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Output Tracking Control via Adaptive Input-Output Linearization: A Backstepping Approach

M. Rios-Bolívar, H. Sira-Ramírez and A. S. I. Zinober

Applied Mathematics Section
School of Mathematics and Statistics
University of Sheffield, Sheffield S10 2TN, UK
A.Zinober@sheffield.ac.uk

Abstract

The output tracking problem of a class of observable minimum-phase uncertain nonlinear systems is considered, and a solution based on a suitable combination of input-output linearization and the adaptive backstepping control design procedure is proposed. Dynamical adaptive controllers arise from dynamical input-output linearization, by using a general non-overparameterized adaptive backstepping algorithm without explicit transformation of the controlled system into parametric-pure or parametric-strict feedback forms. The validity of the proposed approach is tested through digital computer simulations.

1 Introduction

Output tracking and regulation problems of linear and nonlinear systems under parametric uncertainty conditions have been extensively studied in recent years. The outstanding backstepping approach developed in [1] provides an efficient control design procedure for both regulation and tracking problems of uncertain systems. This approach is based upon a systematic procedure for the design of feedback control strategies suitable for a large class of feedback linearizable nonlinear systems exhibiting constant, but unknown, parameter values, and guarantees global regulation and tracking for the class of nonlinear systems transformable into the parametric-strict feedback form.

Recently, a recursive procedure has been reported by Sira-Ramírez *et al* ([2]), which implements the fundamental ideas related to the adaptive backstepping algorithm, developed by Krstić *et al* ([3]), in combination with dynamical input-output linearization (see Fliess [4]). This scheme has been also used from a Sliding Mode Control perspective (see [5]) to design dynamical sliding mode output tracking controllers for uncertain nonlinear plants. Here we develop a general algorithm to design output tracking control of a class of observable minimum-phase uncertain nonlinear systems via a non-overparameterized feedback control strategy, which is based upon a combination of the adaptive backstepping algorithm and dynamical input-output linearization. The results presented are locally valid and their main advantage is associated with their applicability to uncertain nonlinear systems without explicit transformation into parametric-pure or parametric-strict feedback forms, while inputs are allowed to appear at intermediate steps of the procedure and control input derivatives are invariably present at the final step of the proposed algorithm.

In Section 2 the algorithm yielding dynamical adaptive output tracking controllers is presented. An application example and digital computer simulations are carried out in Section 3. Section 4 contains the conclusions and suggestions for further work in this area.

2 Dynamical input-output linearization via adaptive backstepping control

In this section we describe a systematic algorithm for dynamical adaptive output tracking controllers from a backstepping perspective. The steps leading to the adaptive controller differ from the traditional considerations, associated with the parametric-pure and parametric-strict feedback forms, since transformations into these canonical forms are not required and, moreover, the control input and their derivatives may appear at intermediate steps of the recursive design procedure. The adopted computational procedure becomes equivalent to the traditional adaptive backstepping algorithm ([3]) when the output corresponds to a "linearizing function" of the system. This approach is suitable for a large class of observable minimum-phase nonlinear systems, dynamically input-output linearizable and with constant but unknown parameters. This class of systems can be represented through the following dynamical system

$$\begin{aligned} \dot{x} &= f_0(x) + \theta^T \gamma(x) + (g_0(x) + \theta^T \varphi(x))u \quad (1) \\ y &= h(x) \end{aligned}$$

where $x \in \Re^n$ is the state, $u \in \Re$ the control input, $y \in \Re$ the output and $\theta \in \Re^p$ an unknown parameter vector. We assume that f_0, g_0, h and the components of γ and φ are smooth vector fields on $\Re^n$, ρ is the relative degree of (1) with respect to u, the relative degree with respect to θ is 1, and the whole state is available for feedback.

The control objective is to drive the system output $y(t)$ to track asymptotically a desired reference signal $y_r(t)$. We also assume that $y_r(t)$ and its derivatives up to order n are bounded and sufficiently smooth functions of t.

