SLIDING MODE CONTROL AND DIFFERENTIAL ALGEBRA.

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Summary Sliding mode control of nonlinear systems is reformulated from a differential algebraic perspective. Input-dependent sliding surfaces, possibly including time derivatives of the input signal, naturally arise from elementary differential algebraic results as applied to nonlinear controlled systems. This viewpoint directly leads to consider the possibilities of dynamically generated sliding mode controllers, characterizing sliding motions generically devoid of undersirable input chattering. A definite connection between controllability and the possibility of creation of "higher order" sliding regimes is established via differential algebra.

Key words Nonlinear Systems, Sliding Regimes, Differential Algebraic Systems.

1. INTRODUCTION

Sliding mode control of dynamical systems has a long history of theoretical and practical developments. A thorough chronological collection of journal articles, and conference presentations, has been gathered by Professor S.V. Emelyanov (1989), (1990a), while detailed surveys have been produced over the years by Utkin (1977), (1984), (1987). Background on the subject may be acquired from the books written by Emelyanov (1967), Utkin (1978), (1992), Itkis (1976) and Bühler (1986).

Recent developments in nonlinear systems include the use of *Differential Algebra* (see Kolchin 1973) for the formulation, understanding, and conceptual solution of long standing problems in automatic control. Developments in this area are fundamentally due to Prof. M. Fliess (1986,1988, 1989a, 1989b,1989c,1990a, 1990b). Some other pioneering contributions were also independently presented by Pommaret (1983, 1986). Sliding mode control, and discontinuous feedback control, in general, have also benefited from this new trend. A seminal contribution in the use of differential algebraic results to sliding mode control was given by Fliess and Messager (1990). In that article, an example was presented in which no continuous feedback controller can achieve asymptotic stability to the origin, while a discontinuous feedback controller, based on sliding modes, does result in asymptotically stable behavior. These basic results were later extended, and used, in several case studies, by Sira-Ramirez *et al* (1992), Sira-Ramirez and Lischinsky-Arenas (1991) and by Sira-Ramirez (1992a-1992d). A most interesting article dealing with multivariable linear systems and the regulation of non-minimum phase systems is that of Fliess and Messager (1991). Extensions to pulse-width-modulation and pulse-frequency-modulation control strategies have been contributed by Sira-Ramirez (1991b, 1992e, 1992f).

This article is an attempt to present a reapproachment to sliding mode control theory, from the differential algebraic viewpoint. Section 2 of this article is devoted to present an application example which utilizes a sliding surface which not only depends on the state of the system, but also on the system's input. A dynamical sliding mode controller is then synthesized which produces smoothed feedback signals to the plant. This example motivates the need for the more general class of sliding surfaces and the several formalizations that follow. Section 3 presents some implications of the differential algebraic approach in sliding mode controller synthesis. In particular, a clear connection is established between "higher order" sliding motions and controllability. Some suggestions for further research are pointed at the end, in the conclusions section.

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2. A DYNAMICAL SLIDING MODE CONTROLLER EXAMPLE

A rather general model for nonlinear single-input single-output nonlinear systems is constituted by the following n-dimensional analytic system, in Kalman form :

$$\dot{x} = f(x,u) \qquad (2.1)$$

Suppose a smooth feedack control law of the form:

$$u = -k(x) \qquad (2.2)$$

renders a desired closed loop behavior for the given system.

Consider now an auxiliary output function y which measures the "implementation" error of the above feedback controller:

$$y = u + k(x) \qquad (2.3)$$

One may then impose the condition of zeroing, in finite time, the auxiliary output function y by imposing on its evolution the following autonomous discontinuous dynamics:

$$\dot{y} = -W \,\mathrm{sign}(y) \qquad (2.4)$$

Computing the required control signal u one obtains, after using (2.1) (2.2) and (2.4):

$$[\frac{\partial k}{\partial x}]\, f(x,u) + \frac{du}{dt} = -W \,\mathrm{sign}[u + k(x)] \qquad (2.5)$$

which may be viewed as a first order, time-varying, ordinary differential equation for u, with discontinuous right hand side :

$$\frac{du}{dt} = -[\frac{\partial k}{\partial x}]\, f(x,u) - W \,\mathrm{sign}[u + k(x)] \qquad (2.6)$$

The ideal sliding mode behavior, obtained on the input-dependent manifold $y = u + k(x) = 0$, is represented by the desired closed loop system (2.1), (2.2).

