On the Sliding Mode Control of Wheeled Mobile Robots *

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Abstract

In this article a robust multivariable sliding mode controller solution is proposed for tracking problems associated to well known types of mobile robots. Differential flatness of the underlying nonlinear kynematics and dynamics of the robot is suitably exploited for the multivariable sliding controller synthesis tasks. Natural control input smoothing is shown to be provided by the sought decoupled nature of the closed loop system through dynamic compensation

Keywords : Sliding Mode Control, Mobile Robots, Differentially flat systems.

1 Introduction

In recent times the theory of nonlinear control systems has enormously benefited from the general considerations derived from Differential Algebra methods applied to systems of controlled differential equations. Fundamental contributions in this area are due to Prof. M. Fliess and his coworkers (see Fliess *et al* [1]–[3]. An outstanding class of nonlinear systems is constituted by the so called "Differentially Flat" systems. These are multivariable systems that can be exactly linearized by either static or dynamical feedback. Such systems are characterized by the existence of a set of independent "linearizing outputs". These in turn satisfy the property that every variable in the system, including the control inputs, can be expressed as *differential functions* of such outputs. We recall that a differential function of a set of indeterminates is any function which has as arguments the indeterminates and a finite number of their time derivatives. Differentially flat systems have been studied by Fliess and colleagues in [4]–[6]. To pinpoint the importance of differentially flat systems one may say that they are the simplest exponents of a classification of nonlinear systems characterized by the fact that they are linearizable by means of *endogenous* feedback controls. Regulation laws that require no external variables in their definition. They are the simplest possible extension of the concept of controllable linear systems to the nonlinear systems domain. A well known example of differentially flat systems is constituted by the large class of nonholonomically constrained systems which are transformable to "chained" canonical form (see Murray and Sastry [7].

Mobile robots of various kinds constitute an interesting example of nonholonomically constrained nonlinear multivariable systems which belong to the general class of differentially flat systems (see the work of d'Andrea Novel *et al* [8], Campion *et al* [9] and also, Sira-Ramirez [10]).

In this article, using the descriptions of mobile robots found in [8], [9] we exploit their differential flatness properties in the systematic specification of multivariable sliding mode controllers solving tracking problems defined on the two dimensional working space of several types of mobile robots. The fundamental idea consists in imposing closed loop discontinuous dynamics on the tracking error variables which guarantee their asymptotic convergence to zero. Sliding surfaces are therefore constituted by stable linear decoupled differential polynomials in the error variables which, thanks to flatness, are easily expressible also as corresponding coupled discontinuous dynamics of the linearizing coordinates. Since the control inputs are themselves expressible as differential functions of the linearizing coordinates, the controller expression is immediately obtained by substituting on these expressions the higher derivatives of the linearizing coordinates as obtained from the decoupled stabililzing error dynamics.

Section 2 presents the derivation of multivariable sliding mode controllers for wheeled robots of type (2,0), (2,1) and (3,0) (see [8]) and the corresponding simulations. All such robots are differentially flat. Section 3 contains the conclusions.

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2 Multivariable Sliding Mode Control of Mobile Robots

In this section we present a systematic approach for the design of multivariable sliding mode controllers for the regulation of mobile robots of several types. The control objective will be constituted by the feedback solution of a tracking problem constituted by the need to follow specific curves in the working plane of the robot. In particular we explore the task of following circles and straight lines. The method, however, is of general value.

2.1 Mobile Robot Type (2,0)

Consider the kynematic and dynamic model of a mobile robot type (2,0):

$$
\begin{aligned}
\dot{x} &= -\eta_1 \sin\theta \\
\dot{y} &= \eta_1 \cos\theta \\
\dot{\theta} &= \eta_2 \\
\dot{\eta}_1 &= \nu_1 \\
\dot{\eta}_2 &= \nu_2
\end{aligned}
\tag{2.1}
$$

where x and y are the position coordinates on the plane of the center of mass of the robot and θ is the heading angle. η_1 is the instantaneous velocity in the heading direction while η_2 is the instantaneous angular velocity of the coordinates fixed to the body of the robot. The control inputs v_1 and v_2 represent, respectively, linear acceleration or angular accelerations (equivalently, pushing forces and rotation torques).

