DYNAMICAL DISCONTINUOUS FEEDBACK CONTROL OF NON-LINEAR SYSTEMS

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Summary

Discontinuous feedback stabilization of nonlinear systems, expressed in Generalized State representation form, is accomplished by zeroing of a suitable input-dependent manifold. Pulse-Frequency-Modulation, Pulse Width-Modulation and Sampled Sliding Mode Control strategies are treated from a unified viewpoint which naturally arises from fundamental results of the Differential Algebraic approach to systems dynamics and control. The approach naturally leads to dynamical discontinuous feedback policies resulting in (chattering-free) smoothed constrained linearization and induced robust asymptotic output error stabilization. The results are applicable in a variety of nonlinear control problems, including stabilization, tracking, and model matching. Illustrative examples from non-traditional application areas, such as chemical process control and hydraulic systems control, are presented with simulations.

Key Words: Discontinous Feedback Control, Generalized State Space Systems.

1. INTRODUCTION

Recently, results from the *differential algebraic* approach to control theory, pioneered by Prof. Michel Fliess [1]-[2], have greatly improved the applicability of discontinuous feedback strategies, specially those of the *sliding mode* (SM) type, leading to asymptotic stabilization, and tracking, in nonlinear systems (see Sira-Ramirez [3]-[4] for application examples in mechanical and electro-mechanical systems). Some of the traditional disadvantages of sliding mode control policies are fundamentally related to the chattering of input and state variables response signals (See Utkin [5]). These difficulties are easily circumvented via *dynamical* sliding mode controllers while retaining the outstanding robustness, and simplicity, of this class of feedback control schemes.

In this article, Fliess's Generalized Observability Canonical Form (GOCF) is shown to naturally allow for dynamical feedback controller design based on pulse-frequency-modulation (PFM) strategies, pulse-width-modulation (PWM) policies and sampled sliding modes (SSM). The obtained control input signals are substantially smoothed with respect to their corresponding static alternative and, hence, chattering-free discontinuously controlled responses are generated. The obtained PWM and PFM controller designs do not resort to traditional approximation schemes, based on (infinte frequency) *average* models, of the discontinuously controlled systems (see, Sira-Ramirez [6]). These features are particularly important in the regulation of mechanical and chemical systems, in which large and fast input vibrations, or jump discontinuities, cannot be simply allowed on the actuators, while a need still exists for certain degree of robustness and precision of the proposed control scheme.

The synthesis of the several dynamical discontinuous regulators, here proposed, is entirely based on Fliess' *Generalized Observability Canonical Form* (GOCF) for nonlinear systems (See [2]). In Section 2 of this article, we briefly address the dynamical SM control solution to the output stabilization problem and present the basic results of the PFM the PWM and the SSM controller design schemes. In section 3, we present non-traditional application examples on which we test each one of the proposed discontinuous feedback control techniques previously mentioned. The first application example, taken from Parrish and Brosilov [7], is concerned with the total concentration regulation in an isothermal Continuously Stirred Tank Reactor (CSTR). A second example deals with liquid level control in a coupled tank system. The presented examples include computer simulations. Concluding remarks are collected at the end of the article, in Section 4.

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2. DYNAMICAL DISCONTINUOUS FEEDBACK CONTROL OF NONLINEAR SYSTEMS

The results of this section are easily extended to tracking problems (see [3],[4]) and to multivariable cases.

2.1 Fliess's Generalized Observability Canonical Form.

It has been shown in [2] (see also Conte *et al* [8]) that a nonlinear, single-input single-output n- dimensional system given in *generalized state* representation form:

$$\begin{aligned} \dot{x} &= f(x,u,\dot{u},\cdots,u^{(\beta)}) \\ y &= h(x,u,\dot{u},\cdots,u^{(\beta)}) \end{aligned} \tag{2.1}$$

can be locally transformed, via an input-dependent state coordinate transformation of the form :

$$z = \Phi(x,u,\dot{u},\ldots,u^{(\alpha-1)}) \tag{2.2}$$

into a system of the form:

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ &\cdots \\ \dot{z}_n &= c(z,u,\dot{u},\ldots,u^{(\alpha)}) \\ y &= z_1 \end{aligned} \tag{2.3}$$

provided the following "observability" matrix of the system (2.1) is full rank:

$$\begin{bmatrix} \dfrac{\partial h(x,u,\dot{u},\cdots,u^{(\beta)})}{\partial x} \\ \dfrac{\partial h^{(1)}(x,u,\dot{u},\cdots,u^{(\beta+1)})}{\partial x} \\ \cdots \\ \dfrac{\partial h^{(n-1)}(x,u,\dot{u},\ldots,u^{(\alpha-1)})}{\partial x} \end{bmatrix} \tag{2.4}$$

In (2.3), α is assumed to be a strictly positive integer. The results can be extended to *systems exactly linearizable by static state feedback*, i.e., for systems in which $\alpha = 0$ (see Sira-Ramirez [9], and the second example presented in Section 3).

It must be remarked, however, that, in general, (2.3) may not be, necessarily, n-dimensional.

