A Lagrangian Approach to Modeling of DC–to–DC Power Converters *

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Abstract

A Lagrangian approach is used for the average modeling of switch regulated DC–to–DC Power Converters of the "Boost" type undergoing Pulse–Width–Modulation (PWM) feedback strategies characterized by an arbitrary duty ratio. The Euler–Lagrange (EL) parameters of the circuit, corresponding to each one of the two possible switch position values, are first established and then averaged in accordance with the PWM regulation policy, using the duty ratio as a suitable "modulation" parameter. An ideal circuit interpretation of the derived average model is also furnished, which replaces the switching device by an ideal lossless transformer.

Keywords Euler-Lagrange Systems, DC–to–DC Power Converters

1 Introduction

In this article a Lagrangian dynamics approach is used for establishing a physically motivated model of the average behavior of dc–to–dc power converters of the "Boost" type. The approach consists in establishing the Euler–Lagrange (EL) parameters of the circuit associated with each one of the topologies corresponding to the two possible positions of the regulating switch. This consideration leads one to realize that some EL parameters remain *invariant* under the switching actions while some others parameters are definitely modified by the addition of certain quantities. The set of non–invariant EL parameters can then be averaged, over time, by "modulating" the added quantities in accordance with the current duty ratio function. This is done in such a manner that extreme saturation values of the duty ratio function, namely, 0 or 1, recover, from the proposed average EL parameter, the original EL parameters of the circuit corresponding to each saturated value of the duty ratio function (i.e., to a particular switch position value).

The average EL parameter considerations immediately lead, through use of the classical EL equations, to systems of continuous differential equations, describing the average converter behavior. These equations are interpretable in terms of an ideal equivalent circuit realization. The proposed average PWM model entirely coincides with the well known *state average model* of the "Boost" converter introduced in [1].

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2 Lagrangian Modeling of DC Power Supplies

Consider the switch–regulated "Boost" converter circuit of figure 1. The differential equations describing the circuit are given by

$$\begin{aligned} \dot{x}_1 &= -(1-u)\,\frac{1}{L}x_2 + \frac{E}{L} \\ \dot{x}_2 &= (1-u)\,\frac{1}{C}x_1 - \frac{1}{RC}x_2 \end{aligned} \qquad (2.1)$$

where x_1 and x_2 represent, respectively, the input inductor current and the output capacitor voltage variables. The positive quantity E represents the constant voltage value of the external voltage source. The variable u denotes the switch position function, acting as a control input. Such a control input takes values in the discrete set $\{0, 1\}$.

A PWM based regulating policy for the switch position function may be specified as follows,

$$u = \begin{cases} 1 \text{ for } & t_k \le t < t_k + \mu(t_k)T \\ 0 \text{ for } & t_k + \mu(t_k)T \le t < t_k + T \end{cases}$$
$$t_{k+1} = t_k + T \quad ; \quad k = 0, 1, \dots \tag{2.2}$$

where t_k represents a sampling instant; the parameter T is the fixed sampling period, also called the *duty cycle*; the sampled values of the state vector $x(t)$ of the converter are denoted by $x(t_k)$. The function, $\mu(\cdot)$, is the *duty ratio function* acting as a truly feedback policy. The value of the duty ratio function, $\mu[t_k]$, determines, at every sampling instant, t_k, the width of the upcoming "pulse" (switch at the position $u = 1$) as $\mu[t_k]T$. The actual duty ratio function, $\mu(\cdot)$, is evidently a function limited to the closed interval $[0, 1]$ on the real line.

In order to use standard notation we refer to the input current x_1 in terms of the derivative of the circulating charge q_L, as $\dot{q}_L$. Also the capacitor voltage x_2 will be written as q_C/C where q_C is the electrical charge stored in the output capacitor.

Consider then $u = 1$. The resulting circuit is as shown in Figure 2. In this case two separate, or decoupled, circuits are clearly obtained and the corresponding lagrangian dynamics formulation can be carried out as follows.

Define $T_1(\dot{q}_L)$ and $V_1(q_C)$ as the kynetic and potential energies of the circuit respectively. We denote by $D_1(\dot{q}_C)$ the Raleigh dissipation function of the circuit. These quantities are readily found to be

$$\begin{aligned} T_1(\dot{q}_L) &= \frac{1}{2}L(\dot{q}_L)^2 \\ V_1(q_C) &= \frac{1}{2C}q_C^2 \\ D_1(\dot{q}_C) &= \frac{1}{2}R(-\dot{q}_C)^2 \\ F_{q_L} &= E \quad ; \quad F_{q_C} = 0 \end{aligned} \tag{2.3}$$

where F_{q_L} and F_{q_C} are the *generalized forcing* functions associated with the coordinates q_L and q_C, respectively.