The system (2.1) is differentially flat with linearizing outputs given by the body position coordinates x and y. Indeed,

$$
\begin{aligned}
\theta &= -\arctan\left(\frac{\dot{x}}{\dot{y}}\right) \\
\eta_1 &= \sqrt{\dot{x}^2+\dot{y}^2} \\
\eta_2 &= \frac{\dot{x}\ddot{x}+\dot{y}\ddot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}} \\
\nu_1 &= \dot{\eta}_1 = \frac{\dot{x}\ddot{x}+\dot{y}\ddot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}} \\
\nu_2 &= \dot{\eta}_2 = \frac{\left(\dot{x}y^{(3)}-\dot{y}x^{(3)}\right)\left(\dot{x}^2+\dot{y}^2\right)}{\left(\dot{x}^2+\dot{y}^2\right)^2} \\
&\quad -2\frac{(\dot{x}\ddot{y}-\dot{y}\ddot{x})(\dot{x}\ddot{x}+\dot{y}\ddot{y})}{\left(\dot{x}^2+\dot{y}^2\right)^2}
\end{aligned}
\tag{2.2}
$$

The system (2.1) is found not to be decouplable by means of static feedback. It is easy to see that by performing a dynamic extension of the control input ν_1, the extended system becomes decouplable by static feedback. In other words the dynamical extension of ν_1 renders identical values for the *vector relative degree* and the vector of the *essential orders* of the extended system (see Glumnieau and Moog [11]). Consider then the extended model of the type (2,0) mobile robot,

$$
\begin{aligned}
\dot{x} &= -\eta_1 \sin\theta \\
\dot{y} &= \eta_1 \cos\theta \\
\dot{\theta} &= \eta_2 \\
\dot{\eta}_1 &= \nu_1 \\
\dot{\eta}_2 &= \nu_2 \\
\dot{\nu}_1 &= u_1
\end{aligned}
\tag{2.3}
$$

The new control input variable u_1 can also be placed in terms of the linearizing coordinates x and y. For this one simply derives the previously found expression in (2.2) for the original control input ν_1. One then obtains:

$$
\begin{aligned}
u_1 &= \dot{\nu}_1 = \frac{\left(\ddot{x}^2+\dot{x}x^{(3)}+\dot{y}y^{(3)}\right)\left(\dot{x}^2+\dot{y}^2\right)}{\sqrt{\left(\dot{x}^2+\dot{y}^2\right)^3}} \\
&\quad -\frac{\left(\dot{x}\ddot{x}+\dot{y}\ddot{y}\right)^2}{\sqrt{\left(\dot{x}^2+\dot{y}^2\right)^3}}
\end{aligned}
\tag{2.4}
$$

The extended system (2.3) is now decouplable by means of static feedback. For this note that the vector relative degree of the linearizing coordinates is given by $[3,3]$ and the essential orders are also $[3,3]$.

It is desired to build a regulator which makes the robot automatically track a circle of arbitrary but fixed radius R at a constant angular velocity $\dot{\psi} = \Omega$ with respect to the center of the circle. Evidently $\psi = \arctan(x/y)$. The set of sliding surfaces, written in terms of the linearizing outputs, that accomplish the described tracking task is simply given by,

$$
\begin{aligned}
s_1(x,y) &= x^2+y^2-R^2 = 0 \\
s_2(x,y) &= \frac{y\dot{x}-x\dot{y}}{x^2+y^2} - \Omega = 0
\end{aligned}
\tag{2.5}
$$

The relative degree of s_1 is three while that of s_2 is only 2. A third order discontinuous dynamics must then be imposed on s_1 while a second order discontinuous dynamics must be imposed on s_2 which guarantee decoupled asymptotic convergence to zero of such coordinates.