Step 1. Define the output variable error as

$$z_1 = y - y_r(t) = h(x) - y_r(t) \tag{2}$$

and according to the system model equations (1), the time derivative of the output error z_1 is given by

$$\begin{aligned} \dot{z}_1 &= h^{(1)}(x,\theta) - \dot{y}_r(t) \\ &= \frac{\partial h}{\partial x}\left[f_0 + \theta^T\gamma + (g_0 + \theta^T\varphi)u\right] - \dot{y}_r(t) \end{aligned} \tag{3}$$

If the relative degree ρ with respect to u is greater than one,

$$\frac{\partial h}{\partial x}\left((g_0(x) + \theta^T\varphi(x))u\right) = 0 \tag{4}$$

is satisfied. By adding to and subtracting from the actual value of the parameters θ their estimated values $\hat{\theta}$, the expression (3) can be rewritten as

$$\dot{z}_1 = \hat{h}^{(1)}(x,\hat{\theta}) - \dot{y}_r(t) + (\theta - \hat{\theta})^T\omega_1 \tag{5}$$

with

$$\hat{h}^{(1)}(x,\hat{\theta}) = \frac{\partial h}{\partial x}\Big((f_0(x) + \hat{\theta}^T\gamma(x))\Big) \tag{6}$$

$$\omega_1 = \frac{\partial h}{\partial x}\gamma(x) \tag{7}$$

Let us consider the quadratic Lyapunov function

$$V_1 = \frac{1}{2}z_1^2 + \frac{1}{2}(\theta - \hat{\theta})^T\Gamma^{-1}(\theta - \hat{\theta}) \tag{8}$$

where $\Gamma = \Gamma^T > 0$ is a matrix of adaptation gains. The time derivative of V_1 is

$$\dot{V}_1 = z_1\left(\hat{h}^{(1)}(x,\hat{\theta}) - \dot{y}_r(t)\right) + (\theta-\hat{\theta})^T\Gamma^{-1}(-\dot{\hat{\theta}} + \Gamma z_1\omega_1) \tag{9}$$

We can achieve $\dot{V}_1 = -c_1 z_1^2$, with c_1 a positive scalar design constant, by choosing the tuning function

$$\dot{\hat{\theta}} = \tau_1 = \Gamma z_1\omega_1 \tag{10}$$

if the expression

$$\hat{h}^{(1)}(x,\hat{\theta}) - \dot{y}_r(t) = -c_1 z_1 \tag{11}$$

is satisfied. The expression (11) represents a desired algebraic relation by which an effective stabilization of the output error would be possible in combination with the estimation update law (10). However, since (11) is not valid from the outset, we take its difference

$$z_2 = \hat{h}^{(1)}(x,\hat{\theta}) - \dot{y}_r(t) + c_1 z_1 \tag{12}$$

as our second error variable, obtaining the closed-loop form for $\dot{z}_1$ as

$$\dot{z}_1 = -c_1 z_1 + z_2 + (\theta - \hat{\theta})^T\omega_1 \tag{13}$$

and $\dot{V}_1$ yields

$$\dot{V}_1 = -c_1 z_1^2 + z_1 z_2 + (\theta - \hat{\theta})^T\Gamma^{-1}(-\dot{\hat{\theta}} + \tau_1) \tag{14}$$

By proceeding this way successively, we obtain the following j-th generic step, which characterizes the first steps previous to the explicit appearance of the control input in the transformed dynamical system.

Step j ($2 \le j \le \rho - 1$)

$$\begin{aligned} \dot{z}_j = {} & \hat{h}^{(j)}(x,\hat{\theta},t) - y_r^{(j)}(t) + \frac{\partial\alpha_{j-1}}{\partial x}(f_0 + \hat{\theta}^T\gamma) \\ & + \frac{\partial\alpha_{j-1}}{\partial\hat{\theta}}\tau_j + \frac{\partial\alpha_{j-1}}{\partial t} + (\theta - \hat{\theta})^T\omega_j \\ & + \left(\frac{\partial\hat{h}^{(j-1)}}{\partial\hat{\theta}} + \frac{\partial\alpha_{j-1}}{\partial\hat{\theta}}\right)(\dot{\hat{\theta}} - \tau_j) \end{aligned} \tag{15}$$

with

$$\begin{aligned} \hat{h}^{(j)}(x,\hat{\theta},t) = {} & \frac{\partial\hat{h}^{(j-1)}}{\partial x}(f_0 + \hat{\theta}^T\gamma) + \frac{\partial\hat{h}^{(j-1)}}{\partial\hat{\theta}}\tau_j \\ & + \frac{\partial\hat{h}^{(j-1)}}{\partial t} \end{aligned} \tag{16}$$

$$\omega_j = \left(\frac{\partial\hat{h}^{(j-1)}}{\partial x} + \frac{\partial\alpha_{j-1}}{\partial x}\right)\gamma(x) \tag{17}$$

and τ_j is the corresponding tuning function defined at this step. By augmenting the Lyapunov function of the form

$$V_j = V_{j-1} + \frac{1}{2}z_j^2 = \frac{1}{2}\sum_{i=1}^{j} z_i^2 + \frac{1}{2}(\theta - \hat{\theta})^T\Gamma^{-1}(\theta - \hat{\theta}) \tag{18}$$