The input-dependent state coordinate transformation (2.2) is evidently given by the local diffeomorphism:

$$z=\Phi(x,u,u,...,u^{(\alpha-1)})=\begin{bmatrix} h(x,u,\dot{u},\cdots,u^{(\beta)}) \\ h^{(1)}(x,u,\dot{u},\cdots,u^{(\beta+1)}) \\ \cdots \\ h^{(n-1)}(x,u,u,...,u^{(\alpha-1)}) \end{bmatrix} \quad (2.5)$$

Suppose $u = U$, $x = X(U)$ describes a constant equilibrium point for the original system (2.1), such that $h(X(U),U, 0,...,0)$ is zero, then $z = 0$ is an equilibrium point of (2.3). The autonomous dynamics described by:

$$c(0,u,\dot{u},...,u^{(\alpha)}) = 0 \quad (2.6)$$

is the *zero dynamics* (see Fliess [10]). The stability nature of an equilibium point $u = U$ of (2.6) determines the *minimum* or *non-minimum phase* character of the system at the corresponding equilibrium point. We denote the above constant equilibrium point for system (2.1) as $(X(U),U,0)$.

2.2 A GOCF Approach to Dynamical Discontinuous Feedback Controller Design for Nonlinear Systems.

Consider the following auxiliary output function $s : R^n \to R$, defined in terms of the transformed variables z,

$$\sigma(z)=\left(\sum_{i=1}^{n-1}\gamma_i z_i\right)+z_n \quad (2.7)$$

such that the following corresponding polynomial in the complex variable λ is Hurwitz:

$$\sum_{i=1}^{n-1}\gamma_i\lambda^{i-1}+\lambda^{n-1} \quad (2.8)$$

It is easy to see that, provided the system is locally minimum phase, if (2.7) is forcefully constrained to zero (whether in finite time, or in an asymptotically stable fashion) by means of appropriate control actions (possibly of discontinuous nature), the resulting controlled dynamics locally evolves in accordance with:

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ &\cdots \\ \dot{z}_{n-1} &= -\sum_{i=1}^{n-1}\gamma_i z_i \\ y &= z_1 \end{aligned} \quad (2.9)$$

which is asymptotically stable to zero.

Two of the dynamical discontinuous feedback controller design schemes, here proposed, rely on inducing an asymptotically stable linear time invariant controlled dynamics such as (2.9), with eigenvalues placeable at will. This is done by driving the proposed auxiliary output function $\sigma(z)$ to zero. SM controllers can always accomplish such a task in finite time, PFM and PWM controllers, on the other hand, can only accomplish this task in an asymptotially stable fashion, while SSM control can only do it approximately.

Dynamical Sliding Mode Control of Nonlinear Systems

Proposition 2.1 Let W be a strictly positive quantity and let "sgn" stand for the *signum* function. The one dimensional discontinuous system :

$$\dot{\sigma} = -W \operatorname{sgn} \sigma \quad (2.10)$$

globally exhibits a sliding regime on $\sigma = 0$. Furthermore, any trajectory starting on the value $\sigma = \sigma(0)$, at time 0, reaches the condition $\sigma = 0$ in finite time T, given by : $T = W^{-1}|\sigma(0)|$.

Proof Immediate upon checking that globally: $\sigma\, d\sigma/dt < 0$ for $\sigma \neq 0$, which is a well known condition for sliding mode existence [5]. The second part follows trivially from the fact that $|\sigma(t)| = -Wt + |\sigma(0)|$ ■

Proposition 2.2 A minimum phase nonlinear system of the form (2.1) is locally asymptotically stabilizable to the equilibrium point $(U,X(U),0)$ if the control action u is specified as a dynamical SM control policy given by the solution of the following implicit, time-varying, nonlinear discontinuous differential equation :

$$c(z,u,\dot{u},\cdots,u^{(\alpha)}) = -\sum_{i=1}^{n}\gamma_{i-1}z_i - W\operatorname{sgn}[\sum_{i=1}^{n-1}\gamma_i z_i + z_n] \quad (2.11)$$

where $\gamma_0 = 0$.

Proof Immediate upon imposing on the auxiliary output function $\sigma(z)$ in (2.7) the discontinuous dynamics defined by (2.10). ■

We assume that in (2.11) the quantity $\partial c/\partial u^{(\alpha)}$ is locally nonzero and, hence, no singularities need to be locally considered.

Controller (2.11) is easily represented in terms of the original state space coordinates x by using the input dependent state coordinate transformation (2.5).

