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A Module Theoretic Approach to Sliding Mode Control in Linear Systems *

Michel Fliess
C.N.R.S–E.S.E., Laboratoire des Signaux et Systèmes
Plateau de Moulon
91192 Gif-sur-Yvette Cedex, France
and
Hebertt Sira-Ramírez
Departamento Sistemas de Control
Universidad de Los Andes
Mérida 5101, Venezuela.

Abstract

Module Theory is shown to generalize the traditional results of Sliding Mode Control of linear systems, while introducing new possibilities for applications. The algebraic approach clearly shows that any desirable output dynamics, of arbitrary order, is synthesizable by minimum phase sliding mode control, independently of the order of the given plant and of any structural conditions of the matching type. Clear relations are also established with invertibility, nonminimum phase situations, controllability and observability

Keywords : Sliding Regimes, Module Theory, Linear Systems

1 Introduction

The theory and applications of Sliding Mode Control has received the contributions of many authors during the years. Its timeliness can be measured by the number of books entirely devoted the the subject (Utkin [1], [2], Itkis [3], Bühler [4] and Zinober [5]), or books containing introductory chapters (Slotine and Li [6]).

The theory of sliding regimes has recently undergone a trend of developments related to the *differential algebraic* approach to control systems (see Fliess and Messager [7], [8], and Sira-Ramírez [9], [10] and [11]). This has resulted in robust dynamical sliding mode controllers with smoothed chattering. However, the theoretical aspects related to the algebraic approach to sliding mode control for linear systems has only been recently addressed by Fliess and Sira-Ramírez in [12] from a module theoretic viewpoint.

In this article we address the algebraic approach to sliding mode control of linear systems. We first provide some background definitions of the relevant topics in module theory and their relation to linear systems theory. The reader is referred to Fliess [13], and Fliess [14], for further details on the subject.

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2 Background to Module Theory and Linear Systems

In this section we provide some background definitions on Modules. The reader is referred to the book by Adkins and Weintraub [17] for the proofs of fundamental issues.

Definition 2.1 *A ring* $(R,+,\cdot)$ *is a set* R *with two binary operations*

$$+ \quad : R \to R (addition)$$
$$\cdot \quad : R \to R (multiplication)$$

such that $(R,+)$ *is an abelian group with a zero. Multiplication and addition satisfy the usual properties of associativity and distributivity.*

Example 2.2 *The set 2Z of even integers is a ring without an identity. The set of all polynomials in an indeterminate x with coefficients in some commutative field is also a ring.*

Here we shall be dealing only with *commutative rings with identity.*

Definition 2.3 *Let R be an arbitrary ring with identity. An R-module is an abelian group M together with a scalar multiplication map*

$$\cdot : R \times M \to M$$

which satisfies the following axioms: $\forall a, b \in R,\ m, n \in M$

$$\begin{aligned} a(m+n) &= am + an \\ (a+b)m &= am + bm \\ (ab)m &= a(bm) \\ 1m &= m. \end{aligned}$$

Example 2.4 *Let F be a field, then an F-module V is called a vector space over F.*

Any subgroup $N \subset M$ which is closed under scalar multiplication by elements in R is itself a module, called a submodule of M.

If $S \subset M$, then $[S]$ denotes the intersection of all submodules of M containing S. We may say that $[S]$ is the "smallest" submodule, with respect to inclusion, containing the set S. The submodule $[S]$ is also called the *submodule of M generated by S.*

Definition 2.5 *M is* finitely generated *if $M = [S]$ for some finite subset S of M. The elements of S are called "generators" of M.*

We denote by $\Re\left[\frac{d}{dt}\right]$ the commutative principal ideal *ring of finite linear differential operators* with real coefficients. These are operators of the following form

$$\sum_{finite} a_\alpha \frac{d^\alpha}{dt^\alpha}, \quad a_\alpha \in \Re$$

This necessarily restricts the class of problems treated to linear, time–invariant, systems. The results, however, can be extended to time–varying systems by using rings defined over noncommutative *principal ideal rings* (see Fliess [13] [14]).

