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Adaptive Dynamical Sliding Mode Control via Backstepping *

Hebertt Sira-Ramírez
Departamento Sistemas de Control
Universidad de Los Andes
Mérida 5101, Venezuela.
and

Orestes Llanes–Santiago
Postgrado en Ingeniería de Control Automático
Facultad de Ingeniería
Universidad de Los Andes
Mérida, Venezuela

Abstract

An advantageous combination of chattering–free dynamical sliding mode control and the adaptive backstepping technique is proposed for the regulation of exactly linearizable systems, placeable in *parametric–pure feedback form* and in *paremetric–strict feedback form*.

1 Introduction

In recent times special attention has been devoted to adaptive control of nonlinear parametric uncertain systems which are nominally fully state, or input–output, linearizable by state coordinate transformations and feedback (see Sastry and Isidori [1], the collection of articles in the volume edited by Kokotovic [2] and the work of Kanellakopoulos *et al* [3], Kristic *et al* [6], etc). In all these works *continuous* control laws are being proposed for the adaptive stabilization of dynamical systems placeable in special canonical forms. Some of the more recent such canonical forms being addressed as *parameteric pure* and *parametric strict feedback forms* ([3], [6]).

Sliding regimes are frequently used as a robust discontinuous feedback control alternative which replaces, to some extent, adaptive feedback controllers. Discontinuous feedback control strategies frequently act also as a high performance stabilizing technique with great potential for practical implementation (see Slotine and Li [4]). Sliding mode control naturally deals with parameteric uncertain systems (see [4]) without need for establishing a parallel process dealing with parameter updating, charcteristic of all avilable adaptive feedback control schemes. However, only under very special circumstances can sliding mode control be truly regarded as an actual substitute, or alternative, for adaptive feedback control. First of all, somewhat stringent structural conditions, known as the *matching conditions*, related to the structural properties of the parametric uncertainties, must be precisely satisfied in order to obtain a robust performance of the sliding mode controller policy. Secondly, when the stabilizing sliding surface itself contains unknown (but bounded) parameters, the sliding mode control scheme must be effectively turned into an adaptive one. This last issue is particularly the prevailing case in dynamical sliding mode control alternatives (see Sira-ramírez [7],[8]). The procedure for synthesizing such an adaptive sliding surface entitles producing an on line estimate of the stabilizing switching surface and stringent assumptions on its local uniform stabilizing features (see Sira-Ramírez *et al* [5]).

The use of dynamically generated sliding mode control has been recently proposed as a means of obtaining a smoother controlled response than those achievable throuhg traditional discontinuous, or variable structure, feedback regulation schemes (see [7], [8]). All chattering is thus completely relegated to the state space of the dynamical feedback controller. As a consequence of this, a sufficiently smooth control input signal is generated. It is this particular smoothing feature that we would like to retain while avoiding explicit sliding surface adaptation and sliding surface error assessment. Bactstepping techniques provide a natural and sufficiently

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simple way to obtain a more direct sliding surface adaptation procedure while avoiding the excessively restrictive stabilizing features of the adaptive sliding surface.

In this article it is shown, through the simplest possible example, how to conveniently combine the many useful features of adaptive backstepping techniques and chattering-free dynamical sliding mode control. The class of systems to which this proposal applies is restricted to systems placeable in *parametric pure feedback canonical form* and also for those transformable to *parametric strict feedback canonical form* (see Kristić *et al* [6]). The backstepping adaptive algorithm for the second class of systems does not require overparametrization, a feature always present in the first class of systems.

In contradisctinction to existing adaptive sliding mode control techniques ([4], [5]), the proposed approach does not concern itself with convergence of the estimated (adapted) sliding surface and switchings take place on such an estimated surface quite independently of its convergence, or not, to the nominal sliding surface.

2 Sliding Regimes and Backstepping in Parametric Uncertain Systems

Here we shall illustrate through a simple first order example how to combine the idea of dynamical sliding mode control and backstepping for the adaptive control of parametric uncertain systems which are placeable in parametric pure feedback canonical form (see [3]) or parametric strict feedback canonical form (see citeKristic).. Some simulations will also be shown that depict the performance of the proposed adaptive controllers.