Dynamical PFM Control of Nonlinear Systems

Consider the scalar PFM controlled dynamical system, in which the constants r_1, r_2, r_3 and W, are all strictly positive quantities.

$$\dot{\sigma} = -W v$$

$$v = PFM_{\tau,T}(\sigma) = \begin{cases} \operatorname{sgn}\sigma(t_k) & \text{for } t_k \le t < t_k+\tau[\sigma(t_k)]T[\sigma(t_k)] \\ 0 & \text{for } t_k+\tau[\sigma(t_k)]T[\sigma(t_k)] \le t < t_k+T[\sigma(t_k)] \end{cases}$$

$$\tau[\sigma(t)] = \begin{cases} 1 & \text{for } |\sigma(t)| > \frac{1}{r_1} \\ r_1|\sigma(t)| & \text{for } |s(t)| \le \frac{1}{r_1} \end{cases}$$

$$T[\sigma(t)] = \begin{cases} T_{max} & \text{for} |\sigma(t)| \ge \frac{1}{r_2} \\ T_{min} + \frac{r_2 r_3}{r_3 - r_2}[T_{max}-T_{min}](\sigma(t)-\frac{1}{r_3}) & \text{for } \frac{1}{r_3} < |\sigma(t)| < \frac{1}{r_2} \\ T_{min} & \text{for} |\sigma(t)| \le \frac{1}{r_3} \end{cases}$$

$$k = 0,1,2,... \quad ; \quad t_{k+1} = t_k + T[\sigma(t_k)]. \quad (2.12)$$

where it is assumed that $r_2 < r_1 < r_3$. The t_k's represent irregularly spaced sampling instants, determined by the sampled values of the *duty cycle function*, denoted here by $T[\sigma(t_k)]$. The duty cycle function, $T[\sigma(t)]$, takes values on the closed interval $[T_{min}, T_{max}]$ and it varies linearly with respect to $\sigma(t)$ in the region $|\sigma| < 1/r_2$. The duty cycle, or sampling period, saturates to T_{max} for large values of σ, and remains fixed at the constant lower bound T_{min} for small values of σ. At each sampling instant, t_k, the value of the width of the sign-modulated, unit amplitude, control pulse is determined by the sampled value of the *duty ratio function*, represented by $\tau[\sigma(t_k)]$. In general, the duty cycle and the duty ratio functions may be quite independent of each other. The function "sgn" stands for the *signum* function.

The following proposition establishes a sufficient condition for the asymptotic stability to zero of the PFM controlled system (2.12).

Proposition 2.3 The PFM controlled system (2.12) is asymptotically stable to $\sigma = 0$, if

$$0 < r_3 W T_{max} < 2 \quad (2.13)$$

Proof Due to the piecewise constant nature of the control input and the linearity of the continuous system, it suffices to study the stability of the discretized version of (2.12) at the sampling instants. An exact discretization of the PFM controlled system (2.12) yields :

$$\sigma(t_k+T) = \sigma(t_k) - W\operatorname{sgn}[\sigma(t_k)]\,\tau[\sigma(t_k)]\,T[\sigma(t_k)] \quad (2.14)$$

Suppose the initial condition $\sigma(0)$ is chosen deep into the region $|\sigma| > 1/r_2$. The evolution of the sampled values of $\sigma(t)$ obey, according to (2.14):

$$\sigma(t_k+T)=\sigma(t_k) - W\,T_{max} \text{ for } \sigma(t_k) > 0$$
$$\sigma(t_k+T)=\sigma(t_k) + W\,T_{max} \text{ for } \sigma(t_k) < 0 \qquad (2.15)$$

Hence, given an arbitrary initial condition $\sigma(0)$ for σ, it is obvious from (2.15) that the condition : $0 < r_3WT_{max} < 2$ is sufficient to ensure that the value of $\sigma(t_k)$ will be eventually found within the bounded region $|\sigma| < 1/r_2$. This is due to the fact that the controlled increments taken by $\sigma(t_k)$, in the considered region $|\sigma| > 1/r_2$, are of width WT_{max} and, therefore, the condition: $WT_{max} < 2/r_3$ also guarantees that $WT_{max} < 2/r_2$. It follows that $\sigma(t_k)$ can not "jump" over the band $|\sigma| < 1/r_2$ and, hence, $\sigma(t_k)$ will land on this region for sufficiently large k. Two possibilities arise then : either $\sigma(t_k)$ is found in the "band" $1/r_3 < |\sigma(t_k)| < 1/r_2$, or $\sigma(t_k)$ satisfies $|\sigma(t_k)| < 1/r_3$. Suppose first that: $1/r_3 < |\sigma(t_k)| < 1/r_2$, for some k. In this region, the value of $|\sigma(t_k)|$ can only further decrease, as it is easily seen from (2.13). Indeed, the increments: $\Delta\sigma(t_k) = \sigma(t_{k+1}) - \sigma(t_k)$, taken by σ in the region $1/r_3 < |\sigma| < 1/r_2$, satisfy: $WT_{min} < |\Delta\sigma(t_k)| < WT_{max}$. Since, by assumption, $WT_{max} < 2/r_3$, then one has : $WT_{min} < |\Delta\sigma(t_k)| < WT_{max} < 2/r_3 < 2|\sigma(t_k)|$. It follows that $|\sigma(t_k)|$ further decreases and that the controlled evolution of $\sigma(t_k)$ will eventually reach the region: $|\sigma(t_k)| < 1/r_3$. In this last region the sampled values of σ evolve satisfying :

$$\sigma(t_k+T)= \sigma(t_k) - r_1 W\,T_{min}\ \text{sgn}[\sigma(t_k)]|\sigma(t_k)| = (1-r_1W\,T_{min})\,\sigma(t_k)$$

which is asymptotically stable to zero, if and only if : $0 < r_1WT_{min} < 2$. This last condition is evidently equivalent to $WT_{min} < 2/r_1$. Notice, however, that from the assumptions about the parameters in (2.12) : $WT_{min} < WT_{max} < 2/r_3 < 2/r_1$. i.e., the condition (2.13) implies the asymptotical stability requirement for (2.12). The result follows. ■