Definition 2.6 *Let M be an $\Re\left[\frac{d}{dt}\right]$-module. An element $m \in M$ is said to be* torsion *if and only if there exists $\pi \in \Re\left[\frac{d}{dt}\right]$, $\pi \neq 0$, such that $\pi m = 0$. In other words, m satisfies a homogeneous linear differential equation.*

Definition 2.7 *A module T such that all its elements are torsion is said to be a torsion module.*

Definition 2.8 *The set of all torsion elements of a module M is a submodule T called the* torsion submodule *of M.*

Definition 2.9 *A finite set of elements in a $\Re[\frac{d}{dt}]$-module M constitutes a* basis *if every element in the module may be uniquely expressed as a $\Re[\frac{d}{dt}]$-linear combination of such elements. A module M is said to be* free *if it has a basis. Two basis possess the same number of elements. Such number constitutes the* rank *of the module M.*

Propostion 2.10 *Let M be a finitely generated left $\Re\left[\frac{d}{dt}\right]$-module. M is torsion if and only if the dimension of M as a k-vector space is finite .*

Theorem 2.11 *A finitely generated module M is free if and only if its torsion submodule is trivial.*

Theorem 2.12 *Any finitely generated $\Re\left[\frac{d}{dt}\right]$-module M can be decomposed into a direct sum*

$$M = T \oplus \Phi$$

where T is the torsion submodule and Φ is a free submodule.

2.1 Quotient modules

Let M be an R-module and let $N \subset M$ be a submodule of M, then N is a subgroup of the abelian group M and we can form the *quotient group M/N* as the set of all *cosets*

$$M/N = \{m + N \ ; \ \text{for} \ m \in M\} \qquad (1)$$

They evidently accept the operation of addition as a well defined (commutative) operation

$$(m+N) + (p+N) = (m+p) + N$$

The elements $m + N$ of M/N can now be endowed with an R-module structure by defining scalar products in a manner inherited from M, namely,

$$a(m+N) = am + N \ ; \ \forall a \in R \ \text{and} \ m \in M$$

The elements $m' = m(\text{mod}N)$ are called the *residues* of M in M/N. The map $M \to M/N$, $m \mapsto m' = m + N$, is called the *canonical projection.*

2.2 Linear Systems and Modules

Linear systems enjoy a particularly appealing characterization from the algebraic viewpoint. This has been long recognized since the work of Kalman [18]. More recently Fliess [13] has provided a rather different approach to such characterization which still uses modules but in a rather different context. Note that the later viewpoint is related [14] to Willems' *behavioral* approach [15].

Definition 2.13 *A* linear system *is a finitely generated left $\Re\left[\frac{d}{dt}\right]$-module Λ.*

Example 2.14 *Consider a system Σ as a finite set of quantities $w = (w_1, \ldots, w_q)$ which are related by a set of homogeneous linear differential equations with coefficients in $\Re$;*

$$E_\alpha \left(w_i^{(v_j)}\right) = \sum_{finite} a_{\alpha,i,j} w_i^{(v_j)} = 0, \ (a_{\alpha,i,j} \in \Re)$$

Consider the free $\Re\left[\frac{d}{dt}\right]$-module $\mathcal{F}$ spanned by $\overline{w} = (\overline{w}_1, \ldots, \overline{w}_q)$ and let $\Xi \subset \mathcal{F}$ be the submodule spanned by

$$\xi_\alpha = E_\alpha \left(\overline{w}_i^{(v_j)}\right) = \sum_{finite} a_{\alpha,i,j} \overline{w}_i^{(v_j)}, \ (a_{\alpha,i,j} \in \Re)$$

The quotient module $\Lambda = \mathcal{F}/\Xi$ is the module corresponding to the system. It is easy to see that the residue $\overline{w}$ of w in $\mathcal{F}/\Xi$ satisfies the system equations.

2.3 Unperturbed Linear Dynamics

Definition 2.15 *A linear dynamics $\mathcal{D}$ is a linear system $\mathcal{D}$ where we distinguish a finite set of quantities, called the inputs $u = (u_1, \ldots, u_m)$, such that the module $\mathcal{D}/[u]$ is torsion.*

The set of inputs u are said to be *independent* if and only if $[u]$ is a *free* module. An output vector $y = (y_1, \ldots, y_p)$ is a finite set of elements in $\mathcal{D}$.