2.1 Adaptive sliding mode control through backstepping for systems in parametric–pure–form

Consider the parametric uncertain scalar system

$$\dot{x} = u + \theta x^2 \tag{1}$$

where θ is assumed to be constant but unknown. A static sliding mode controller may be readily proposed as $u = -k \text{ sign } x$ for $k > 0$, where "sign" stands for the *signum* function. It is clear that for sufficiently high values of k a *local* sliding regime can be created, in finite time, on $x = 0$ by means of this simple control policy. Indeed, if the parameter θ satisfies the bounding constraint $|\theta| \leq M$ then the region of initial states $x(0)$, for which the existence of a sliding regime is guaranteed, may be determined as $|x(0)| \leq \sqrt{k/M}$. This implies that in order to enlarge the region of initial conditions for sliding mode existence, the design gain k must be made significantly large, thus implying increased chattering responses for the control variable x. Figure 1 depicts the simulated behaviour of the above nonadaptive discontinuous feedback control strategy for an (unknown) value of $\theta = 1$ and $k = 0.5$. The controlled system trjectories become unstable for $|x(0)| > 0.707$. They converge to zero otherwise.

A means of easing the need for large bang-bang control input signals, while only guaranteeing asymptotic convergence to zero of the regulated variable x, can be obtained by devising a discontinuous feedback control which also relies on a "tunnable" estimate $\hat{\theta}$ of the unknown parameter θ. Such a control law is proposed as $u = -k \text{ sign} x - \hat{\theta} x^2$. The closed loop system is then given by

$$\dot{x} = -k \text{ sign } x + (\theta - \hat{\theta})x^2 = -k \text{ sign } x + \phi x^2 \tag{2}$$

where ϕ is the estimation error $\theta - \hat{\theta}$. A standard Lyapunov stability analysis performed on the function $V(x, \phi) = \frac{1}{2}x^2 + \frac{1}{2}\phi^2$ yields the following adaptation law

$$\dot{\hat{\theta}} = x^3 \tag{3}$$

The resulting time derivative of the Lyapunov function $V(x, \phi)$ is given by $\dot{V} = -k\ |x|$, which means that V is bounded and that, hence, x is absolutely integrable. Since the derivative of x is also bounded this guarantees asymptotic convergence of x to zero for any positive value of the gain k, no matter how small such a positive value is. However, if the sign of the unknown parameter θ is known *a priori*, and the estimate $\hat{\theta}$ is consistent with such knowledge, then the variable structure controller gain k can be safely reduced down to values above a new lower bound, represented by $|M - \hat{\theta}|$, while still having finite time convergence to zero of the variable x. Figure 2 depicts the simulated trajectories for the adaptive sliding-mode controlled state x, and the parameter estimate trajectories $\hat{\theta}(t)$. In the presented simulations, the sliding mode controller gain was again set to be $k = 0.5$ and the "true" value of the constant parameter was set to be $\theta = 1$. Evidently the set of initial states for which the controlled trajectories converge to zero is now the entire real line.

In the preceeding sliding mode–adaptive control scheme, the contol input evidently exhibits a bang-bang behavior which may be deemed as countereffective in many instances, specially when the regulated system is of the mechanical type. Our fundamental objective in the next section will be to remove the bang–bang nature of the control input by resorting to system extension and letting the added integration attenuate the discontinuities associated the input while relegating them to the first time derivative of the input. The backstepping alternative will be systematically used, in pursuing such an objective, by prescribing (possibly overparametrized) parameter update laws and discontinuous feedback control strategies in an "interlaced" manner.

Consider now the *extended* system

$$\begin{aligned} \dot{x} &= u + \theta x^2 \\ \dot{u} &= v \end{aligned} \tag{4}$$

where v is an auxiliary input representing the time derivative of the original input u.