Consider now the case $u = 0$. The resulting circuit is as shown in Figure 3. The corresponding lagrangian dynamics formulation is carried out in the next paragraphs.

Define $T_0(\dot{q}_L)$ and $V_0(q_C)$ as the kynetic and potential energies of the circuit, respectively. We denote by $D_0(\dot{q}_L, \dot{q}_C)$ the Raleigh dissipation function of the circuit. These quantities are readily found to be,

$$\begin{aligned} T_0(\dot{q}_L) &= \frac{1}{2}L(\dot{q}_L)^2 \\ V_0(q_C) &= \frac{1}{2C}q_C^2 \\ D_0(\dot{q}_L, \dot{q}_C) &= \frac{1}{2}R(\dot{q}_L - \dot{q}_C)^2 \\ F_{q_L} &= E \quad ; \quad F_{q_C} = 0 \end{aligned} \tag{2.4}$$

where, as before, F_{q_L} and F_{q_C} are the *generalized forcing* functions associated with the coordinates q_L and q_C, respectively.

The EL parameters of the two situations generated by the different switch position values result in identical kynetic and potential energies. The switching action merely changes the Raleigh dissipation function between the values $D_0(\dot{q}_C)$ and $D_1(\dot{q}_L, \dot{q}_C)$.

Note that, according to the PWM switching policy (2.2), at every sampling interval of period T, the Raleigh dissipation function $T_1(\dot{q}_C)$ is valid only $\mu(t_k)$ percent of the time while the Raleigh dissipation function $T_0(\dot{q}_L, \dot{q}_C)$ is valid $(1 - \mu(t_k))$ percent of the time.

We propose the following set of EL parameters, for the average circuit behaviour,

$$\begin{aligned} T_\mu(\dot{q}_L) &= \frac{1}{2}L(\dot{q}_L)^2 \\ V_\mu(q_C) &= . \frac{1}{2C}q_C^2 \\ D_\mu(\dot{q}_L, \dot{q}_C) &= \frac{1}{2}R[(1-\mu)\dot{q}_L - \dot{q}_C]^2 \\ F_{q_L} &= E \quad ; \quad F_{q_C} = 0 \end{aligned} \tag{2.5}$$

Note that in the cases where μ takes the extreme values $\mu = 1$ and $\mu = 0$, one recovers, respectively, the dissipation functions $D_1(\dot{q}_C)$ in (2.3) and $D_0(\dot{q}_L, \dot{q}_C)$ in (2.4) from the proposed dissipation function, $D_\mu(\dot{q}_L, \dot{q}_C)$, of equation (2.5).

Define the lagrangian function associated with the above defined EL parameters as,

$$\mathcal{L}_\mu = T_\mu(\dot{q}_L) - V_\mu(q_C) = \frac{1}{2}L(\dot{q}_L)^2 - \frac{1}{2C}q_C^2 \quad (2.6)$$

Using the EL equations on (2.6), one obtains the differential equations which correspond to the proposed average EL parameters (2.5),

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}_\mu}{\partial \dot{q}_L}\right) - \frac{\partial \mathcal{L}_\mu}{\partial q_L} &= -\frac{\partial D_\mu}{\partial \dot{q}_L} + F_{q_L} \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}_\mu}{\partial \dot{q}_C}\right) - \frac{\partial \mathcal{L}_\mu}{\partial q_C} &= -\frac{\partial D_\mu}{\partial \dot{q}_C} + F_{q_C} \end{aligned} \quad (2.7)$$

One obtains the following set of differential equations

$$\begin{aligned} L\ddot{q}_L &= -(1-\mu)R\left[(1-\mu)\dot{q}_L - \dot{q}_C\right] + E \\ \frac{q_C}{C} &= R\left[(1-\mu)\dot{q}_L - \dot{q}_C\right] \end{aligned} \quad (2.8)$$

which can be rewritten, after substitution of the second equation into the first, as

$$\begin{aligned} \ddot{q}_L &= -(1-\mu)\frac{q_C}{LC} + \frac{E}{L} \\ \dot{q}_C &= -\frac{1}{RC}q_C + (1-\mu)\dot{q}_L \end{aligned} \quad (2.9)$$

Using $z_1 = \dot{q}_L$ and $z_2 = q_C/C$ one obtains

$$\begin{aligned} \dot{z}_1 &= -(1-\mu)\frac{1}{L}z_2 + \frac{E}{L} \\ \dot{z}_2 &= (1-\mu)\frac{1}{C}z_1 - \frac{1}{RC}z_2 \end{aligned} \quad (2.10)$$

where we denote by z_1 and z_2 the *average input current* and the *average output capacitor voltage*, respectively, of the PWM regulated "Boost" converter.