the time derivative of V_j satisfies

$$\begin{aligned} \dot{V}_j = {} & -\sum_{i=1}^{j-1} c_i z_i^2 + z_j\left(\frac{\partial\hat{h}^{(j-1)}}{\partial\hat{\theta}} + \frac{\partial\alpha_{j-1}}{\partial\hat{\theta}}\right)(\dot{\hat{\theta}} - \tau_j) \\ & + \left(\sum_{i=2}^{j-1} z_i\frac{\partial\hat{h}^{(i-1)}}{\partial\hat{\theta}} + \sum_{i=3}^{j-1} z_i\frac{\partial\alpha_{i-1}}{\partial\hat{\theta}}\right)(\dot{\hat{\theta}} - \tau_{j-1}) \\ & + (\theta - \hat{\theta})^T\Gamma^{-1}(-\dot{\hat{\theta}} + \tau_{j-1} + \Gamma z_j\omega_j) \\ & + z_j\left[z_{j-1} + \hat{h}^{(j)}(x,\hat{\theta},t) - y_r^{(j)}(t) + \frac{\partial\alpha_{j-1}}{\partial t}\right. \\ & \left. + \frac{\partial\alpha_{j-1}}{\partial\hat{\theta}}\tau_j + \frac{\partial\alpha_{j-1}}{\partial x}(f_0 + \hat{\theta}^T\gamma)\right] \end{aligned} \tag{19}$$

We can eliminate $(\theta - \hat{\theta})$ from $\dot{V}_j$ by choosing the tuning function

$$\dot{\hat{\theta}} = \tau_j = \tau_{j-1} + \Gamma z_j \omega_j \quad (20)$$

and noting that

$$\dot{\hat{\theta}} - \tau_{j-1} = \dot{\hat{\theta}} - \tau_j + \tau_j - \tau_{j-1} = \dot{\hat{\theta}} - \tau_j + \Gamma z_j \omega_j \quad (21)$$

we rewrite $\dot{V}_j$ as

$$\begin{aligned} \dot{V}_j = & -\sum_{i=1}^{j-1} c_i z_i^2 + (\theta - \hat{\theta})^T \Gamma^{-1}(-\dot{\hat{\theta}} + \tau_j) \\ & + \left(\sum_{i=2}^{j} z_i \frac{\partial \hat{h}^{(i-1)}}{\partial \hat{\theta}} + \sum_{i=3}^{j} z_i \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}} \right) (\dot{\hat{\theta}} - \tau_j) \\ & + z_j \left[\left(\sum_{i=2}^{j-1} z_i \frac{\partial \hat{h}^{(i-1)}}{\partial \hat{\theta}} + \sum_{i=3}^{j-1} z_i \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}} \right) \Gamma \omega_j \right. \\ & + \hat{h}^{(j)}(x, \hat{\theta}, t) - y_r^{(j)}(t) + \frac{\partial \alpha_{j-1}}{\partial x}(f_0 + \hat{\theta}^T \gamma) \\ & \left. + \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}} \tau_j + \frac{\partial \alpha_{j-1}}{\partial t} + z_{j-1} \right] \end{aligned} \quad (22)$$

We can achieve $\dot{V}_j = -\sum_{i=1}^{j} c_i z_i^2$, with the c_i's being positive scalar design constants, if

$$\begin{aligned} & \left(\sum_{i=2}^{j-1} z_i \frac{\partial \hat{h}^{(i-1)}}{\partial \hat{\theta}} + \sum_{i=3}^{j-1} z_i \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}} \right) \Gamma \omega_j \\ & + \hat{h}^{(j)}(x, \hat{\theta}, t) - y_r^{(j)}(t) + \frac{\partial \alpha_{j-1}}{\partial x}(f_0 + \hat{\theta}^T \gamma) \\ & + \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}} \tau_j + \frac{\partial \alpha_{j-1}}{\partial t} + z_{j-1} = -c_j z_j \end{aligned} \quad (23)$$

is satisfied. Again, since (23) is not valid from the outset, we take its difference as our $(j+1)$-th error variable

$$z_{j+1} = \hat{h}^{(j)}(x, \hat{\theta}, t) - y_r^{(j)}(t) + \alpha_j(x, \hat{\theta}, t) \quad (24)$$

with

$$\begin{aligned} \alpha_j = & z_{j-1} + \left(\sum_{i=2}^{j-1} z_i \frac{\partial \hat{h}^{(i-1)}}{\partial \hat{\theta}} + \sum_{i=3}^{j-1} z_i \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}} \right) \Gamma \omega_j \\ & + \frac{\partial \alpha_{j-1}}{\partial x}(f_0(x) + \hat{\theta}^T \gamma(x)) + \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}} \tau_j \\ & + \frac{\partial \alpha_{j-1}}{\partial t} + c_j z_j \end{aligned} \quad (25)$$