The control input discontinuies associated to the sliding mode control scheme would be now adscribed to the first time derivative of the control input. Hence, the input u does not exhibit the traditional bang-bang chattering associated to static sliding modes. The dynamical discontinuous feedback control scheme represented by (2.6) is depicted in figure 1. Figure 2 shows a reinterpretation of (2.6) by adscribing the feedback integrator to each feedback path. This clearly explains why the proposed dynamical scheme is deemed to be robust with respect to feedback input errors.

We illustrate the above developments by means of a simple first order mechanical system example in which a static discontinuous feedback controller is not technically convenient and the above dynamical discontinuous feedback alternative is seen to be specially advantageous.

Example (Dynamical sliding mode control of a pressure tank system).

Consider a steam tank with controlled charge, operating under subcritical flow conditions (see Keckman, 1988). The system is described by the following first order controlled ordinary differential equation:

$$\frac{dP}{dt} = -\frac{R\tau}{V} K_0A_0\sqrt{P_0(P-P_0)} + \frac{R\tau}{V} u \qquad (2.7)$$

where P is the preasure of the steam inside the tank, which is to be regulated to an equilibrium value P(U) depending on the constant value U of the flow rate u, acting as a control variable. R is the gas constant, τ is the temperature, assumed to be constant throughout (i.e., the process is assumed to be isothermal). P_0 is the downstream pressure, which is also assumed to be constant. V is the volume of the tank and A_0 is the constant cross-sectional area through which the gas flows, while K_0 is a constant associated to the valve and gas characteristics. For a given constant input flow rate U the corresponding equilibrium pressure P(U) is, simply given by:

$$P(U) = P_0 + \frac{U^2}{K_0^2A_0^2P_0} \qquad (2.8)$$

The assumption regarding a subcritical flow condition roughly implies that $P < 2\ P_0$. Hence, under equilibrium conditions, it must also be true that:

$$\frac{U^2}{K_0^2A_0^2P_0} < P_0 \quad \Leftrightarrow \quad U < P_0K_0A_0 \qquad (2.9)$$

A sliding mode controller is easily designed by considering as an output of (2.7) the pressure error: $y = P - P(U)$. Imposing on y the discontinuous dynamics in (2.4), one obtains:

$$u = -\frac{WV}{R\tau} \text{sign}\,[P-P(U)] + K_0A_0\sqrt{P_0(P-P_0)} \qquad (2.10)$$

Simulations of the controlled response of (2.7), (2.10) are shown in figure 3. The tank pressure and the control input trajectory are portrayed in this figure. The excesive chattering appearing in the input trajectory makes the approach rather impractical.

It is easy to see that the following continuous feedback controller achieves exact linearization of the tank pressure dynamics with asymptotically stable response toward the equilibrium value P(U).

$$u = -\frac{V}{R\tau}\lambda[\,P - P(U)] + K_0A_0\sqrt{P_0(P-P_0)} \qquad (2.11)$$

where λ is a positive design constant. Indeed, substitution of (2.11) into (2.7) leads to:

$$\frac{dP}{dt} = -\lambda[P-P(U)] \qquad (2.12)$$

Consider now the auxiliary output function given by :

$$y = u + \frac{V}{R\tau}\lambda[\,P - P(U)] - K_0A_0\sqrt{P_0(P-P_0)} \qquad (2.13)$$

Imposing on the new output function y the dynamics in (2.4), one obtains the following dynamical sliding mode controller:

$$\begin{aligned} \dot{u} + \lambda u - \frac{K_0A_0P_0}{2\sqrt{P_0(P-P_0)}}\frac{R\tau}{V}u = \\ -\frac{K_0^2A_0^2P_0}{2}\frac{R\tau}{V} + \lambda K_0A_0\sqrt{P_0(P-P_0)} \\ -W\,\text{sign}\left[u + \frac{V}{R\tau}\lambda[\,P - P(U)] - K_0A_0\sqrt{P_0(P-P_0)}\right] \end{aligned} \qquad (2.14)$$

Simulations of the controlled response of (2.7), (2.14) are shown in figure 4. In this figure the tank pressure is shown to asymptotically converge to the desired equilibrium while the smoothed (non-chattering) control input trajectory also converges to the prescribed constant flow rate.