The above extended system is already in *pure parameter feedback* form [3]. We now apply the first few steps in the design procedure proposed in [3]. Let let $\hat{\theta}_1$ denote a first estimate of the unknown parameter θ. Define a new state variable, or input dependent sliding surface coordinate function, s as $s = c_1 x + u + \hat{\theta}_1 x^2$ where c_1 is a positive design constant. This choice result in the following expression of the dynamics of the regulated system variable x

$$\dot{x} = -c_1 x + s + (\theta_1 - \hat{\theta}_1)\, x^2 \tag{5}$$

A Lyapunov function candidate of the form $V_1 = \frac{1}{2}x^2 + \frac{1}{2}(\theta_1 - \hat{\theta}_1)^2$ exhibits a time derivative given by

$$\begin{aligned} \dot{V}_1 &= -c_1 x^2 + xs + (\theta - \hat{\theta}_1)(x^3 - \dot{\hat{\theta}}_1) \\ &= -c_1 x^2 + xs + \phi(x_1^3 - \dot{\hat{\theta}}_1) \end{aligned} \tag{6}$$

We eliminate the term containing the parameter estimation error $\phi = \theta - \hat{\theta}$ by adopting the following parameter update law

$$\dot{\hat{\theta}}_1 = x^3 \tag{7}$$

and let, temporarily, the Lyapunov function stand as $\dot{V}_1 = -c_1 x^2 + xs$. The evolution of the adaptive sliding surface s is computed directly from its definiton and the system equations as

$$\dot{s} = v + \psi_1\,(x, s, \hat{\theta}_1) + \theta\, \psi_2\,(x, s, \hat{\theta}_1) \tag{8}$$

where

$$\psi_1\,(x, s, \hat{\theta}_1) = (c_1 + 2\hat{\theta}_1 x)(-c_1 x + s - \hat{\theta}_1 x^2) + x^5 \tag{9}$$

and

$$\psi_2\,(x, s, \hat{\theta}_1) = (c_1 x^2 + 2\hat{\theta}_1 x^3) \tag{10}$$

The discontinuous feedback controller is obtained from,

$$v = -\psi_1\,(x, s, \hat{\theta}_1) - x - \hat{\theta}_2\, \psi_2\,(x, s, \hat{\theta}_1) - k \operatorname{sign} s \tag{11}$$

where $\hat{\theta}_2$ is a *new* estimate of the unknown parameter θ.

The new parameter update law is now assessed from a stability analysis based on the new Lyapunov function candidate

$$V_2 = V_1 + \frac{1}{2}s^2 + \frac{1}{2}(\theta - \hat{\theta}_2)^2 \tag{12}$$

The time derivative of V_2, computed along the trayectories of the discontinuously controlled closed loop systems is given by

$$\begin{aligned} \dot{V}_2 &= -c_1 x^2 - k|s| + (\theta - \hat{\theta}_2) \\ &\quad (-\dot{\hat{\theta}}_2 + s\, x^2\, (c_1 + 2\hat{\theta}_1 x)) \end{aligned} \tag{13}$$

One then simply choses

$$\dot{\hat{\theta}}_2 = s\, x^2\, (c_1 + 2\hat{\theta}_1 x) \tag{14}$$

The resulting Lyapunov function derivative is

$$\dot{V}_2 = -c_1 x^2 - k|s| \;\leq\; 0 \tag{15}$$

The obtained adaptive sliding mode controller is indeed a dynamical controller as it easily seen when it is summarized in terms of the original control variable u and its time derivative $\dot{u} = v$

$$\begin{aligned} \dot{u} &= -(c_1 + 2\hat{\theta}_1 x)u - x^5 \\ &\quad -\hat{\theta}_2\, (c_1 + 2\hat{\theta}_1 x)x^2 \\ &\quad -x - k \operatorname{sign} s \\ s &= c_1 x + u + \hat{\theta}_1 x^2 \\ \dot{\hat{\theta}}_1 &= x^3 \\ \dot{\hat{\theta}}_2 &= s\, x^2\, (c_1 + 2\hat{\theta}_1 x) \end{aligned} \tag{16}$$

The simulated adaptive sliding mode controlled system trajectories, shown in figures 3 and 4, were obtained for $k = 0.5$, $c_1 = 2$ and with the (unknown) nominal parameter value $\theta = 1$. The state trajectory is now quite smooth and asymptotically converges to zero, while the control input u is no longer bang-bang but *continuous*. The trajectories associated to the overparametrization estimation laws for the unknow parameter θ. also show asymptotically stable convergence features.