Example 2.16 *Consider the single input single output system*

$$a(\frac{d}{dt})y = b(\frac{d}{dt})u \qquad a, b \in \Re[\frac{d}{dt}] \ , \ a \neq 0$$

Take the free left $\Re[\frac{d}{dt}]$-module $\mathcal{F} = [\overline{u}, \overline{y}]$ spanned by $\overline{u}$, $\overline{y}$. Let $\Xi \subset \mathcal{F}$ be the submodule spanned by $a(\frac{d}{dt})\overline{y} - b(\frac{d}{dt})\overline{u}$. The quotient module $\mathcal{D} = \mathcal{F}/\Xi$ is the system module. Let u, y be the residues of $\overline{u}, \overline{y}$ in $\mathcal{D}$. Then u, y satisfy the system equations. If we let $\underline{y}$ be the residue of y in $\mathcal{D}/[u]$, then $\underline{y}$ satisfies $a(\frac{d}{dt})\underline{y} = 0$: $\underline{y}$ and $\mathcal{D}/[u]$ are torsion.

2.4 Controllability

Definition 2.17 [13], [14] *A linear system is said to be* controllable *if and only if its associated module Λ is* free.

Example 2.18 *The system given by $\dot{w}_1 = w_2$ is controllable since its associated module is $[w_2]$, which is obviously free.*

Definition 2.19 *A linear dynamics $\mathcal{D}$, with input u, is said to be* controllable *if and only if the associated linear system is controllable.*

Remark 2.20 *For an uncontrollable dynamics $\mathcal{D}$, the torsion submodule T corresponds to the Kalman uncontrollable subspace.*

Example 2.21 *The linear dynamics $\dot{x}_1 = u$ is controllable, since its associated linear system is described by a free module.*

2.5 Observability

Definition 2.22 [13] *A linear dynamics $\mathcal{D}$ with input u and output y, is said to be observable if and only if $\mathcal{D} = [u, y]$. The quotient module $\mathcal{D}/[u, y]$ is trivial.*

Example 2.23 *The linear dynamics $\dot{x}_1 = x_2$; $\dot{x}_2 = u$; $y = x_1$ is observable since $x_1 = y$; $x_2 = \dot{y}$.*

If the system is unobservable then $[u, y] \subset \mathcal{D}$ and the quotient module $\mathcal{D}/[u, y]$ is torsion.

Example 2.24 *The linear dynamics $\dot{x}_1 = x_1$; $\dot{x}_2 = u$; $y = x_2$ is unobservable since $x_1 \notin [u, y]$ and the residues $\overline{x}_1$, $\overline{x}_2$ in the quotient module $\mathcal{D}/[u, y]$ satisfy $\dot{\overline{x}}_1 - \overline{x}_1 = 0$ and $\dot{\overline{x}}_2 = 0$ which is torsion but nontrivial.*

3 A Module Theoretic Approach to Sliding Regimes in Linear Systems

Here we will introduce the basic elements that allow us to treat sliding mode control of perturbed linear systems from an algebraic viewpoint. The basic developments and details may also be found in Fliess and Sira-Ramírez [12]

3.1 Linear Perturbed Dynamics

Definition 3.1 *A linear perturbed dynamics $\overline{\mathcal{D}}$ is a module where we distinguish a control input vector $\overline{u} = (\overline{u}_1, \ldots, \overline{u}_m)$ and perturbation inputs $\overline{\xi} = (\overline{\xi}_1, \ldots, \overline{\xi}_m)$ such that*

$$\overline{\mathcal{D}}/[\overline{u}, \overline{\xi}] = torsion.$$

Control and perturbation inputs are not assumed to interact, thus the condition

$$[\overline{\xi}] \cap [\overline{u}] = \{0\}$$

appears to be quite natural. It will be assumed furthermore assumed that $[\overline{u}]$ is free. This means that we are essentially considering linear systems with unrestricted control inputs. Note, however, that perturbations are not necessarily independent in the sense that they might indeed satisfy some (unknown) set of differential equations. For this reason we assume here that $[\overline{\xi}]$ is not necessarily free, i.e. it may contain torsion elements. Consider the canonical epimorphism

$$\phi : \overline{\mathcal{D}} \rightarrow \overline{\mathcal{D}}/[\overline{\xi}] = \mathcal{D}$$

Since $[\overline{u}] \cap [\overline{\xi}] = 0$, then $\phi\mid_{[\overline{u}]}$ and $\phi\mid_{[\overline{\xi}]}$ are isomorphisms, i.e.

$$[\overline{u}] \simeq [u] \quad ; \quad [\overline{\xi}] \simeq [\xi]$$

This means that we should not distinguish between "perturbed" and "unperturbed" versions of the control input (i.e. between $\overline{u}$ and u), nor between similar versions of the perturbation input ($\overline{\xi}$ and ξ). Since $\mathcal{D}/[u]$ is torsion, we call $\mathcal{D}$ the unperturbed linear dynamics with u being the unperturbed control. It is also reasonable to assume that the unperturbed version of the system, $\mathcal{D}$ is controllable, i.e. $\mathcal{D}$ is free. Regulation of uncontrollable systems is only possible in quite limited and unrealistic cases.