3. A DIFFERENTIAL ALGEBRAIC APPROACH TO SLIDING MODE CONTROL OF NONLINEAR SYSTEMS

3.1 Nonlinear Controlled Dynamics and Sliding Regimes

In this section we closely follow Fliess's differential algebraic approach to systems dynamics (see Fliess 1986,1988, 1989a, 1989b,1989c,1990a, 1990b) for the basic definitions and results.

Definition 3.1 Consider an ordinary differential field k of characteristic zero. A *system* is any finitely generated differential field extension of k, denoted by K/k.

Let u be a *transcendence element* of the system K/k. u is then a *differential indeterminate* representing the input to the system. By itself, u is then assumed not to satisfy any algebraic differential equation with coefficients in k. We say that u qualifies as a *differential transcendence element , or basis* , of K/k.

The field extension k<u> denotes the smallest differential field containing, both, k and u. The field extension k<u> is also referred to as the field *generated* by k and u.

Definition 3.2 A *dynamics* is defined as a *finitely generated differentially algebraic extension* K/k‹u› of the differential field k‹u›.

It is well known that if u is a differential transcendence basis of K/k then, the extension K/k<u> is differentially algebraic.

Proposition 3.3 Suppose $x = (x_1,x_2,\ldots,x_n)$ is a *nondifferential transcendence basis* of K/k‹u›, then, the derivatives dx_i/dt (i=1,...,n) are k‹u›-algebraically dependent on the components of x.

Proof: immediate. ■

One of the consequence of the last result, drawn by Fliess (1990a), is that a more general and natural representation of nonlinear systems requires *implicit algebraic differential equations*. Indeed, from the above proposition, it follows that there exist exactly n polynomial differential equations with coefficients in k, of the form:

$$P_i(\dot{x}_i, x, u, \dot{u}, \cdots, u^{(\alpha)}) = 0\ ;\ i=1,\cdots,n \qquad (3.1)$$

implicitly describing the controlled dynamics.

It has been shown by Fliess and Hassler (1990) that such implicit representations are not entirely unusual in physical examples. The more traditional form of the state equations, known as *normal form* is recovered, in a local fashion, under the assumption that such polynomials locally satisfy the following rank condition :

$$\text{rank}\begin{bmatrix} \frac{\partial P_1}{\partial \dot{x}_1} & 0 & 0 \\ \vdots & \ddots & \vdots \\ 0 & 0 & \frac{\partial P_n}{\partial \dot{x}_n} \end{bmatrix} = n$$

The time derivatives of the x_i's may then be, locally, solved for as:

$$\dot{x}_i = p_i(x,u,\dot{u},\cdots,u^{(\alpha)}) = 0 \ ; \ i = 1,\cdots,n \qquad (3.2)$$

The representation (3.2) is now known as the *generalized state representation* of a nonlinear dynamics.

Consider a (nonlinear) dynamics K/k<u>. Let, furthermore, $\zeta = (\zeta_1,...,\zeta_n)$ be a *non-differential transcendence baisis* for K, i.e., the (non-differential) *transcendence degree* of K/k<u> is, then, assumed to be n.

Definition 3.4 A *first order sliding surface* is any element σ of the dynamics K/k<u> such that its time derivative $d\sigma/dt$ is k<u>-algebraically dependent on ζ. That is, there exists a polynomial S over k such that :

$$S(\dot{\sigma}, \zeta, u, \dot{u},...,u^{(\beta)}) = 0 \qquad (3.3)$$

Remark A more traditional definition of sliding surface coordinate function is related to the fact that no input signals, nor any of its time derivatives, were customarily allowed to be part of the expression defining such a sliding surface candidate. In this unnecessarily restricted sense, the sliding surface was allowed to be an (algebraic) function of the state components only. Moreover, for systems in "Kalman form", described by a state vector ξ, the time derivative of the sliding surface was required to be algebraically dependent only on ξ and u. Hence, all the resulting sliding mode controllers were, necessarily, of static nature. The differential algebraic approach naturally points to the possibilities of dynamical sliding mode controllers, specially in nonlinear systems where elimination of input derivatives may not be possible at all (see Fliess *et al* 1991, for a physical example of this nature). ■

One generalizes the above definition by considering "higher order" sliding surface candidates.