2.2 Adaptive sliding mode control through backstepping for systems in parametric-strict-form

Consider again the extended system treated in the previous section. This system is also in parametric-strict-feedback form ([6]). We now proceed, following the steps proposed in [6], to obtain a dynamical adaptive sliding mode controller which only requires one adaptation law for the unknown parameter. Define the adaptive sliding surface coordinate function s exactly as before. Consider the Lyapunov function

$$V = \frac{1}{2}x^2 + \frac{1}{2}(\theta - \hat{\theta})^2 \qquad (17)$$

Then $\dot{V}$ is given by

$$\dot{V} = -c_1 x^2 + sx + (\theta - \hat{\theta})(-\dot{\hat{\theta}} + x^3) \qquad (18)$$

Instead of choosing, as before, $\dot{\hat{\theta}} = x^3$, we let the term $\dot{\hat{\theta}}$ stand in the expression for $\dot{V}$. The dynamics for x now obeys

$$\dot{x} = -c_1 x + s + (\theta - \hat{\theta})x^2 \qquad (19)$$

The dynamics for the aaptive sliding surface variable s is obtained in a slightly different fashion to the previous section. The difference is that we do not yet define an adaptation law

$$\begin{aligned} \dot{s} &= v + (c_1 + 2\hat{\theta}_1 x)\, u \\ &\quad + \dot{\hat{\theta}}\, x^2 + (c_1 + 2\hat{\theta}_1 x)\, \theta x^2 \end{aligned} \qquad (20)$$

We now consider the Lyapunov function candidate

$$\begin{aligned} V_2 &= \frac{1}{2}x^2 + \frac{1}{2}(\theta - \hat{\theta})^2 + \frac{1}{2}s^2 \\ &= V + \frac{1}{2}s^2 \end{aligned} \qquad (21)$$

The time derivative of V_2, along the solutions of the system, is now given by

$$\begin{aligned} \dot{V}_2 &= -c_1 x^2 + s\Big[x + v + (c_1 + 2\hat{\theta}x)u \\ &\quad + \dot{\hat{\theta}}x^2 + (c_1 + 2\hat{\theta}x)\,\theta\, x^2\Big] \\ &\quad + (\theta - \hat{\theta})\Big[-\dot{\hat{\theta}} + x^3\Big] \end{aligned} \qquad (22)$$

Choosing now

$$\begin{aligned} v &= -x - (c_1 + 2\hat{\theta}x)\,(u + \hat{\theta}\, x^2) \\ &\quad - \dot{\hat{\theta}}x^2 - k \text{ sign } s \end{aligned} \qquad (23)$$

and

$$\dot{\hat{\theta}} = x_1^3 + s\, x^2\, (c_1 + 2\hat{\theta}x) \qquad (24)$$

the time derivative of the Lyapunov function V_2 simply becomes

$$\dot{V}_2 = -c_1 x^2 - k\, |s| \;\leq\; 0 \qquad (25)$$

and the states x and s converge to zero. A summary of the adaptive controller in terms of the original control variable u is as follows

$$\begin{aligned} \dot{u} &= -x - (c_1 + 2\hat{\theta}x)(u + \hat{\theta}x^2) \\ &\quad - \dot{\hat{\theta}}x^2 - k \text{ sign } s \\ \dot{\hat{\theta}} &= x^3 + s\, x^2\, (c_1 + 2\hat{\theta}x) \\ s &= u + c_1 x + \hat{\theta}x^2 \end{aligned} \qquad (26)$$

Figure 5 depicts the closed loop behaviour of the dynamical adptive sliding mode controlled system obtained without overparametrization. In this figure, the controlled state variable x is shown along with the smoothed dynamical controller output signal u. Figure 6 shows the trajectory of the estimated parameter θ.

3 Conclusions

Sliding mode control can be advantageously combined with adaptive backstepping techniques for the regulation of parametric uncertain systems. The combination allows to design either static discontinuous feedback controllers, or dynamical discontinuous controllers. The last option can be accomplished, usually, through straightforward state extention procedures, or by resorting to Fliess' Generalized Observability Canonical Forms [7]. The advantages reside in the substantially smoothed controller output and reduction of the otherwise characteristic "chattering" responses of the associated state variables. The sliding mode controller also bestows robustness with respect to the class of matched external additive perturbation inputs of the unknown but bounded nature.