Proposition 2.4 A minimum phase nonlinear system of the form (2.1) is locally asymptotically stabilizable to the equilibrium point (U,X(U),0) if the control action u is specified as a dynamical PFM control policy given by the solution of the following implicit, time-varying, nonlinear discontinuous differential equation :

$$c(z,u,\dot{u},\cdots,u^{(\alpha)}) = -\sum_{i=1}^{n}\gamma_{i-1}z_i - W\,PFM_{\tau,T}[\sum_{i=1}^{n-1}\gamma_i z_i + z_n] \qquad (2.16)$$

where $\gamma_0 = 0$.

Proof Immediate upon imposing on the auxiliary output function $\sigma(z)$ in (2.7) the asymptotically stable discontinuous dynamics defined by (2.12). ■

Dynamical PWM Control of Nonlinear Systems

Consider the scalar PWM controlled system, in which $r > 0$ and $W > 0$:

$$\dot{\sigma} = -W\,v$$
$$v = PWM_\tau(\sigma) = \begin{cases} \text{sgn}\,\sigma(t_k) & \text{for } t_k \le t < t_k + \tau[\sigma(t_k)]T \\ 0 & \text{for } t_k+\tau[\sigma(t_k)]T \le t < t_k + T \end{cases} \qquad (2.17)$$
$$\tau[\sigma(t)] = \begin{cases} 1 & \text{for } |\sigma(t)| > \frac{1}{r} \\ r|\sigma(t)| & \text{for } |\sigma(t)| \le \frac{1}{r} \end{cases}$$
$$k = 0,1,2,\dots \quad ; \quad t_{k+1} = t_k + T.$$

where the t_k's represent regularly spaced sampling instants and "sgn" stands for the *signum* function.

It is easy to see that (2.17) is just a particular case of the PFM controlled system (2.12) in which the duty cycle function $T(\sigma(t_k))$ is now taken as a constant of value T. The following results follow immediately from this fact.

Proposition 2.5 The PWM controlled system (2.17) is asymptotically stable to $\sigma = 0$ if and only if:

$$0 < rWT < 2 \qquad (2.18)$$

Proof Sufficiency is clear from the preceeding proposition. Necesity follows from the fact that (2.18) is necessary to have $\sigma(t_k)$ lie in the region $|\sigma(t_k)| \le 1/r$, for some k, independently of the initial condition. In this region, the PWM controlled dynamics adopts the form $\sigma(t_{k+1}) = (1-rWT)\,\sigma(t_k)$. The result follows. ■

Proposition 2.6 A minimum phase nonlinear system of the form (2.1) is locally asymptotically stabilizable to the equilibrium point (U,X(U),0) if the control action u is specified as a dynamical PWM control policy given by the solution of the following implicit, time-varying, nonlinear discontinuous differential equation :

$$c(z,u,\dot{u},\cdots,u^{(\alpha)}) = -\sum_{i=1}^{n}\gamma_{i-1}z_i - W\,PWM_\tau[\sum_{i=1}^{n-1}\gamma_i z_i + z_n] \qquad (2.19)$$

where $\gamma_0 = 0$.

Proof Immediate upon imposing on the auxiliary output function $\sigma(z)$ in (2.7) the asymptotically stable discontinuous dynamics defined by (2.17). ■

Dynamical Sampled Sliding Mode Control of Nonlinear Systems

Proposition 2.7 Consider the following one-dimensional Sampled Sliding Mode controlled system:

$$\dot{\sigma} = -W\,v$$
$$v = SSM[\sigma(t)] = \text{sign}\,[\sigma(t_k)] \quad \text{for } t_k < t < t_k+T \qquad (2.20)$$
$$k = 0,1,\dots \ ; \quad t_{k+1} = t_k + T$$

Then, given an $\varepsilon > 0$, there exist a sampling interval $T(\varepsilon) = \varepsilon / W$ for which the trajectories satisfy the condition $|\sigma(t)| \le 2\,\varepsilon$ for all $t > T(\varepsilon)\,/\,|\sigma(0)|$.

Proof The proof is immediate from the exact discretization of (2.20):

$$\sigma(t_k+T) = \sigma(t_k) - W\,T\,\text{sign}[\,\sigma(t_k)\,]$$

hence,

$$|\,\sigma(t_k+T) - \sigma(t_k)\,| = W\,T$$

The first part follows by letting $WT = \varepsilon$. The second part is immediate from the linearity of the system and the fact that for all $t \ge 0$, $|\,d\sigma/dt\,| = W$ ■

Chattering of σ, around the value $\sigma = 0$, can be made of arbitrarily small amplitude, according to the width of the sampling interval $T(\varepsilon)$. As $T \to 0$, the response of σ to a SSM strategy asymptotically converges to the response of a SM policy.