Definition 3.5 A *p-th order sliding surface* is any element σ of the dynamics K/k<u> such that its p-th order time derivative is k<u>-algebraically dependent on ζ. That is, there exists a polynomial *S* over k such that :

$$S(\sigma^{(p)}, \zeta, u, \dot{u},...,u^{(\gamma)}) = 0 \qquad (3.4)$$

This definition gives rise to the possibilities of a smoothed asymptotic approach to the zero "level set" of the sliding surface σ through a discontinuous feedback policy. The implications will be explored in detail in Section 3.3, below. Notice that the integer p is not necessarily the first higher order time derivative of σ for which a k<u>- algebraic dependence on ζ may be establised. Thus, a p-th order sliding surface candidate might have also qualified as a lower order sliding surface candidate.

Suppose σ is a first order sliding surface candidate. Imposing on σ a discontinuous sliding dynamics of the form:

$$\dot{\sigma} = -W \ \mathrm{sign}(\sigma) \qquad (3.6)$$

one obtains, from (3.3), an *implicit dynamical sliding mode controller* given by :

$$S(-W \ \mathrm{sign}(\sigma), \zeta, u, \dot{u},...,u^{(\beta)}) = 0 \qquad (3.7)$$

which is to be viewed as an implicit, time-varying, discontinous ordinary differential equation for the control input u.

The two "structures" associated to the underlying variable structure control system are represented by the pair of implicit dynamical controllers:

$$\begin{aligned} &S(W, \zeta, u, \dot{u},...,u^{(\beta)}) = 0 \ ; \ \text{for } \sigma > 0 \\ &S(-W, \zeta, u, \dot{u},...,u^{(\beta)}) = 0 \ ; \ \text{for } \sigma < 0 \end{aligned} \qquad (3.8)$$

each one valid, respectively, on one of the "regions" : $\sigma > 0$ and $\sigma < 0$. Precisely on the condition $\sigma = 0$ neither one of the control structures is valid.

We formally define the *equivalent control dynamics* as the dynamical state feedback control law obtained by letting $d\sigma/dt$ become zero in (3.3), and considering the resulting implicit differential equation for u :

$$S(0, \zeta, u_{EQ}, \dot{u}_{EQ},...,u_{EQ}^{(\beta)}) = 0 \qquad (3.9)$$

Suppose now that in (3.3) $\partial S/\partial(d\sigma/dt) \neq 0$, then one locally obtains:

$$\dot{\sigma} = s(\zeta, u, \dot{u},...,u^{(\beta)}) \qquad (3.10)$$

and the corresponding dynamic sliding mode controller, complying with (3.6), is given by:

$$s(\zeta, u, \dot{u},...,u^{(\beta)}) = -W \ \mathrm{sign}(\sigma) \qquad (3.11)$$

If, furthermore, $\partial s/\partial u^{(\beta)}$ is non zero, one locally obtains an explicit time-varying state space representation for the dynamical sliding mode controller (3.11), in the form:

$$\begin{aligned} &\dot{u}_1 = u_2 \\ &\dot{u}_2 = u_3 \\ &\ \vdots \\ &\dot{u}_\beta = \theta(u_1,\cdots,u_\beta, \zeta, W\mathrm{sign}(\sigma)) \\ &u = u_1 \end{aligned} \qquad (3.12)$$

All discontinuities arising from the bang-bang control policy are seen to be confined to the highest derivative of the control input through the nonlinear function θ. The output u of the dynamical controller is clearly the outcome of β integrations performed on such discontinuous time derivative of u and, for this reason, the signal u, emerging from the controller, is sufficiently smoothed out.

3.2 Dynamical sliding regimes based on Fliess's Generalized Controller Canonical Form.

The following theorem plays a fundamental role in the study of systems dynamics from the differential algebraic approach (Fliess, 1990a).

Theorem 3.6 Let K/k<u> be a dynamics. Then, there exists an element $\xi \in K$ such that $K = k\langle u,\xi\rangle$ i.e., such that K coincides with the smallest field generated by the indeterminates u and ξ.