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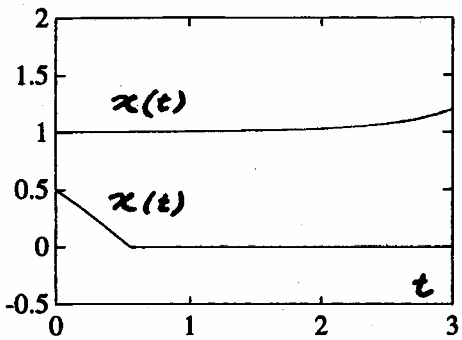

Figure 1: Static sliding mode controlled state responses for first order parametrically uncertain system.

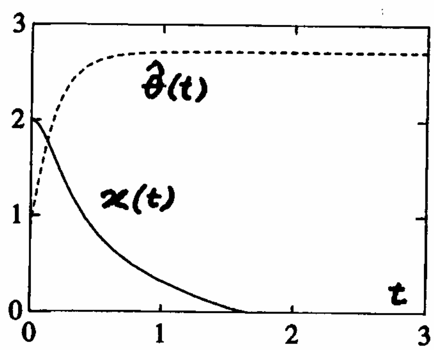

Figure 2: Static adaptive sliding mode controlled response for first order parametrically uncertain system.

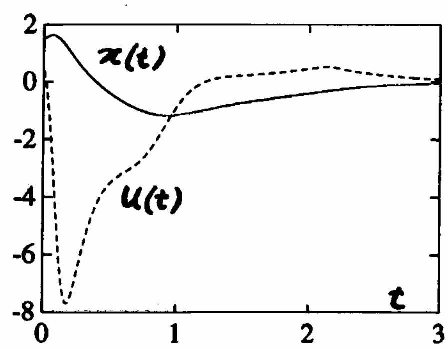

Figure 3: Dynamical adaptive sliding mode controlled state and smoothed input responses obtained through overparametrized backstepping

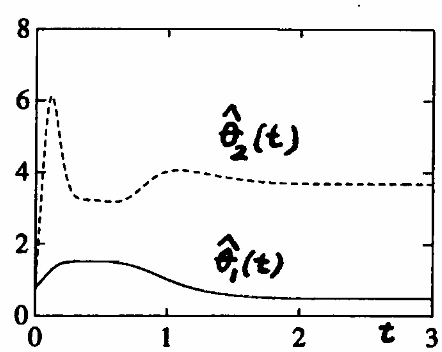

Figure 4: Parameter estimates evolution for dynamical adaptive sliding mode controlled system obtained through overparametrized backstepping

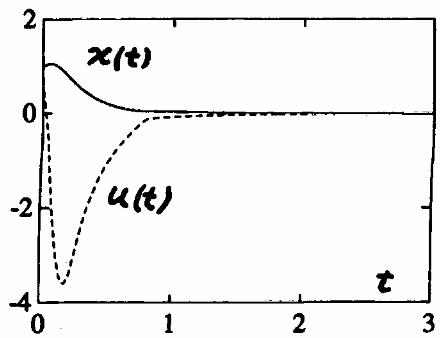

Figure 5: Controlled state variable and smoothed input response for dynamical adaptive sliding mode controlled system, obtained through backstepping without overparametrization

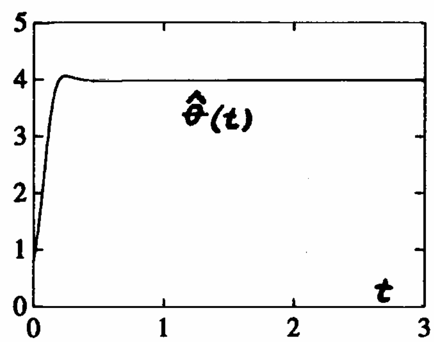

Figure 6: Parameter estimate evolution for dynamical adaptive sliding mode controlled system, obtained through backstepping without overparametrization