Proposition 2.8 A minimum phase nonlinear system of the form (2.1) is locally stabilizable around the equilibrium point (U,X(U),0), modulo some small chattering, if the control action u is specified as a dynamical SSM control policy given by the solution of the following implicit, time-varying, nonlinear discontinuous differential equation :

$$c(z,u,\dot{u},\cdots,u^{(\alpha)}) = -\sum_{i=1}^{n}\gamma_{i-1}z_i - W\,SSM[\sum_{i=1}^{n-1}\gamma_i z_i + z_n] \qquad (2.21)$$

where $\gamma_0 = 0$.

Proof Immediate upon imposing on the auxiliary output function $\sigma(z)$ in (2.7) the discontinuous dynamics defined by (2.20). ■

A sampled sliding mode control policy may also be viewed as a particular case of a PWM control policy in which the pulse width $\tau[\sigma(t_k)]T$ is saturated to the value of the sampling interval T (i.e., the duty ratio, $\tau[\sigma(t_k)]$, is equal to 1 for all k).

3. SOME APPLICATION EXAMPLES

3.1 Dynamical Discontinuous Feedback Policies in Concentration Control for an Exothermic Continuously Stirred Tank Reactor.

Consider the following nonlinear dynamical controlled model of an exothermic reaction occuring inside a CSTR (see Parrish and Brosilov [7]), where the control objective is to regulate the outlet concentration through manipulation of the water jacket temperature :

$$\dot{x}_1 = \frac{F}{V}(c_0 - x_1) - a x_1 e^{-b/x_2}$$
$$\dot{x}_2 = \frac{F}{V}(T_0 - x_2) + \frac{aL}{c_p} x_1 e^{-b/x_2} - \frac{h}{Vc_p}(x_2 - u) \tag{3.1}$$
$$y = x_2 - T$$

Where x_1 represents the product concentration. The state variable x_2 represents the reactor temperature. The control variable u is the water jacket temperature. F is the reactor throughput in lb/hr, c_0 is the inlet flow concentration in lb/lb, T_0 is the inlet flow temperature measured in deg.R, c_p is the material heat capacity in BTU/lb.R while V and L are, respectively, the reactor holdup (in lb.) and the heat of the reaction (in BTU/lb.) . The constant h is the heat transfer parameter (in BTU/hr.R) , b is the activation constant (in deg R) and a is the pre-exponential factor in hr^{-1} . A constant temperature T is to be stably maintained to indirectly control the product concentration x_1 to its constant equilibrium value X_1.

A stable constant equilibrium point for this system is then given by :

$$x_2 = T \; ; \; x_1 = X_1(T) = \frac{c_0}{1 + \frac{V}{F} a e^{-b/T}} \; ;$$
$$u = U(T) = T - \frac{c_p F}{h}(T_0 - T) - \frac{aLV}{h} \frac{c_0 e^{-b/T}}{1 + \frac{V}{F} a e^{-b/T}} \tag{3.2}$$

We next summarize the design procedure leading to a dynamical stabilizing PFM controller for system (3.1), based on the GOCF of the system. As it is easily verified the relative degree of the system (3.1) is equal to one and, hence, the dimension of the zero dynamics is also one.

The input-dependent state coordinate transformation taking system (3.1) into GOCF is given by:

$$z_1 = x_2 - T$$
$$z_2 = \frac{F}{V}(T_0 - x_2) + \frac{aL}{c_p} x_1 e^{-b/x_2} - \frac{h}{Vc_p}(x_2 - u) \tag{3.3}$$

$$x_1 = \frac{c_p}{aL} e^{b/(z_1+T)} \left\{ z_2 - \frac{F}{V}(T_0 - T - z_1) + \frac{h}{Vc_p}(z_1 + T - u) \right\}$$
$$x_2 = z_1 + T \tag{3.4}$$

The resulting GOCF is then :

$$\dot{z}_1 = z_2$$
$$\dot{z}_2 = \frac{aL}{c_p} e^{-b/(z_1+T)} \Big[\frac{F}{V} c_0 - \frac{c_p}{aL} e^{b/(z_1+T)} \frac{F}{V}\left(1 + \frac{V}{F} a e^{-b/(z_1+T)} - \frac{bV}{F}\frac{z_2}{(z_1+T)^2}\right)$$
$$\left(z_2 - \frac{F}{V}(T_0 - T - z_1) + \frac{h}{Vc_p}(z_1 + T - u)\right) \Big] \tag{3.5}$$
$$-\left(\frac{F}{V} + \frac{h}{Vc_p}\right) z_2 + \frac{h}{Vc_p}\dot{u}$$
$$y = z_1$$