The (nondifferential) transcendence degree n of K/k<u> is the smallest integer n such that $\xi^{(n)}$ is k<u>-algebraically dependent on $\xi, d\xi/dt,...., d^{(n-1)}\xi/dt^{(n-1)}$. We let $q_1 = \xi$, $q_2 = d\xi/dt$,...,$q_n = d^{(n-1)}\xi/dt^{(n-1)}$. It follows that $q = (q_1,...,q_n)$ also qualifies as a (non-differential) transcendence basis of K/k<u>. One, hence, obtains a nonlinear generalization of the controller canonical form, known as the Global Generalized Controller Canonical Form (GGCCF) :

$$\begin{aligned} &\frac{d}{dt} q_1 = q_2 \\ &\frac{d}{dt} q_2 = q_3 \\ &\ \vdots \\ &\frac{d}{dt} q_{n-1} = q_n \\ &C(\dot{q}_n, q, u, \dot{u},...,u^{(\alpha)}) = 0 \end{aligned} \qquad (3.13)$$

where C is a polynomial with coefficients in k. If one can locally solve for the time derivative of q_n in the last equation, one locally obtains an explicit system of first order differential equations, known as the *Local Generalized Controller Canonical Form* (LGCCF):

$$\begin{aligned} &\frac{d}{dt} q_1 = q_2 \\ &\frac{d}{dt} q_2 = q_3 \\ &\ \vdots \\ &\frac{d}{dt} q_{n-1} = q_n \\ &\frac{d}{dt} q_n = c(q, u, \dot{u}, \ddot{u},\ldots,u^{(\alpha)}) \end{aligned} \qquad (3.14)$$

Remark We assume throughout that $\alpha \geq 1$. The case $\alpha = 0$ corresponds to that of exactly linearizable systems under state coordinate transformations and static state feedback. One may still obtain the same smoothing effect of the dynamical sliding mode controllers we propose in this article by considering a suitable *prolongation* of the input space. This is accomplished by succesively considering the "extended system" (see Nijmeijer and

Van der Schaft, 1990) of the original one, and proceeding to use the same differential primitive element yielding the Generalized Controller Canonical Form of the given smaller dimensional system. ■

The preceeding general results on canonical forms for nonlinear systems have an immediate consequence in the definition of sliding surfaces for stabilization and tracking problems in nonlinear systems.

Consider the following *sliding surface coordinate function*, expressed in the generalized phase coordinates q previously defined :

$$\sigma = c_1 q_1 + \ldots + c_{n-1} q_{n-1} + q_n \qquad (3.15)$$

where the scalar coefficients c_i (i=1,...,n-1) are chosen in such a manner that the following polynomial, $p(\lambda)$, in the complex variable λ, is Hurwitz:

$$p(\lambda) = c_1 + c_2\lambda + \cdots + c_{n-1}\lambda^{n-2} + \lambda^{n-1} \qquad (3.16)$$

Imposing on the sliding surface coordinate function σ the discontinuous dynamics:

$$\dot{\sigma} = -W\,\mathrm{sign}(\sigma) \qquad (3.17)$$

then, the trajectories of σ are seen to exhibit, in finite time T given by $T = W^{-1}|\sigma(0)|$, a sliding regime on $\sigma = 0$. Substituting on (3.17) the expression (3.15) for σ, and using (3.14), one obtains, after some straightforward algebraic manipulations, the following dynamical implicit sliding mode controller :

$$\begin{aligned} c(q,u,\dot{u},\ddot{u},\ldots,u^{(\alpha)}) = & -c_1 q_2 - \ldots - c_{n-1} q_n \\ & -W\mathrm{sign}[c_1 q_1 + \ldots + c_{n-1} q_{n-1} + q_n] \end{aligned} \qquad (3.18)$$

Evidently, under ideal sliding conditions $\sigma = 0$, the variable q_n no longer qualifies as a state variable for the system since it is expressible as a linear combination of the remaining states and, hence, q_n is no longer a non-differentially transcendental element of the field extension K. The ideal (autonomous) closed loop dynamics may then be expressed in terms of a reduced non-differential transcendence basis K/k which only includes the remaining n-1 phase coordinates associated to the original differential primitive element. This leads to the following *ideal sliding dynamics*:

$$\begin{aligned} \frac{d}{dt} q_1 &= q_2 \\ \frac{d}{dt} q_2 &= q_3 \\ &\vdots \\ \frac{d}{dt} q_{n-1} &= -c_1 q_1 - \cdots - c_{n-1} q_{n-1} \end{aligned} \qquad (3.19)$$

The characteristic polynomial of (3.19) is evidently given by (3.16) and, hence, the (reduced) autonomous closed loop dynamics is asymptotically stable to zero. Notice that by virtue of (3.15), the condition $\sigma = 0$, and the asymptotic stability of (3.19), that q_n also tends in an asymptotically stable fashion to zero.