obtaining the closed-loop form for $\dot{z}_j$ as

$$\begin{aligned} \dot{z}_j = & -z_{j-1} - c_j z_j + z_{j+1} + (\theta - \hat{\theta})^T \omega_j \\ & - \left(\sum_{i=2}^{j-1} z_i \frac{\partial \hat{h}^{(i-1)}}{\partial \hat{\theta}} + \sum_{i=3}^{j-1} z_i \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}} \right) \Gamma \omega_j \\ & + \left(\frac{\partial \hat{h}^{(j-1)}}{\partial \hat{\theta}} + \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}} \right) (\dot{\hat{\theta}} - \tau_j) \end{aligned} \quad (26)$$

and $\dot{V}_j$ yields

$$\begin{aligned} \dot{V}_j = & -\sum_{i=1}^{j} c_i z_i^2 + z_j z_{j+1} + (\theta - \hat{\theta})^T \Gamma^{-1}(-\dot{\hat{\theta}} + \tau_j) \\ & + \left(\sum_{i=2}^{j} z_i \frac{\partial \hat{h}^{(i-1)}}{\partial \hat{\theta}} + \sum_{i=3}^{j} z_i \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}} \right) (\dot{\hat{\theta}} - \tau_j) \end{aligned} \quad (27)$$

We now summarize the steps containing the control input and its derivatives in the following generic step.

Step k $(\rho \le k \le n-1)$

$$\begin{aligned} \dot{z}_k = & \hat{h}^{(k)}(x, \hat{\theta}, u, \ldots, u^{(k-\rho)}, t) - y_r^{(k)}(t) \\ & + \frac{\partial \alpha_{k-1}}{\partial t} + \frac{\partial \alpha_{k-1}}{\partial \hat{\theta}} \tau_k \\ & + \frac{\partial \alpha_{k-1}}{\partial x}[f_0 + \hat{\theta}^T \gamma + (g_0 + \hat{\theta}^T \varphi) u] \\ & + \sum_{i=1}^{k-\rho} \frac{\partial \alpha_{k-1}}{\partial u^{(i-1)}} u^{(i)} + (\theta - \hat{\theta})^T \omega_k \\ & + \left(\frac{\partial \hat{h}^{(k-1)}}{\partial \hat{\theta}} + \frac{\partial \alpha_{k-1}}{\partial \hat{\theta}} \right) (\dot{\hat{\theta}} - \tau_k) \end{aligned} \quad (28)$$

with

$$\begin{aligned} \hat{h}^{(k)}(x, \hat{\theta}, u, \ldots, u^{(k-\rho)}, t) = & \frac{\partial \hat{h}^{(k-1)}}{\partial \hat{\theta}} \tau_k \\ & + \frac{\partial \hat{h}^{(k-1)}}{\partial x}[f_0 + \hat{\theta}^T \gamma + (g_0 + \hat{\theta}^T \varphi) u] \\ & + \sum_{i=1}^{k-\rho} \frac{\partial \hat{h}^{(k-1)}}{\partial u^{(i-1)}} u^{(i)} + \frac{\partial \hat{h}^{(k-1)}}{\partial t} \end{aligned} \quad (29)$$

$$\omega_k = \left(\frac{\partial \hat{h}^{(k-1)}}{\partial x} + \frac{\partial \alpha_{k-1}}{\partial x} \right) (\gamma + \varphi u) \quad (30)$$

and τ_k is a tuning function. By augmenting the Lyapunov function of the form

$$V_k = V_{k-1} + \frac{1}{2} z_k^2 = \frac{1}{2} \sum_{i=1}^{k} z_i^2 + \frac{1}{2} (\theta - \hat{\theta})^T \Gamma^{-1} (\theta - \hat{\theta}) \quad (31)$$