For discontinuous feedback stabilization purposes we propose, in transformed coordinates, the following auxiliary output function:

$$\sigma = z_2 + \gamma_1 z_1 \quad ; \quad \gamma_1 > 0 \tag{3.6}$$

Evidently, the restricted asymptotically sable motions of the controlled dynamics when σ is forced to zero are governed by:

$$\dot{z}_1 = -\gamma_1 z_1 \tag{3.7}$$

The dynamical PFM controller, in transformed coordinates, is given by:

$$\dot{u} = \frac{Vc_p}{h} \Bigg\{ -\frac{aL}{c_p} e^{-b/(z_1+T)} \Big[\frac{F}{V} c_0 - \frac{c_p}{aL} e^{b/(z_1+T)} \frac{F}{V}\left(1 + \frac{V}{F} a e^{-b/(z_1+T)} - \frac{bV}{F}\frac{z_2}{(z_1+T)^2}\right)$$
$$\left(z_2 - \frac{F}{V}(T_0 - T - z_1) + \frac{h}{Vc_p}(z_1 + T - u)\right) \tag{3.8}$$
$$+ \left(\frac{F}{V} + \frac{h}{Vc_p} - \gamma_1\right) z_2 - W\,PFM_{\tau,T}[\sigma(z,u)] \Big] \Bigg\}$$

The zero dynamics results in an asymptotically stable system of the form:

$$\dot{u} = -\frac{F}{V}\left(1 + \frac{V}{F} a e^{-b/T}\right) \left[u - T + \frac{c_p F}{h}(T_0 - T) + \frac{aLV}{h} \frac{c_0 e^{-b/T}}{1 + \frac{V}{F} a e^{-b/T}} \right] \tag{3.9}$$

In original coordinates the dynamical PFM controller and the corresponding auxiliary output function are obtained as:

$$\dot{u} = \frac{Vc_p}{h} \Bigg\{ \left(\frac{F}{V} + \frac{h}{Vc_p} - \frac{abL}{c_p} e^{-b/x_2} \frac{x_1}{x_2^2} - \gamma_1 \right) \frac{F}{V}(T_0 - x_2) +$$
$$\frac{aL}{c_p} x_1 e^{-b/x_2} - \frac{h}{Vc_p}(x_2 - u) - \frac{aL}{c_p} e^{-b/x_2}\left(\frac{F}{V}(c_0 - x_1) - a x_1 e^{-b/x_2} \right) \tag{3.10}$$
$$- W\,PFM_{\tau,T}[\sigma(x,u)] \Bigg\}$$

$$\sigma(x,u) = \frac{F}{V} T_0 - \gamma_1 T - \left(\frac{F}{V} + \frac{h}{Vc_p} - \gamma_1 \right) x_2 + \frac{aL}{c_p} x_1 e^{-b/x_2} + \frac{h}{Vc_p} u \tag{3.11}$$

Dynamical PWM Controller Design

The dynamical PWM controller is readily obtained by imposing on the auxiliary output function, previously defined in (3.6), or (3.11), the closed loop PWM regulated dynamics in (2.17). This results in :

$$\dot{u} = \frac{Vc_p}{h} \Bigg\{ \left(\frac{F}{V} + \frac{h}{Vc_p} - \frac{abL}{c_p} e^{-b/x_2} \frac{x_1}{x_2^2} - \gamma_1 \right) \frac{F}{V}(T_0 - x_2) +$$
$$\frac{aL}{c_p} x_1 e^{-b/x_2} - \frac{h}{Vc_p}(x_2 - u) - \frac{aL}{c_p} e^{-b/x_2}\left(\frac{F}{V}(c_0 - x_1) - a x_1 e^{-b/x_2} \right) \tag{3.12}$$
$$- W\,PWM_{\tau}[\sigma(x,u)] \Bigg\}$$

Dynamical Sampled Sliding Mode Controller Design

Similarly, a dynamical SSM controller is obtained as:

$$\dot{u} = \frac{Vc_p}{h} \Bigg\{ \left(\frac{F}{V} + \frac{h}{Vc_p} - \frac{abL}{c_p} e^{-b/x_2} \frac{x_1}{x_2^2} - \gamma_1 \right) \frac{F}{V}(T_0 - x_2) +$$
$$\frac{aL}{c_p} x_1 e^{-b/x_2} - \frac{h}{Vc_p}(x_2 - u) - \frac{aL}{c_p} e^{-b/x_2}\left(\frac{F}{V}(c_0 - x_1) - a x_1 e^{-b/x_2} \right) \tag{3.13}$$
$$- W\,SSM[\sigma(x,u)] \Bigg\}$$

where $\sigma(x,u)$ is as in (3.11) and the SSM control operation is defined in (2.20).

3.2 Simulation Results

Simulations were performed for a discontinuously controlled exothermic CSTR (3.1) using the PFM, the PWM and the SSM controllers obtained in (3.10), (3.12)and (3.13), with the same input-dependent stabilizing manifold (3.11) for the three controllers. The CSTR is characterized by the following parameters, taken after [7]:

$F = 2000$ lb/hr ; $c_0 = 0.50$ lb/lb ; $V = 2400$ lb. ;
$a = 7.08 \times 10^{10}\ hr^{-1}$; $b = 15080$ deg R ; $T_0 = 5320$ deg.R ;

$L = 600$ BTU/lb. ; $c_p = 0.75$ BTU/lb.R ;
$h = 15000$ BTU/hr.R ;

For such parameter values, the equilibrium point (3.2) of system (3.1) results in :

$$X_2(T) = T = 600 \text{ deg R} ; \; u = U(600) = 107.679 \text{ deg. R} ;$$
$$X_1(600) = 0.246 \text{ lb/lb}.$$

The PFM controller parameters were chosen as:
$\gamma_1 = 8$, $W = 50$, , $T_{max} = 0.4$ hr, $T_{min} = 5 \times 10^{-2}$hr ,
$r_1 = 4 \times 10^{-2}$, $r_2 = 2 \times 10^{-2}$, $r_3 = 0.3$.