Let the desirable dynamics for s_1 and s_2 be respectively given by,

$$
\begin{aligned}
&s_1^{(3)} + \beta\ddot{s}_1 + \lambda\dot{s}_1 + W_1 \operatorname{sign}\sigma_1 = 0 \\
&\ddot{s}_2 + \phi\dot{s}_2 + W_2 \operatorname{sign}\sigma_2
\end{aligned}
\tag{2.6}
$$

where

$$\begin{aligned}\sigma_1 &= \ddot{s}_1 + \beta \dot{s}_1 + \lambda s_1 \quad ; \quad \beta, \lambda > 0 \\ \sigma_2 &= \dot{s}_2 + \phi \dot{s}_2 \quad ; \quad \phi > 0 \end{aligned} \tag{2.7}$$

It is easy to see that if σ_1 and σ_2 converge to zero in finite time then both s_1 and s_2 asymptotically converge to zero with dynamical features entirely determined by the design coefficients β, λ and ϕ. That such is the case may be immediately realized since equations (2.6) are equivalent to

$$\dot{\sigma}_1 = -W \text{ sign } \sigma_1 \quad ; \quad \dot{\sigma}_2 = -W \text{ sign } \sigma_2 \tag{2.8}$$

Substituting (2.5) into (2.6), and using the extended system equations (2.3), one obtains the feedback expressions for u_1 and ν_2 in terms of the linearizing coordinates x, y, and their first and second order time derivatives (i.e., position, velocity and accelerations must be measured along the x and y axis).

The multivariable feedback controller components may be readily found from the expressions (2.2) and (2.4) after substitution of the highest derivatives of x and y obtained from (2.6). In order to save space we shall not reproduce the corresponding expressions here.

2.1.1 Simulation results

Simulations were performed to assess the closed loop behavior of the robotic system. The simulated system was constituted by a perturbed version of the system model in conjunction with the derived discontinuous feedback controller.

In order to test the robustness of the proposed feedback control scheme, an unmodelled, bounded, stochastic perturbation signal v_2 was added to the original system (2.1)

$$\begin{aligned}\dot{x} &= -\eta_1 \sin\theta \\ \dot{y} &= \eta_1 \cos\theta \\ \dot{\theta} &= \eta_2 \\ \dot{\eta}_1 &= \nu_1 \\ \dot{\eta}_2 &= \nu_2 + v_1 \end{aligned} \tag{2.9}$$

A second perturbation signal v_2 was also included in the equation for the extended control input $\dot{\nu}_1$, i.e.,

$$\dot{\nu}_1 = u_1 + v_2 \tag{2.10}$$

It should be pointed out that the uncertain perturbation signals, here considered, belong to the class of *matched* uncertain signals affecting the system behavior. It may be shown, however, that even if the perturbation inputs are not matched, as long as the time derivatives of such perturbation signals remain bounded, then the above discontinuous feedback control scheme still exhibits remarkable robustness features. This is due to the fact that the proposed control scheme is actually based on a particular class of input-output representation of the system related to the linearizing outputs. It is known that in such a kind of context the traditional state space matching conditions have no further meaning.

Figure 1 shows the controlled perturbed evolution of the position coordinates of the robot converging towards the specified circle on the plane. The tracking errors s_1 and s_2, shown in Figure 2 are seen to also converge to zero in an asymptotic and robust manner.

2.2 Mobile Robot Type (2,1)

Consider the dynamical model of a type (2,1) mobile robot

$$\begin{aligned}\dot{x} &= -\eta_1 \sin(\theta + \beta) \\ \dot{y} &= \eta_1 \cos(\theta + \beta) \\ \dot{\eta}_1 &= \nu_1 \\ \dot{\theta} &= \eta_2 \\ \dot{\eta}_2 &= \nu_2 \\ \dot{\beta} &= \delta_1 \\ \dot{\delta}_1 &= \nu_3 \end{aligned} \tag{2.11}$$