3.2 A Module–Theoretic Characterization of Sliding Regimes

The work presented here follows [12], where an algebraic characterization of sliding regimes is presented in terms of module theory.

Definition 3.2 *Let $\overline{\mathcal{D}}$ be a linear perturbed dynamics, such that $\mathcal{D}$ is controllable. We define a submodule $\overline{S}$ of $\overline{\mathcal{D}}$ as a sliding submodule if the following conditions holds*

1. *The sliding module does not contain elements which are driven exclusively by the perturbations. This condition is synthesized by $[\overline{S}] \cap [\overline{\xi}] = \{0\}$*
2. *The canonical image S of $\overline{S}$ in $\mathcal{D} = \overline{\mathcal{D}}/[\overline{\xi}]$ is a rank m free submodule, i.e. the quotient module*

   $\mathcal{D}/S$ is torsion.

The second condition means that all the control effort is spent in making the system behave as elements that are found in S.

It is convenient to assume that the unperturbed version of the system is observable; $\mathcal{D} = [u, y]$. This guarantees that elements in the sliding module S may be obtained, if necessary, from asymptotic estimation procedures.

$\mathcal{D}/S$ is the *unperturbed (residual) sliding dynamics* while $\overline{\mathcal{D}}/\overline{S}$ is the *perturbed sliding dynamics*. The canonical image of $\overline{u}$ in $\overline{\mathcal{D}}/\overline{S}$ is the *perturbed equivalent control*, denoted by $\overline{u}_{eq}$. The canonical image of u on $\mathcal{D}/S$ is addressed simply as the *equivalent control*, u_{eq}. Note that $\overline{u}_{eq}$ generally depends on the perturbation inputs $\overline{\xi}$, while u_{eq}, is perturbation independent.

Example 3.3 *Consider the linear perturbed dynamics $\dot{\overline{y}} = \overline{u} + \overline{\xi}$. In this case $\overline{\mathcal{D}} = [\tilde{u}, \tilde{y}, \tilde{\xi}]/[\tilde{e}]$, with $\tilde{e} = \dot{\tilde{y}} - \tilde{u} - \tilde{\xi}$. The module $\overline{\mathcal{D}}/[\overline{u}, \overline{\xi}]$ is torsion and $\mathcal{D}$ is rank 1, with y being a basis; $\mathcal{D}$ is also controllable. The condition $\dot{\overline{y}} = -\overline{y}$ may be regarded as a* desirable *asymptotically stable dynamics. Consider $\overline{S} = [\overline{s}] = [\overline{y} + \overline{u}]$. It is easy to see that $\overline{S} \subset \overline{\mathcal{D}}$ with rank $S = 1$, while $\overline{S} \cap [\overline{\xi}] = 0$. Finally, the residue $\underline{y}$ of y in $\mathcal{D}/[y+u]$ satisfies : $\dot{\underline{y}} = -\underline{y}$, which is torsion. Note that the unperturbed equivalent control satisfies $\dot{u}_{eq} + u_{eq} = 0$, while the perturbed equivalent control satisfies $\dot{\overline{u}}_{eq} + \overline{u}_{eq} = -\overline{\xi}$.*

3.3 The Switching Strategy

Let $z = (z_1, \ldots, z_m)$ be a *basis* of S and $\overline{z} = (\overline{z}_1, \ldots, \overline{z}_m)$ be a *basis* of $\overline{S}$, such that z is the image of $\overline{z}$ under $\phi|_{\overline{S}}$. The input-output system relating u to z is *right* and *left invertible*, and hence *decouplable* [16]. Therefore the multivariable case reduces to the single-input single-output case. The switching strategy is, therefore, the same adopted for single input systems. Note that the basis z (resp. $\overline{z}$) is unique up to a constant factor.