The PWM controller parameters were set to be:

$$\gamma_1 = 8, \; W = 30, \; T = 0.1 \text{ hr}, \; r_1 = 0.1$$

Design parameters for the SSM controller were chosen as:

$$\gamma_1 = 8, \; W = 35$$

Figure 1 to 3 portray the time responses of the dynamical discontinuously controlled state variables, x_1 and x_2, the chattering-free (smoothed) continuous control input trajectory, $u(t)$, and the evolution of the auxiliary output function $\sigma(x,u)$, for the PFM, the PWM and the SSM controlled system, respectively.

3.3 Dynamical PFM Feedback Control in the Regulation of Liquid Level in a Coupled Tank System.

The dynamical (normalized) state space model of two coupled tanks, a conical one discharging liquid on a cylindrical one placed below, is given, according to [11], by the following system:

$$\begin{aligned} \dot{x}_1 &= -x_1^{-3/2} + x_1^{-2} u \\ \dot{x}_2 &= x_1^{1/2} - x_2^{1/2} \\ y &= x_2 \end{aligned} \qquad (3.14)$$

where x_1 and x_2 are, respectively, the heigths of the liquid levels on the upper and lower tanks. The control input u represents the input flow to the first tank.

System (3.14) is a globally feedback linearizable system, and, as such, only static (i.e., chattering) discontinuous feedback controllers may be designed when using the techniques proposed in Section 2. From a practical stand point a bang-bang behavior for the input flow u is quite unacceptable. To circumvent this limitation, we develop below a dynamical discontinuous feedback policy for system (3.14), by first resorting to the "extended system" (See Nijmeijer and van der Schaft [12]). We shall only present the dynamical PFM controller case.

The extended system, with auxiliary input v, is given by:

$$\begin{aligned} \dot{x}_1 &= -x_1^{-3/2} + x_1^{-2} x_3 \\ \dot{x}_2 &= x_1^{1/2} - x_2^{1/2} \\ \dot{x}_3 &= v \\ y &= x_2 - X \end{aligned} \qquad (3.15)$$

where X is the desired equlibrium level for the liquid in the second tank.

The GOCF of system (3.15) is obtained as :

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ \dot{z}_3 &= 2\{z_2+[z_1+X]^{1/2}]^2 \left(z_3+\tfrac{1}{2}[z_1+X]^{-1/2}z_2\right) \times \\ &\Big[\left(z_2+[z_1+X]^{1/2}\right)^{-7} + \tfrac{1}{4}[z_1+X]^{-1/2} - \tfrac{5}{2}\left(z_2+[z_1+X]^{1/2}\right)^{-3} \\ &\left(z_3+\tfrac{1}{2}\left(z_2+[z_1+X]^{1/2}\right)^{-4} + \tfrac{1}{2}\left(z_2+[z_1+X]^{1/2}\right)[z_1+X]^{-1/2} - \tfrac{1}{2}\right)\Big] \\ &+\tfrac{1}{2}\left(z_2+[z_1+X]^{1/2}\right)^{-5} v + \tfrac{1}{4}\left(z_2+[z_1+X]^{1/2}\right)[z_1+X]^{-3/2} z_2 \\ y &= z_1 \end{aligned} \qquad (3.16)$$

The stabilizing auxiliary output function, in transformed and original coordinates, is given by :

$$\sigma(z) = \gamma_1 z_1 + \gamma_2 z_2 + z_3 \qquad (3.17)$$

$$\begin{aligned} \sigma(x,u) &= \gamma_1(x_2 - X) + \gamma_2(x_1^{1/2} - x_2^{1/2}) + \\ &\tfrac{1}{2} - \tfrac{1}{2} x_1^{1/2} x_2^{-1/2} - \tfrac{1}{2} x_1^{-2}(1 - x_1^{-1/2} u) \end{aligned} \qquad (3.18)$$

Imposing on (3.17) the closed loop stable PFM dynamics in (2.12) we obtain a static PFM controller for the auxiliray input function v. This static controller, however, constitutes a dynamical PFM feedback controller when written in original coordinates x, u:

$$\begin{aligned} v &= -4\left(z_3+\tfrac{1}{2}[z_1+X]^{-1/2}z_2\right) \times \\ &\left[1 + \tfrac{1}{4}\xi^7[z_1+X]^{-1/2} - \tfrac{5}{2}\xi^4\left(z_3+\tfrac{1}{2}\xi^{-4} + \tfrac{1}{2}\xi[z_1+X]^{-1/2} - \tfrac{1}{2}\right)\right] + \\ &+\xi^5\left[-\tfrac{1}{2}\xi\,[z_1+X]^{-3/2}z_2 - 2\gamma_2 z_3 - 2\gamma_1 z_2 - 2W\,\mathrm{PFM}_{\tau,T}[\sigma(z)]\right] \end{aligned} \qquad (3.19)$$

with:

$$\xi = z_2 + [z_1 + X]^{1/2}$$

The zero dynamics, associated to the dynamical closed loop system, is readily obtained from (3.16) as:

$$\dot{u} = 0 \qquad (3.20)$$

Eventhough the system is not minimum phase, the discontinuous nature of the dynamic PFM controller is seen to drive to zero any incipient deviation from the equilibrium values of the state and control input. It is in this sense that roubustness against perturbations is preserved by the dynamic discontinuous controller.