the time derivative of V_k satisfies

$$\begin{aligned} \dot{V}_k = & z_k \left(\frac{\partial \hat{h}^{(k-1)}}{\partial \hat{\theta}} + \frac{\partial \alpha_{k-1}}{\partial \hat{\theta}} \right) (\dot{\hat{\theta}} - \tau_k) - \sum_{i=1}^{k-1} c_i z_i^2 \\ & + \left(\sum_{i=2}^{k-1} z_i \frac{\partial \hat{h}^{(i-1)}}{\partial \hat{\theta}} + \sum_{i=3}^{k-1} z_i \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}} \right) (\dot{\hat{\theta}} - \tau_{k-1}) \\ & + (\theta - \hat{\theta})^T \Gamma^{-1} (-\dot{\hat{\theta}} + \tau_{k-1} + \Gamma z_k \omega_k) \\ & + z_k \left[z_{k-1} + \hat{h}^{(k)}(x, \hat{\theta}, u, \ldots, u^{(k-\rho)}, t) - y_r^{(k)} \right. \\ & + \frac{\partial \alpha_{k-1}}{\partial \hat{\theta}} \tau_k + \sum_{i=1}^{k-\rho} \frac{\partial \alpha_{k-1}}{\partial u^{(i-1)}} u^{(i)} + \frac{\partial \alpha_{k-1}}{\partial t} \\ & \left. + \frac{\partial \alpha_{k-1}}{\partial x}[f_0 + \hat{\theta}^T \gamma + (g_0 + \hat{\theta}^T \varphi) u] \right] \end{aligned} \quad (32)$$

We can eliminate $(\theta - \hat{\theta})$ from $\dot{V}_k$ by choosing the tuning function

$$\dot{\hat{\theta}} = \tau_k = \tau_{k-1} + \Gamma z_k \omega_k \tag{33}$$

and noting that

$$\dot{\hat{\theta}} - \tau_{k-1} = \dot{\hat{\theta}} - \tau_k + \tau_k - \tau_{k-1} = \dot{\hat{\theta}} - \tau_k + \Gamma z_k \omega_k \tag{34}$$

we rewrite $\dot{V}_k$ as

$$\begin{aligned}
\dot{V}_k = &-\sum_{i=1}^{k-1} c_i z_i^2 + (\theta - \hat{\theta})^T \Gamma^{-1}(-\dot{\hat{\theta}} + \tau_k) \\
&+ \left(\sum_{i=2}^{k} z_i \frac{\partial \hat{h}^{(i-1)}}{\partial \hat{\theta}} + \sum_{i=3}^{k} z_i \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}} \right) (\dot{\hat{\theta}} - \tau_k) \\
&+ z_k \Bigg[\left(\sum_{i=2}^{k-1} z_i \frac{\partial \hat{h}^{(i-1)}}{\partial \hat{\theta}} + \sum_{i=3}^{k-1} z_i \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}} \right) \Gamma \omega_k \\
&+ \hat{h}^{(k)}(x, \hat{\theta}, u, \ldots, u^{(k-\rho)}, t) - y_r^{(k)}(t) \\
&+ \frac{\partial \alpha_{k-1}}{\partial x}[f_0 + \hat{\theta}^T \gamma + (g_0 + \hat{\theta}^T \varphi) u] \\
&+ \frac{\partial \alpha_{k-1}}{\partial \hat{\theta}} \tau_k + \sum_{i=1}^{k-\rho} \frac{\partial \alpha_{k-1}}{\partial u^{(i-1)}} u^{(i)} \\
&+ \frac{\partial \alpha_{k-1}}{\partial t} + z_{k-1} \Bigg]
\end{aligned} \tag{35}$$

We can achieve $\dot{V}_k = -\sum_{i=1}^{k} c_i z_i^2$, with the c_i's being positive scalar design constants, if the expression

$$\begin{aligned}
&z_{k-1} + \left(\sum_{i=2}^{k-1} z_i \frac{\partial \hat{h}^{(i-1)}}{\partial \hat{\theta}} + \sum_{i=3}^{k-1} z_i \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}} \right) \Gamma \omega_k \\
&+ \hat{h}^{(k)}(x, \hat{\theta}, u, \ldots, u^{(k-\rho)}, t) - y_r^{(k)}(t) + \frac{\partial \alpha_{k-1}}{\partial \hat{\theta}} \tau_k \\
&+ \frac{\partial \alpha_{k-1}}{\partial x}[f_0 + \hat{\theta}^T \gamma + (g_0 + \hat{\theta}^T \varphi) u] \\
&+ \sum_{i=1}^{k-\rho} \frac{\partial \alpha_{k-1}}{\partial u^{(i-1)}} u^{(i)} + \frac{\partial \alpha_{k-1}}{\partial t} = -c_k z_k
\end{aligned} \tag{36}$$

is satisfied. However, since (36) is not valid from the outset, we take its difference as our $(k+1)$-th error variable

$$\begin{aligned}
z_{k+1} = &\hat{h}^{(k)}(x, \hat{\theta}, u, \ldots, u^{(k-\rho)}, t) - y_r^{(k)}(t) \\
&+ \alpha_k(x, \hat{\theta}, u, \ldots, u^{(k-\rho)}, t)
\end{aligned} \tag{37}$$