Remark 3.4 *Note that in the adopted framework the "matching conditions" are* always *satisfied. This is particularly clear by realizing that the decoupled subsystems may be always placed in a* Generalized Observability Canonical form *[8].*

Example 3.5 *Consider the previous example, $\dot{\overline{y}} = \overline{u} + \overline{\xi}$, with sliding module S generated by $s = u + y$. The element s is a basis for S, while $\overline{s} = \overline{u} + \overline{y}$ is a basis for $\overline{S}$. The relation between s and u is trivially invertible. A switching strategy is obtained by considering $\dot{s} = -W\text{sign } s$, with $W > 0$ a sufficiently large constant. This choice results in the discontinuous controller, $\dot{u} + u = -W$ sign $(u + y)$. The response of the perturbed basis to the synthesized controller is governed by $\dot{\overline{s}} = \overline{\xi} - W$ sign $\overline{s}$.*

3.4 Relations with Minimum Phase Systems and Dynamical Feedback

Definition 3.6 *Let $[u, S]$ stand for the module generated by u and S. The sliding module S is said to be minimum phase if and only if one of the following conditions are satisfied*

1. *$[u] = S$*
2. *If $[u] \not\subset S$ then the endomorphism τ, defined as $\tau : [u, S]/S \rightarrow [u, S]/S$ has eigenvalues with negative real parts.*

The first condition means that the elements of the vector u can be expressed as a (decoupled) $\Re[\frac{d}{dt}]$-linear combination of the basis elements in S. The second condition means that some Hurwitz differential polynomial, associated with u, can be expressed as a decoupled $\Re[\frac{d}{dt}]$-linear combination of the basis elements in S.

Example 3.7 *In the previous example the basis s for S was taken to be $s = u + y$ and evidently $[u] \not\subset S$, since u is not expressible as a $\Re[\frac{d}{dt}]$ linear combination of s. Definitely $[u] \subset [u, S] = [u, s]$. Because $\dot{s} = \dot{u} + u$, The residue $\underline{u}$ of u in $[u, s]/[s]$ satisfies the linear system equation $\dot{\underline{u}} + \underline{u} = 0$ and therefore the sliding module is minimum phase.*

3.5 Nonminimum phase case

Let S be non-minimum phase. One may replace s by some other output $\sigma \in \mathcal{D}$, which is for instance a basis of $[u, s]$ and such that the transfer function relating u and σ is minimum phase.

It is easy to see, due to linearity, that the convergence of σ ensures that of s. Thus the minimum phase case is recovered. If the resulting numerator of the transfer function, relating σ and u, is not constant, then switchings may be taken by the highest order derivative of the control signal. This gives naturally the possibility of smoothed sliding mode controllers (see [7]-[11]).

3.6 Some Illustrations

Example 3.8 *Consider the perturbed linear dynamics, $\dot{\overline{y}} = \overline{u} + \xi$, and the (desired) unperturbed second order dynamics given by $\ddot{\overline{y}} + 2\zeta\omega_n\dot{\overline{y}} + \omega_n^2\overline{y} = 0$, with $0 < \zeta < 1$ and $\omega_n > 0$. Consider the sliding module $S \subset \mathcal{D}$, generated by $s = \dot{u} + 2\zeta\omega_n u + \omega_n^2 y$. The element s is a basis for S and $\overline{s} = \dot{\overline{u}} + 2\zeta\omega_n\overline{u} + \omega_n^2\overline{y}$ is a basis for $\overline{S}$. The residue $\underline{y}$ of y in $\mathcal{D}/S$ satisfies the relation $\ddot{\underline{y}} + 2\zeta\omega_n\dot{\underline{y}} + \omega_n^2\underline{y} = 0$, which is certainly torsion and asymptotically stable to zero.*

Evidently $[u] \not\subset [s]$. In order to obtain the necessary inclusion, consider the module $[u, s]$. Here one finds that the relationship between u and the basis element s for S, is given by $\dot{s} = \ddot{u} + 2\zeta\omega_n\dot{u} + \omega_n^2 u$. Taking the quotient $[u, s]/[s]$, one is left with the torsion system $\ddot{\underline{u}} + 2\zeta\omega_n\dot{\underline{u}} + \omega_n^2\underline{u} = 0$.

The linear map associated to $\frac{d}{dt}$ is represented by the matrix

$$\tau = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix}$$

which has eigenvalues with negative real parts. The sliding module S is therefore minimum phase.