3.4 Simulation Results

Simulations were performed for a dynamical PFM feedback controlled tank system (3.14) with dynamical controller (3.19) and stabilizing output function (3.18). The parameter values for the system and the controller were chosen as:

$$X = 1.0 \; ; \; \gamma_1 = 0.16 \; , \; \gamma_2 = 0.72 \; ; \; T_{max} = 2 \; ; \; T_{min} = 1$$
$$r_1 = 200 \, , \; r_2 = 150 \, , \; r_3 = 250.$$

The equilibrium point of the controlled system is :

$$X_1(X) = X_2(X) = X = 1 \; ; \; u = U(1) = 1.$$

Figure 4 portrays the asymptotically stable time response of the dynamical PFM controlled state variables, x_1 and x_2, the chattering-free (smoothed) continuous control input trajectory, $u(t)$, and the evolution of the auxiliary output function $\sigma(x,u)$.

4. CONCLUSIONS

The feasibility of effective chattering-free discontinuous feedback controllers of the PFM, PWM and SSM types for robust stabilizaton of nonlinear systems has been demonstrated via use of dynamical feedback control strategies. These strategies are based on stabilization of suitably specified auxiliary output functions defined on the basis of generalized phase variables of Fliess's GOCF. Stabilizing sliding mode controllers, sampled or not, pulse-frequency-modulation, and pulse-width-modulation controller design procedures, for nonlinear dynamical plants, are unified via this technique, which is based on elementary results derived from the differential algebraic approach to system dynamics. The results can be extended to multivariable plants and to other classes of dynamical systems.

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FIGURES

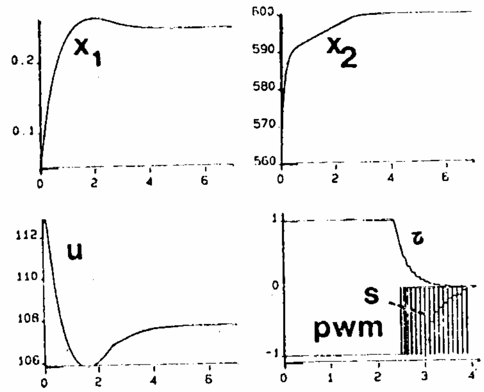

Figure 2. State and input trajectories of dynamically PWM controlled exothermic CSTR system.

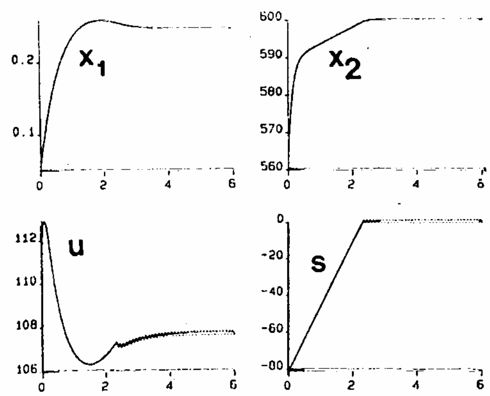

Figure 3. State and input trajectories of dynamically SSM controlled exothermic CSTR system.

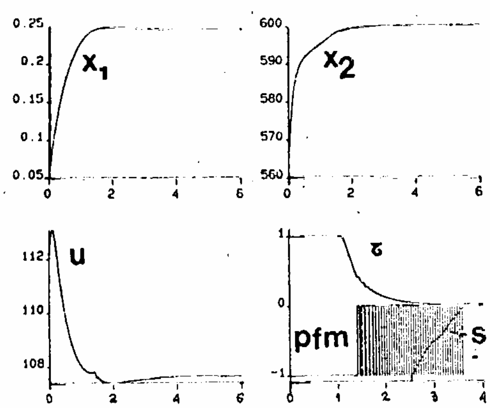

Figure 1. State and input trajectories of dynamically PFM controlled exothermic CSTR system.

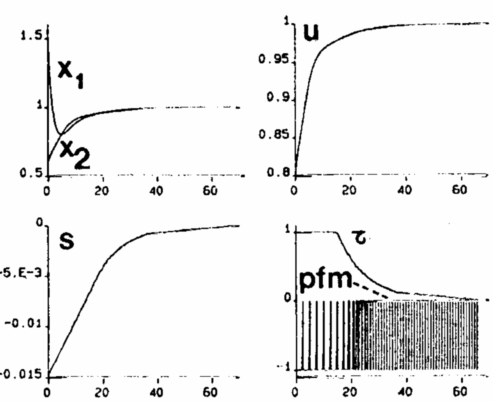

Figure 4. State and input trajectories of dynamically PFM controlled coupled tank system.