The *equivalent control*, denoted by u_{EQ} is a virtual feedback control action achieving ideally smooth evolution of the system on the constraining sliding surface $\sigma = 0$, provided initial conditions are precisely set on such a switching surface. The equivalent control is formally obtained from the condition $d\sigma/dt = 0$. After some simple algebraic manipulations one obtains from (3.15), (3.18) and $\sigma = 0$:

$$\begin{aligned} & c(q,u_{EQ},\dot{u}_{EQ},\cdots,u_{EQ}^{(\alpha)}) = \\ & c_1 c_{n-1} q_1 + (c_2 c_{n-1} - c_1) q_2 + \cdots + (c_{n-2} c_{n-1} - c_{n-3}) q_{n-2} + (c_{n-1} c_{n-1} - c_{n-2}) q_{n-1} \end{aligned} \qquad (3.20)$$

Since q asymptotically converges to zero, the solutions of the above time-varying implicit differential equation, describing the evolution of the equivalent control, asymptotically approach the solutions of the following autonomous implicit differential equation :

$$c(0,u,\dot{u},\cdots,u^{(\alpha)}) = 0 \qquad (3.21)$$

Equation (3.21) constitutes the *zero dynamics* (See Fliess, 1990b) associated to the problem of zeroing the differential primitive element, considered now as an (auxiliary) output of the system. Notice that (3.20) may also be regarded as the *zero dynamics* associated with zeroing of the sliding surface coordinate function σ. If (3.21) locally asymptotically approaches a constant equilibrium point u = U, then the system is said to be locally *minimum phase* around such an equilibrium point, otherwise the system is said to be *non-minimum phase*. The equivalent control is, thus, locally asymptotically stable to U, whenever the underlying input-output system is minimum phase.

3.3 Higher order sliding regimes

In recent times some efforts have been devoted to non-traditional smoothing of sliding regimes through the so called "higher order" sliding regimes (see Chang, 1991 for a second order sliding mode controller example). The ideas behind "binary control systems", as applied to variable structure control, are geard towards obtaining asymptotic convergence towards the sliding surface, in a manner that avoids control input chattering through integration (See Emelyanov, 1987). These two develpments are also closely related to the differential algebraic approach presented here. In the following paragraphs we explain how the same ideas may be formally derived from differential algebra, in all generality.

Let σ be a p-th order sliding surface candidate. i.e.

$$S(\sigma^{(p)}, \zeta, u, \dot{u},\ldots,u^{(\gamma)}) = 0 \qquad (3.22)$$

for some polynomial function S. Let us assume that (3.22) may be locally expressed as:

$$\sigma^{(p)} = s(\zeta, u, \dot{u},\ldots,u^{(\gamma)}) \qquad (3.23)$$

Let M be a positive constant. Moreover, let the set of coefficients $\{m_1,\ldots,m_{p-1}\}$ be such that the polynomial in the complex variable λ :

$$q(\lambda) = \lambda^p + m_{p-1}\lambda^{p-1} + \cdots + m_2\lambda + m_1$$

is Hurwitz. The following dynamical implicit sliding mode controller achieves an asymptotic approach to the zero level set of the sliding surface σ.

$$\begin{aligned} s(\zeta, u, \dot{u},\ldots,u^{(\gamma)}) = & -m_1\dot{\sigma} - m_2\ddot{\sigma} - \ldots - m_{p-1}\sigma^{(p-1)} \\ & -M\,\mathrm{sign}\left[m_1\sigma + m_2\dot{\sigma} + \ldots + m_{p-1}\sigma^{(p-2)} + \sigma^{(p-1)}\right] \end{aligned} \qquad (3.24)$$

Since, generally speaking, the time derivatives of σ are k<u>-algebraically dependent on ζ, the right hand side of the dynamical sliding mode controller (3.24) may be ultimately expressed in terms of the (time-varying) state components.

Remark A differential primitive element of the finitely generated dynamics K/k<u>, with (non-differential) transcendence degree n, always qualifies as a candidate for an n-th order sliding regime. ■

An additional possibility of creating higher order sliding regimes is represented by the consideration of the input-sliding surface system as an input-output system.