Let W be a positive constant parameter. A dynamical sliding mode controller, which is robust with respect to ξ, is given by

$$\ddot{\overline{u}} + 2\zeta\omega_n\dot{\overline{u}} + \omega_n^2\overline{u} = -W\text{sign}(\dot{\overline{u}} + 2\zeta\omega_n\overline{u} + \omega_n^2\overline{y}).$$

Use of the proposed dynamical switching strategy on the system leads to the following regulated dynamics for $\overline{s}$,

$$\dot{\overline{s}} = \dot{\xi} + 2\zeta\omega_n\dot{\xi} + \omega_n^2\xi - W\text{sign}\ \overline{s}.$$

For sufficiently high values of the gain parameter W, the element $\overline{s}$ goes to zero in finite time, and the desired (torsion) dynamics is achieved.

Example 3.9 *Consider the nonminimum phase system $\ddot{\overline{y}} + 2\zeta\omega_n\dot{\overline{y}} + \omega_n^2\overline{y} = \dot{\overline{u}} - \beta\overline{u} + \xi$, (with $\beta > 0$), and the desired dynamics $\dot{\overline{y}} + \alpha\overline{y} = 0$; $\alpha > 0$. Evidently, $s = \dot{y} + \alpha y$ is a basis for the sliding submodule S, and $s = 0$ is deemed to be desirable.*

However, as before, $[u] \not\subset S$. The relationship between s and u is readily obtained as $\ddot{u} + (\alpha - \beta)\dot{u} - \alpha\beta u = \ddot{s} + 2\zeta\omega_n\dot{s} + \omega_n^2 s$. The canonical image $\underline{u}$ of u in $[u, s]/[s]$ leads to the following unstable (torsion) dynamics $\ddot{\underline{u}} + (\alpha - \beta)\dot{\underline{u}} - \alpha\beta\underline{u} = (\frac{d}{dt} + \alpha)(\frac{d}{dt} - \beta)\underline{u} = 0$. The sliding module is therefore nonminimum phase.

Take a new basis σ of S such that $\dot{\sigma} = \beta\sigma + s$. Note that $s = \dot{\sigma} - \beta\sigma$ and $\overline{s} = \dot{\overline{\sigma}} - \beta\overline{\sigma}$. Also, after some algebraic manipulations one can write

$$\sigma = \frac{(\alpha + \beta)(u - \dot{y}) + (\omega_n^2 - 2\zeta\omega_n\alpha - \alpha\beta)y}{\beta^2 + 2\zeta\omega_n\beta + \omega_n^2}$$

which clearly shows that σ is an element in the system module. One now has $\ddot{\sigma} + 2\zeta\omega_n\dot{\sigma} + \omega_n^2\sigma = \dot{u} + \alpha u$. The residue of u in $[u, \sigma]/[\sigma]$ satisfies $\dot{u} + \alpha u = 0$, and the sliding module is now minimum phase.

A robust static (resp. dynamical) sliding mode controller may now be synthesized which guarantees finite time (resp. asymptotic) convergence of $\overline{\sigma}$ to zero. Note that since $\overline{s} = (\frac{d}{dt} - \beta)\ \overline{\sigma}$, by forcing $\overline{\sigma}$ to zero, then $\overline{s}$ also converges to zero. The desired dynamics is, therefore, attainable by means of sliding modes.

4 Conclusions

Module Theory recovers and generalizes all known results of sliding mode control of linear multivariable systems. A more relaxed concept of sliding regimes evolves in this context, as any desirable output dynamics is synthesizable by minimum phase sliding mode control. This statement is independent of the order of the desired dynamics. Generalizations demonstrate, for instance, that *matching conditions* are linked to particular state space realizations, but they have no further meaning from a general viewpoint. This fact has also been corroborated in recent developments in *sliding mode observers* (see Sira-Ramírez and Spurgeon [19]). In this article multivariable sliding mode control problems have been shown to be always reducible to single-input single output problems in a natural manner.

Nonminimum phase problems can be handled by a suitable change of the output variable, whenever physically feasible. The practical implications of this result seem to be multiple (see also Benvenuti et al [20]). Extension of the results here presented to the case of time varying linear systems requires non-conmutative algebra.

An exciting area in which the algebraic approach may be used to full advantage is the area of sliding mode observers for linear systems. An interesting topic for research rests on the extension of sliding mode theory, from an algebraic viewpoint, to nonlinear multivariable sytems. The results so far seem to indicate that the class of systems to which the theory can be extended, without unforseen complications, is constrained to the class of *flat systems* (see Fliess *et al* [21]).

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