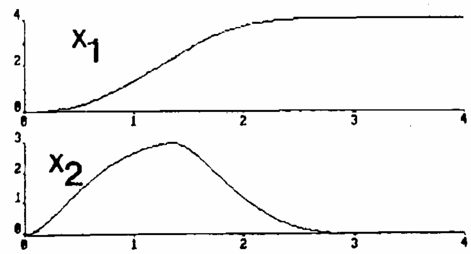

Figure 5. Dynamical SSM controlled response of robotic maniplator

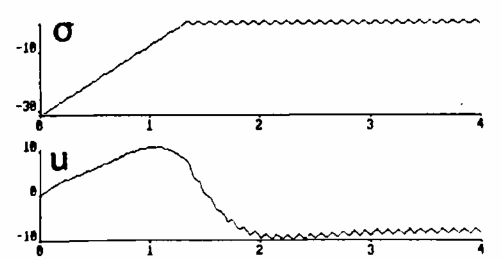

Figure 6. Auxiliary output function and applied input torque.

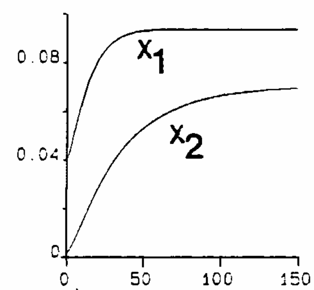

Figure 7. Dynamical PFM controlled response of doble effect evaporator system.

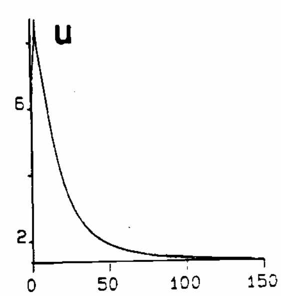

Figure 8. Applied input signal