Consider the differential field extension $k\langle u,\sigma\rangle/k\langle u\rangle$ as an input-output system. Evidently since $k\langle u,\sigma\rangle$ is finitely generated over $k\langle u\rangle$, then $k\langle u,\sigma\rangle$ is differentially algebraic over $k\langle u\rangle$. The sliding surface candidate σ satisfies, then, an implicit algebraic differential equation with coefficients in $k\langle u\rangle$. i.e.,

$$P(\sigma, \dot{\sigma}, \cdots, \sigma^{(v)}, u, \dot{u}, \cdots, u^{(\mu)}) = 0 \qquad (3.25)$$

We may rewrite such an implicit differential equation as the following Global Generalized Observability Canonical Form (GGOCF) (see Fliess, 1988):

$$\begin{aligned} \dot{\sigma}_1 &= \sigma_2 \\ \dot{\sigma}_2 &= \sigma_3 \\ &\vdots \\ P(\sigma_1, \cdots, \sigma_v, \dot{\sigma}_v, u, \dot{u}, \cdots, u^{(\mu)}) &= 0 \\ \sigma &= \sigma_1 \end{aligned} \qquad (3.26)$$

where: $\sigma_i := d^{i-1}\sigma/dt^{i-1}$ (i=1,2,...,v)

As before, an explicit Local Generalized Observability

Canonical Form (LGOCF) can be obtained for the element σ whenever $\partial P/\partial(d\sigma_\nu/dt)\neq 0$:

$$\begin{aligned}\dot\sigma_1 &= \sigma_2\\ \dot\sigma_2 &= \sigma_3\\ &\vdots\\ \dot\sigma_\nu &= p(\sigma_1, \cdots, \sigma_\nu, u, \dot u, \cdots, u^{(\mu)})\\ \sigma &= \sigma_1\end{aligned} \qquad (3.27)$$

One takes a *sliding surface candidate* as any arbitrary (algebraic) function of σ and its time derivatives, up to ν-1-st order. For obvious reasons, the most convenient type of function is represented by a stabilizing linear combination of σ and its time derivatives.

$$s = m_1\sigma_1 + m_2\sigma_2 + \cdots + m_{\nu-1}\sigma_{\nu-1} + \sigma_\nu \qquad (3.28)$$

A first-order sliding motion is then imposed on such a linear combination of generalized "phase variables", by means of the discontinuous sliding mode dynamics:

$$\dot s = -M\,\mathrm{sign}(s) \ \ ; \ \ M > 0 \qquad (3.29)$$

This results in the following implicit dynamical higher order sliding mode controller:

$$\begin{aligned}p(\sigma_1, ..., \sigma_\nu, u, \dot u, ..., u^{(\mu)}) = &-m_1\sigma_2 - ... - m_{\nu-1}\sigma_\nu\\ &-M\,\mathrm{sign}[m_1\sigma_1 + ... + m_{\nu-1}\sigma_{n-1} + \sigma_\nu]\end{aligned} \qquad (3.30)$$

As previously discussed, s goes to zero in finite time and, provided the coefficients in (3.28) are properly chosen, an ideally asymptotically stable motion can be then obtained for σ, as it is ideally governed by the following autonomous linear dynamics:

$$\begin{aligned}\dot\sigma_1 &= \sigma_2\\ \dot\sigma_2 &= \sigma_3\\ &\vdots\\ \dot\sigma_{\rho-1} &= -m_1\sigma_1 - m_2\sigma_2 - \cdots - m_{\rho-1}\sigma_{\rho-1}\\ \sigma &= \sigma_1\end{aligned} \qquad (3.31)$$

3.4 Sliding regimes and the controllability of nonlinear systems

The *differentially algebraic closure* of the ground field k in the dynamics K is defined as the differential field κ, where $K \supseteq \kappa \supseteq k$, consisting of the elements of K which are differentially algebraic over k. The field k is *differentially algebraically closed* if, and only if, $\kappa = k$.

The following definition is taken from Fliess (1991) (see also Pommaret, 1988):

Definition 3.14 The dynamics K/k<u> is said to be *controllable* if, and only if, the groud field k is differentially algebraically closed in K.

Controllability implies, then, that any element of K is necessarily influenced by the input u, since such an element satisfies a differential equation which is not independent of u and of, possibly, some of its time derivatives.