with

$$\begin{aligned}
\alpha_k = &\left(\sum_{i=2}^{k-1} z_i \frac{\partial \hat{h}^{(i-1)}}{\partial \hat{\theta}} + \sum_{i=3}^{k-1} z_i \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}} \right) \Gamma \omega_k \\
&+ z_{k-1} + \sum_{i=1}^{k-\rho} \frac{\partial \alpha_{k-1}}{\partial u^{(i-1)}} u^{(i)} \\
&+ \frac{\partial \alpha_{k-1}}{\partial x}[f_0 + \hat{\theta}^T \gamma + (g_0 + \hat{\theta}^T \varphi) u] \\
&+ \frac{\partial \alpha_{k-1}}{\partial \hat{\theta}} \tau_k + \frac{\partial \alpha_{k-1}}{\partial t} + c_k z_k
\end{aligned} \tag{38}$$

obtaining the closed-loop form for $\dot{z}_k$ as

$$\begin{aligned}
\dot{z}_k = &-z_{k-1} - c_k z_k + z_{k+1} + (\theta - \hat{\theta})^T \omega_k \\
&- \left(\sum_{i=2}^{k-1} z_i \frac{\partial \hat{h}^{(i-1)}}{\partial \hat{\theta}} + \sum_{i=3}^{k-1} z_i \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}} \right) \Gamma \omega_k \\
&+ \left(\frac{\partial \hat{h}^{(k-1)}}{\partial \hat{\theta}} + \frac{\partial \alpha_{k-1}}{\partial \hat{\theta}} \right) (\dot{\hat{\theta}} - \tau_k)
\end{aligned} \tag{39}$$

and $\dot{V}_k$ yields

$$\begin{aligned}
\dot{V}_k = &-\sum_{i=1}^{k} c_i z_i^2 + z_k z_{k+1} \\
&+ \left(\sum_{i=2}^{k} z_i \frac{\partial \hat{h}^{(i-1)}}{\partial \hat{\theta}} + \sum_{i=3}^{k} z_i \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}} \right) (\dot{\hat{\theta}} - \tau_k) \\
&+ (\theta - \hat{\theta})^T \Gamma^{-1}(-\dot{\hat{\theta}} + \tau_k)
\end{aligned} \tag{40}$$

Step n. At this final step we select our actual update law $\dot{\hat{\theta}} = \tau_n$ and the dynamical output tracking controller. This step is carried out in a similar manner to obtain, from the tuning function, the following update law for the unknown parameters

$$\dot{\hat{\theta}} = \tau_n = \tau_{n-1} + \Gamma z_n \omega_n = \Gamma \sum_{i=1}^{n} z_i \omega_i \tag{41}$$

From the desired algebraic relation for the error variable z_n, we obtain the following expression characterizing the output tracking controller

$$\begin{aligned}
&z_{n-1} + \left(\sum_{i=2}^{n-1} z_i \frac{\partial \hat{h}^{(i-1)}}{\partial \hat{\theta}} + \sum_{i=3}^{n-1} z_i \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}} \right) \Gamma \omega_n \\
&+ \hat{h}^{(n)}(x, \hat{\theta}, u, \ldots, u^{(n-\rho)}, t) - y_r^{(n)}(t) \\
&+ \frac{\partial \alpha_{n-1}}{\partial x}[f_0 + \hat{\theta}^T \gamma + (g_0 + \hat{\theta}^T \varphi) u] + \frac{\partial \alpha_{n-1}}{\partial t} \\
&+ \sum_{i=1}^{n-\rho} \frac{\partial \alpha_{n-1}}{\partial u^{(i-1)}} u^{(i)} + \frac{\partial \alpha_{n-1}}{\partial \hat{\theta}} \tau_n = -c_n z_n
\end{aligned} \tag{42}$$

The control function u can be obtained implicitly, as the solution of the nonlinear time-varying differential equation defined by (42). This expression, together with the update law (41), allows us to achieve our goal

$$\dot{V} = \dot{V}_n = -\sum_{i=1}^{n} c_i z_i^2 \leq 0 \tag{43}$$

The convergence of the output to the desired trajectory $y_r(t)$ can be proved by using the LaSalle invariance theorem (see [3]).