Note that the proposed average dynamics coincides with the state average model developed in [1] and with the infinite switching frequency model, or Filippov average model, found in [2]. To obtain the average model (2.10), one simply replaces the switch position function, u, in (2.1) by the duty ratio function μ and the actual state variables x_1, x_2 by their averaged values, z_1, z_2, (see [2]).

It is easy to realize that the average model (2.10) has a circuit-theoretic interpretation by letting the quantity $(1-\mu)z_2$, in the first equation, represent a *controlled voltage source* and letting the quantity $(1-\mu)z_1$, in the second equation, to represent a *controlled input current source*. Figure 4 depicts the ideal equivalent circuit describing the average PWM model. In such a circuit, a quadripole is identified which replaces the switching device.

Consider the isolated quadripole of Figure 4, constituted by the ideal controlled sources. Note that the (average) input power to the quadripole is given by,

$$P_{\text{in}} = \overbrace{z_1}^{\text{input current}} \quad \overbrace{(1-\mu)z_2}^{\text{input voltage}} \quad (2.11)$$

On the other hand, the (average) output power delieverd by the quadripole is given by,

$$P_{\text{out}} = \overbrace{(1-\mu)z_1}^{\text{output current}} \quad \overbrace{z_2}^{\text{output voltage}} \quad (2.12)$$

The quadripole is then a lossless, ideal (average) power transfering device satisfying $P_{\text{in}} = P_{\text{out}}$. The switching element has, thus, been replaced by an *ideal transformer* with *turn ratio* parameter given by $(1-\mu)$.

3 Conclusions

In this article we have shown that the well known average model of the "Boost" DC-to-DC Power Converter is indeed an Euler-Lagrange system. The physical, rather than analytic, nature of the approach is highly appealing and consistent with recent trends in Automatic Control theory (see Ortega *et al* [3] and the references therein).

Euler-Lagrangian formulations of physical control systems have been, so far, restricted to continuously regulated systems. In this article we have given preliminary steps towards the understanding of a Lagrangian Dynamics formulation for discontinuously regulated physical systems. Our approach justifies the use of a passivity-based approach for the design of feedback regulation loops in DC-to-DC power converters.

The results here developed equally apply to other types of DC-to-DC Power Converters, such as the "Buck-Boost", the "Buck" and the "Cúk" converter (see [4] for details). The approach can also be extended to more realistic models of traditional switch-regulated power supplies including parasitic resistances and capacitances.

References

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[4] H. Sira-Ramírez and M. Delgado, "A Lagrangian Approach to Modeling of DC–to–DC Power Converters," IEEE Transactions on Circuits and Systems, (submitted for publication).

FIGURES

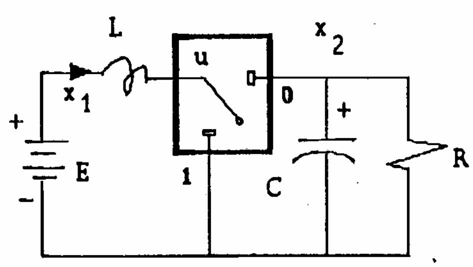

Figure 1: "Boost" Converter Circuit.

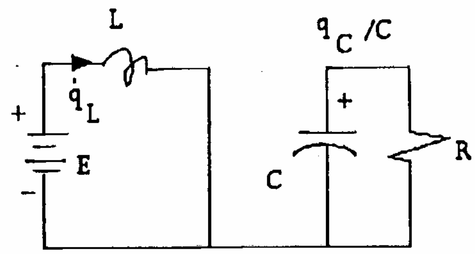

Figure 2: Boost Converter Circuit (u = 1).

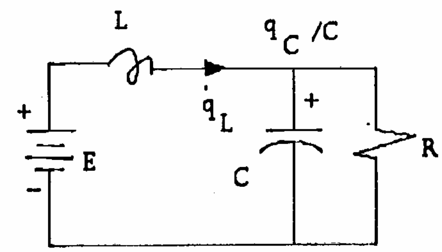

Figure 3: Boost Converter Circuit (u = 0).

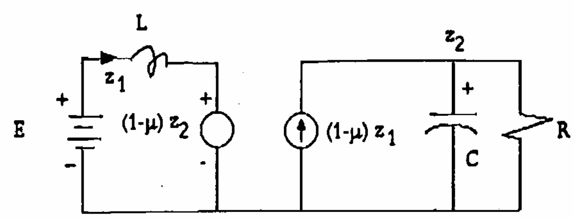

Figure 4: Equivalent Circuit of the Average PWM Model of the "Boost" Converter Circuit.