Proposition 3.15 A higher order sliding regime can be created on any element σ of the dynamics K/k<u> if, and only if, K/k<u> is controllable.

Proof sufficiency is obvious from the fact that controllability implies that σ satisfies a differential equation with coefficients in k<u>. For the necessity of the condition, suppose, contrary to what is asserted, that K/k<u> is not controllable and yet a higher order sliding regime can be created on any element of the differential field extension K/k<u>. Since k is not differentially algebraically closed, then, there are elements in K, which belong to a differential field κ containing k, which satisfy differential equations with coefficients found exclusively in k. Clearly, these elements are not related to the control input u through differential equations. It follows that a higher order sliding regime cannot be created on such elements. A contradiction is established. ■

In this more relaxed notion of a higher order sliding regime, one may say that a sliding regime can be created on any element of the dynamics of the system, if, and only if, the system is controllable. This characterization of sliding mode existence through controlability is a direct consequence of the differential algebraic approach.

4. CONCLUSIONS AND SUGGESTIONS FOR FURTHER RESEARCH

The use of the differential algebraic methods provides a firm theoretical basis to sliding mode control of nonlinear systems. The results are seen to point towards potential practical implications. More general classes of sliding surfaces, which include the presence of inputs and, possibly, their time derivatives, were shown to naturally allow for chattering-free sliding mode controllers of dynamical nature. The theoretical simplicity, and conceptual advantages, stemming from the differential algebraic approach, render new possibilities to the broader area of discontinous feedback control in general. Extensions of the theory, and its implications, to other classes of discontinuous feedback controlled systems, such as pulse-width-modulated control strategies, are entirely possible (see Sira-Ramirez, 1992e). The less explored pulse-frequency-modulated control techniques may be shown to also benefit from this new approach (Sira-Ramirez, 1992f). For other classes of systems, such as infinite dimensional, discrete time and differential-difference systems, the extensions of the sliding mode control theory remain largely unexplored, from this new viewpoint.

It has been shown, in a most elegant manner, by Fliess and Messager (1991), that non-minimum phase linear systems can be asymptotically stabilized using dynamical precompensators in combination with sliding mode controllers. Such result could be extended to the nonlinear systems case with, possibly, some significant additional efforts. This topic, as well as extensions of the theory to nonlinear multivariable sliding systems, deserve some attention in the future.

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FIGURES

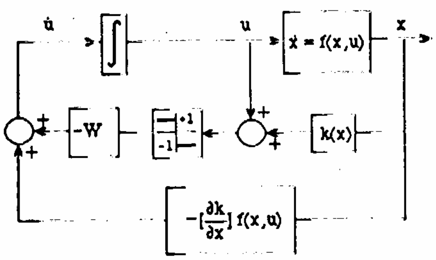

Figure 1. Dynamical Sliding mode controller enforcing a known smooth feedback law.

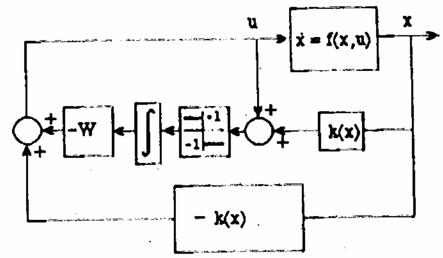

Figure 2. Reinterpretation of dynamical Sliding mode controller enforcing a known smooth feedback law.

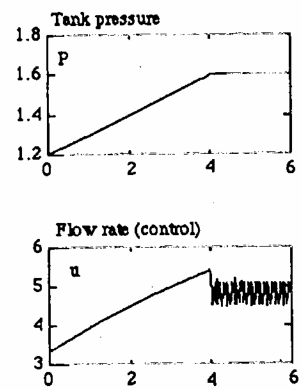

Figure 3. Simulation of statically variable structure controlled pressure tank.

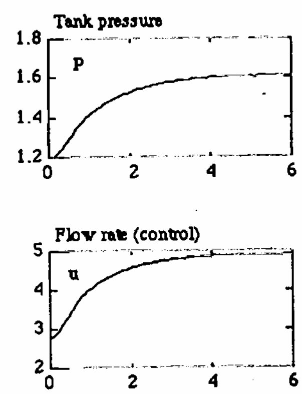

Figure 4. Simulation of dynamically variable structure controlled pressure tank.