3 An application example

Consider the following nonlinear dynamical model of a field controlled DC-motor

$$\begin{aligned} \dot{x}_1 &= -\frac{R_a}{L_a}x_1 - \frac{K}{L_a}x_2 u + \frac{V_a}{L_a} \\ \dot{x}_2 &= -\frac{B}{J}x_2 + \frac{K}{J}x_1 u \\ y &= x_2 \end{aligned} \tag{44}$$

where x_1 represents the armature circuit current and x_2 is the angular velocity of the rotating axis. V_a is a fixed voltage applied to the armature circuit and u is the field winding input voltage, acting as the control input. The constants R_a, L_a, and K represent the resistance, the inductance in the armature circuit and the constant torque, respectively. The parameters J and B are the moment of inertia and the associated viscous damping coefficient of the load. We assume that all parameters are unknown and rewrite (44) as

$$\begin{aligned} \dot{x}_1 &= -\theta_1 x_1 - \theta_2 x_2 u + \theta_3 \\ \dot{x}_2 &= -\theta_4 x_2 + \theta_5 x_1 u \\ y &= x_2 \end{aligned} \tag{45}$$

with

$$\theta_1 = \frac{R_a}{L_a}\ ;\ \theta_2 = \frac{K}{L_a}\ ;\ \theta_3 = \frac{V_a}{L_a}\ ;\ \theta_4 = \frac{B}{J}\ ;\ \theta_5 = \frac{K}{J} \tag{46}$$

We assume that $y_r(t)$ is a known, desired, bounded reference trajectory for the angular velocity x_2 taken to be the output function. By following the algorithm described in the previous section, we design an adaptive controller to track the desired trajectory. Note that the relative degree ρ is one and therefore the time derivative of the control input will appear at the second step of the design algorithm.

Step 1. By defining the output variable error $z_1 = x_2 - y_r(t)$, the time derivative of z_1 is given by

$$\dot{z}_1 = \hat{\theta}^T\omega_1 + (\theta - \hat{\theta})^T\omega_1 - \dot{y}_r(t) \tag{47}$$

with

$$\omega_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -x_2 \\ x_1 u \end{bmatrix}$$

Consider now the Lyapunov function

$$V_1 = \frac{1}{2}z_1^2 + \frac{1}{2}(\theta - \hat{\theta})^T\Gamma^{-1}(\theta - \hat{\theta}) \tag{48}$$

Its time derivative is given by

$$\dot{V}_1 = z_1\left[\hat{\theta}^T\omega_1 - \dot{y}_r(t)\right] + (\theta - \hat{\theta})^T\Gamma^{-1}(-\dot{\hat{\theta}} + \Gamma z_1\omega_1) \tag{49}$$

We can achieve $\dot{V}_1 = -c_1 z_1^2$ with the tuning function

$$\dot{\hat{\theta}} = \tau_1 = \Gamma z_1 \omega_1 \tag{50}$$

if the expression

$$\hat{\theta}^T\omega_1 - \dot{y}_r(t) = -c_1 z_1 \tag{51}$$

is satisfied. However, since (51) is not valid, we define our second error variable as

$$z_2 = \hat{\theta}^T\omega_1 - \dot{y}_r(t) + c_1 z_1 \tag{52}$$

obtaining the following closed-loop form for $\dot{z}_1$

$$\dot{z}_1 = -c_1 z_1 + z_2 + (\theta - \hat{\theta})^T\omega_1 \tag{53}$$

and

$$\dot{V}_1 = -c_1 z_1^2 + z_1 z_2 + (\theta - \hat{\theta})^T\Gamma^{-1}(-\dot{\hat{\theta}} + \tau_1) \tag{54}$$

Step 2. At this step we design the output tracking controller and the updating law for the unknown parameters. The time derivative of the second error variable is

$$\begin{aligned} \dot{z}_2 = &\ \hat{\theta}^T\omega_2 + \dot{\hat{\theta}}^T\omega_1 + \hat{\theta}_5 x_1 \dot{u} - \ddot{y}_r(t) - c_1\dot{y}_r(t) \\ &+ (\theta - \hat{\theta})^T\omega_2 \end{aligned} \tag{55}$$

with

$$\omega_2 = \begin{bmatrix} -\hat{\theta}_5 x_1 u \\ -\hat{\theta}_5 x_2 u^2 \\ \hat{\theta}_5 u \\ -(c1 - \hat{\theta}_4)x_2 \\ (c_1 - \hat{\theta})x_1 u \end{bmatrix}$$

By augmenting the Lyapunov function as

$$V_2 = V_1 + \frac{1}{2}z_2^2 \tag{56}$$

we obtain the following time derivative of V_2

$$\begin{aligned} \dot{V}_2 = &-c_1 z_1^2 + (\theta - \hat{\theta})^T\Gamma^{-1}(-\dot{\hat{\theta}} + \tau_1 + \Gamma z_2\omega_2) \\ &+ z_2\Big[z_1 + \hat{\theta}^T\omega_2 + \dot{\hat{\theta}}^T\omega_1 + \hat{\theta}_5 x_1\dot{u} \\ &- \ddot{y}_r(t) - c_1\dot{y}_r(t)\Big] \end{aligned} \tag{57}$$