It is easy to show that the system is differentially flat with linearizing coordinates represented by x, y and β. Indeed, all variables in the system may be written solely in terms of differential functions of these variables

$$\begin{aligned}\eta_1 &= \sqrt{\dot{x}^2 + \dot{y}^2} \\ \eta_2 &= -\frac{\dot{y}\ddot{x}_2 - \dot{x}\ddot{y}}{\dot{x}^2 + \dot{y}^2} - \dot{\beta} \\ \theta &= \arcsin\left(-\frac{\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}}\right) - \beta \\ \nu_1 &= \frac{\dot{x}\ddot{x}_2 - \dot{x}\ddot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}} \\ \nu_2 &= -\frac{\left(\dot{y}x^{(3)} - \dot{x}y^{(3)}\right)\left(\dot{x}^2 + \dot{y}^2\right)}{\left(\dot{x}^2 + \dot{y}^2\right)^2} \\ &\quad -2\frac{\left(\dot{y}\ddot{x} - \dot{x}\ddot{y}\right)\left(\dot{x}\ddot{x} + \dot{y}\ddot{y}\right)}{\left(\dot{x}^2 + \dot{y}^2\right)^2} - \ddot{\beta} \\ \nu_3 &= \ddot{\beta} \end{aligned} \tag{2.12}$$

The system is not decouplable by means of static state feedback. A dynamical extension of the input ν_1 yields a statically decouplable extended system. We thus consider the system (2.11) along with the integrator equation,

$$\dot{\nu}_1 = \nu_4 \tag{2.13}$$

The new input is also a differential function of the linearizing coordinates,

$$\nu_4 = \frac{(\ddot{x}^2 + \dot{x}x^{(3)} + \ddot{y}_2^2 + \dot{y}y^{(3)})\,(\dot{x}^2 + \dot{y}^2)}{\sqrt{(\dot{x}^2 + \dot{y}^2)^3}} - \frac{(\dot{x}\ddot{x} + \dot{y}\ddot{y})^2}{\sqrt{(\dot{x}^2 + \dot{y}^2)^3}} \tag{2.14}$$

In this instance it is desired to synthesize a regulator such that the mobile robot follows the straight line $y = x$ in the working plane, with constant velocity. The error coordinates that capture the essence of such a tracking problem are given by

$$\begin{aligned} s_1 &= x - y = 0 \\ s_2 &= \dot{x}^2 + \dot{y}^2 - \Omega^2 = 0 \\ s_2 &= \beta = 0 \end{aligned} \tag{2.15}$$

We impose the following sliding mode dynamics on the error surface coordinates

$$\begin{aligned} & s_1^{(3)} + \lambda_1 \ddot{s}_1 + \beta \dot{s}_1 + W_1 \text{ sign } \sigma_1 = 0 \\ & \ddot{s}_2 + \phi \dot{s}_2 + W_2 \text{ sign } \sigma_2 = 0 \\ & \ddot{s}_3 + \alpha \dot{s}_3 + W_3 \text{ sign } \sigma_3 = 0 \end{aligned} \tag{2.16}$$

where

$$\sigma_1 = \ddot{s}_1 + \lambda \dot{s}_1 + \beta s_1 \quad ; \quad \sigma_2 = \dot{s}_2 + \phi s_2 \quad ; \quad \sigma_3 = \dot{s}_3 + \alpha s_3 \tag{2.17}$$

As before, it is easy to see that if σ_1 , σ_2 and σ_3 converge to zero in finite time then the errors s_1, s_2 and s_3 asymptotically converge to zero with dynamical features determined by the design coefficients. It is easy to check that the objective is satisfied since equations (2.16) are equivalent to

$$\begin{aligned} \dot{\sigma}_1 &= -W_1 \text{ sign } \sigma_1 \quad ; \quad \dot{\sigma}_2 = -W_2 \text{ sign } \sigma_2 \\ \dot{\sigma}_3 &= -W_3 \text{ sign } \sigma_3 \end{aligned} \tag{2.18}$$

Substituting (2.15) into (2.16), and using the system equations (2.13), one obtains the feedback expressions for ν_2 and ν_3 and ν_4 in terms of the linearizing coordinates x, y, β and their time derivatives

The multivariable feedback controller components may be readily found from the expressions (2.12) and (2.14) after substitution of the highest derivatives of x and y obtained from (2.16).

2.2.1 Simulation results

Simulations were performed to assess the closed loop behavior of the system. The system equations used for the simulations were again constituted by a matched stochastically perturbed version of the system model in conjunction with the derived sliding mode feedback controller.

Figure 3 shows the controlled perturbed evolution of the position coordinates of the robot converging towards the desired straight line defined on the working plane of the robot. The tracking errors s_1 and s_2, are also shown, in Figure 4 to converge to zero in an asymptotic and robust manner.

2.3 Mobile Robot Type (3,0)

We now summarize the developments leading to a multivariable sliding mode controller design for a mobile robot of type (3,0). The tracking problem consists in following the straigh line $y = x$ on the plane with constant velocity Ω. As before the procedure exploits the differential flatness of such a mobile robotic system,

$$\begin{aligned} \dot{x} &= \eta_1 \cos\theta - \eta_2 \sin\theta \\ \dot{y} &= \eta_1 \sin\theta + \eta_2 \cos\theta \\ \dot{\theta} &= \eta_3 \\ \dot{\eta}_1 &= \nu_1 \\ \dot{\eta}_2 &= \nu_2 \\ \dot{\eta}_3 &= \nu_3 \end{aligned} \tag{2.19}$$

Linearizing Coordinates

$$x \quad ; \quad y \quad ; \quad \theta \tag{2.20}$$

Assessment of differential flatness

$$\begin{aligned} \eta_1 &= \frac{\dot{x}}{\cos\theta} + \tan\theta\,(\dot{y}\cos\theta - \dot{x}\sin\theta) \\ \eta_2 &= \dot{y}\cos\theta - \dot{x}\sin\theta \\ \eta_3 &= \dot{\theta} \\ \nu_1 &= \frac{\ddot{x}\cos\theta + \dot{x}\dot{\theta}\sin\theta + \dot{\theta}\,(\dot{y}\cos\theta - \dot{x}\sin\theta)}{\cos^2\theta} \\ &\quad + \tan\theta\left(\ddot{y}\cos\theta - \dot{y}\dot{\theta}\sin\theta - \ddot{x}\sin\theta - \dot{x}\dot{\theta}\cos\theta\right) \\ \nu_2 &= \ddot{y}\cos\theta - \dot{y}\dot{\theta}\sin\theta - \ddot{x}\sin\theta - \dot{x}\dot{\theta}\cos\theta \\ \nu_3 &= \ddot{\theta} \end{aligned} \tag{2.21}$$

Tracking error coordinates

$$\begin{aligned} s_1 &= x - y \quad ; \quad s_2 = \dot{x}^2 + \dot{y}^2 - \Omega^2 \\ s_3 &= \theta - \pi/4 \end{aligned} \tag{2.22}$$

Discontinuous decopuled closed loop error dynamics

$$\begin{aligned} & \ddot{s}_1 + \lambda \dot{s}_1 + W_1 \text{ sign } \sigma_1 = 0 \\ & \dot{s}_2 + W_2 \text{ sign } \sigma_2 = 0 \\ & \ddot{s}_3 + \sigma \dot{s}_3 + W_3 \text{ sign } \sigma_3 = 0 \end{aligned} \tag{2.23}$$

$$\sigma_1 = \dot{s}_1 + \lambda s_1 \quad ; \quad \sigma_2 = s_2 \quad ; \quad \sigma_3 = \dot{s}_3 + \sigma s_3$$

Existence of sliding motions

$$\dot{\sigma}_1 = -W_1 \,\text{sign}\, \sigma_1 \quad ; \quad \dot{\sigma}_2 = -W_2 \,\text{sign}\, \sigma_2 \quad ;$$
$$\dot{\sigma}_3 = -W_3 \,\text{sign}\, \sigma_3 \qquad (2.24)$$

2.3.1 Simulation results

Figure 5 shows the controlled perturbed evolution of the position coordinates of the robot converging towards the desired straight line defined on the working plane of the robot.

3 Conclusions

Differential flatness properties of multivariable nonlinear mobile robots can be advantageously exploited for the discontinuous feedback design of linearizing controllers. The enhanced robustness features of the closed loop system make the proposed approach particularly attractive. In this article several types of wheeled mobile robots have been shown to be differentially flat and the multivariable sliding mode controller design has been carried out and implemented through computer simulations including bounded stocastic perturbation signals of unmodelled but matched nature.

An interesting area of possible further research is constituted by a study of the possibilities of using sliding mode controllers on the class of non-differentially flat systems.