We now can eliminate $(\theta - \hat{\theta})$ from $\dot{V}_2$ with the update law

$$\dot{\hat{\theta}} = \tau_2 = \tau_1 + \Gamma z_2\omega_2 = \Gamma(z_1\omega_1 + z_2\omega_2) \tag{58}$$

and, finally, we achieve

$$\dot{V} = \dot{V}_2 = -c_1 z_1^2 - c_2 z_2^2 \leq 0 \tag{59}$$

with the control implicitly defined by the following nonlinear time-dependent diferential equation

$$\dot{u} = \frac{1}{\hat{\theta}_5 x_1}\left[-z_1 - \hat{\theta}^T\omega_2 - \tau_2^T\omega_1 + \ddot{y}_r(t) + c_1\dot{y}_r(t) - c_2 z_2\right] \tag{60}$$

Simulations of a tracking task were performed for a DC-motor with the following parameter values

$$R_a = 7\ \Omega \quad ; \quad L_a = 120\ \text{mH} \quad ; \quad V_a = 5\ \text{V}$$

$$K = 1.41 \times 10^2\ \text{N-m/A}$$

$$B = 6.04 \times 10^{-6}\ \text{N-m-s/rad}$$

$$J = 1.06 \times 10^{-6}\ \text{N-m-s}^2\text{/rad}$$

A desired output reference trajectory $y_r(t)$ was considered to allow a smooth transition of the angular velocity x_2 between two operating equilibrium points X_2, X_2^*

$$y_r(t) = \begin{cases} X_2 & 0 \leq t < t_1 \\ X_2^* + (X_2 - X_2^*)\exp(-kt^2) & t \geq t_1 \end{cases} \tag{61}$$

Figures 1 and 2 show the time response achieved.

4 Conclusions

The output tracking problem of a class of observable minimum-phase uncertain nonlinear systems has been solved via adaptive input-output linearization in combination with the backstepping algorithm. The proposed approach achieves the design of dynamical adaptive output tracking controllers and can be applied to a large class of nonlinear systems, including those that are not transformable into the parametric-pure and parametric-strict feedback forms, typically considered in the applications of the backstepping procedure. As an application example, the controlled smooth transition of the angular velocity of a nonlinear DC-motor was presented. As a topic for further research, the use of state observers to estimate unmeasured state variables should be studied.

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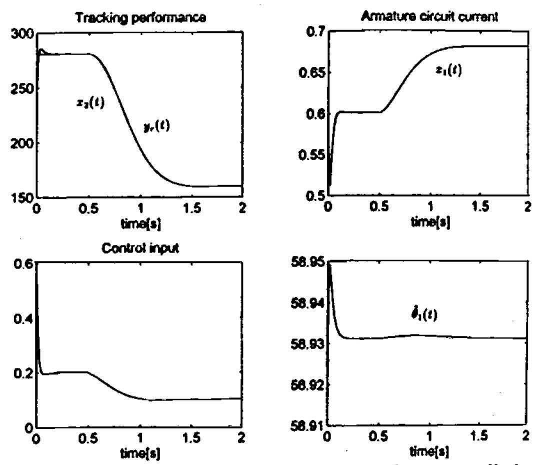

Figure 1: Angular velocity response for controlled tracking task, armature circuit current, control input voltage and parameter estimate of θ_1.

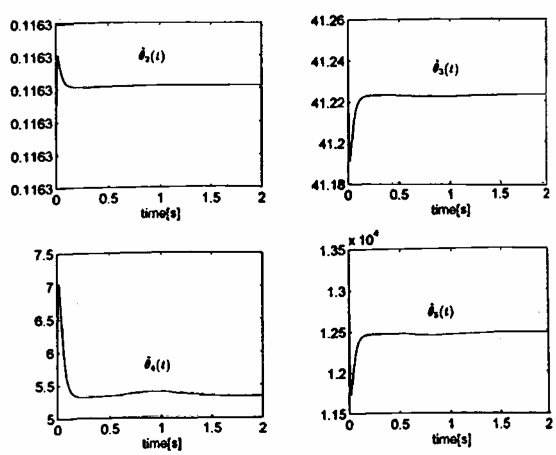

Figure 2: Parameter estimates of $\theta_2, \theta_3, \theta_4$ and θ_5.