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FIGURES

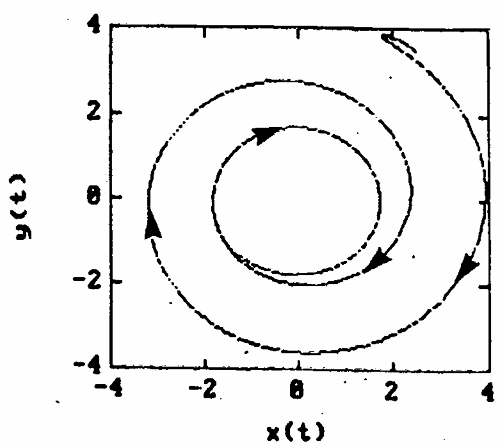

Figure 1: Closed loop trajectory of sliding mode controlled wheeled robot type (2,0)

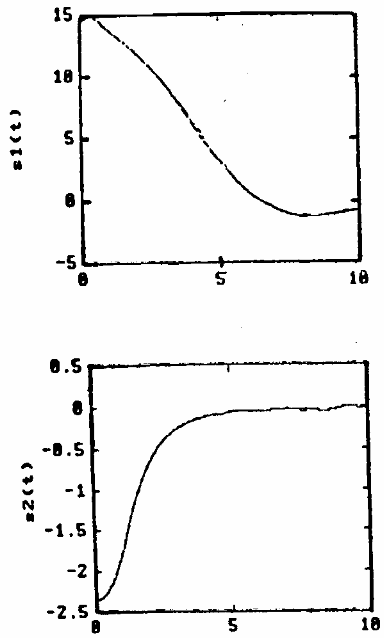

Figure 2: Closed loop behavior of tracking errors for type (2,0) robot

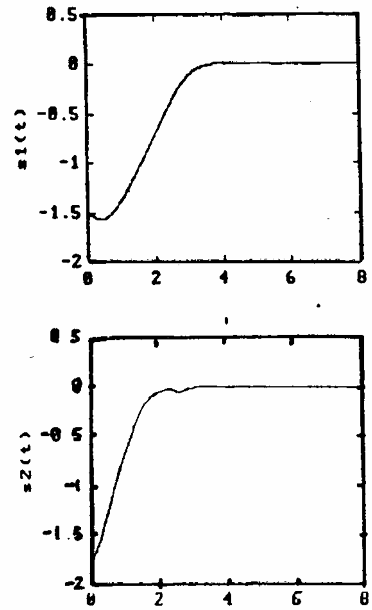

Figure 4: Closed loop behavior of tracking errors for type (2,1) robot

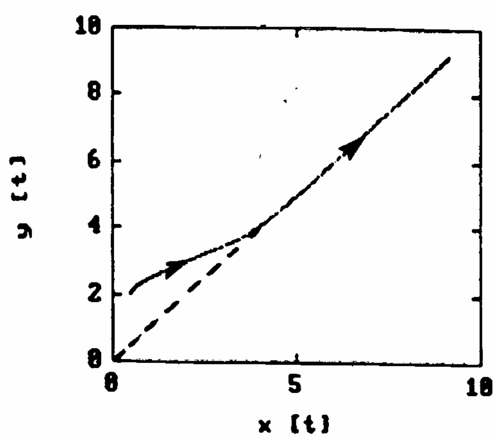

Figure 3: Closed loop trajectory of sliding mode controlled wheeled robot type (2,1) robot

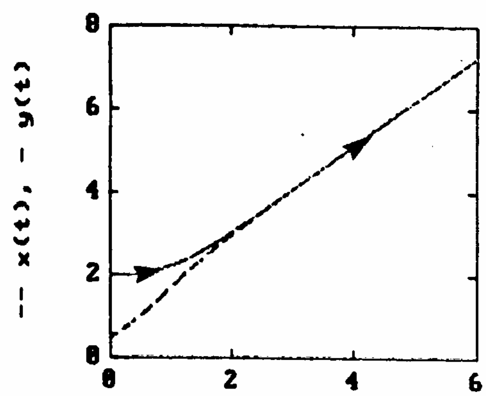

Figure 5: Closed loop trajectory of sliding mode controlled wheeled robot type (3,0) robot