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REGULATION OF DC–TO–DC POWER CONVERTERS: A DIFFERENTIAL FLATNESS APPROACH [1]

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Abstract. Differential Flatness of average models of Pulse-Width-Modulation (PWM) regulated dc-to-dc power converters is demonstrated and exploited for the design of linearizing feedback duty ratio synthesizers achieving local stabilization tasks in converters of the "boost", "buck-boost" and "boost-boost" types. The performance and robustness of the proposed feedback regulators are illustrated by computer simulations that include external perturbation inputs.

Keywords. DC-to-DC Power Converters, Differentially Flat Systems, Pulse-Width-Modulation Control

1. INTRODUCTION

Differentially flat systems constitute a widespread class of dynamical systems which represents the simplest possible extension of controllable linear systems to the nonlinear systems domain. "Flat" systems were first introduced by Fliess and his co-workers in (Fliess *et al.*, 1992) and further developed and characterized in (Fliess *et al.*, 1993*a*). Flat systems (in short) enjoy the property of possessing a finite set of *differentially independent outputs*, i.e. outputs which do not satisfy, by themselves, nonlinear differential equations, called *linearizing* or *flat outputs*. The fundamental characteristics of such linearizing outputs are three: First, the number of linearizing outputs is identical to the number of inputs in the system. Second, all variables in the system, including the control input variables, can be written, exclusively, in terms of *differential functions* of such linearizing outputs, i.e. functions of the linearizing outputs and of a finite number of their time derivatives. Finally, generally speaking, the linearizing outputs can, in turn, be expressed as differential functions of the system state vector components. Therefore, these outputs possibly have explicit dependences on the control inputs and a finite number of their time derivatives. Flat systems are, thus, linearizable to controllable linear ones by means of *endogenous* feedback (see (Martin, 1992)).

In this article we take the alternative view of differential

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flatness for the exact linearization of PWM regulated switchmode dc–to–dc power converters. The topic, using static feedback linearization techniques (see (Isidori, 1989)), has been already treated from a sliding mode control viewpoint in (Sira-Ramírez and Ilic, 1989).

In Section 2 we first demonstrate that average PWM models of dc–to–dc power converters of the "boost" and "buck–boost" types are indeed differentially flat. This fact is used in the design of average linearizing feedback controllers in Section 3. The resulting static feedback regulators achieve local asymptotic stabilization of the converter states by means of perfect closed loop linearization of the average dynamics of suitable energy–related scalar functions, acting as linearizing outputs. We also apply the differential flatness approach to an interesting class of multivariable switchmode power converters constitued by a cascade connection of two "boost" dc–to–dc power converters. Section 4 is devoted to test the performance and robustness of the proposed linearizing controllers by computer simulations. The simulations include external stochastic perturbation inputs of the "unmatched" type. Section 5 is devoted to conclusions and suggestions for further work.

2. DIFFERENTIAL FLATNESS OF AVERAGE PWM MODELS OF DC–TO–DC POWER CONVERTERS

2.1 *The "Boost" Converter*

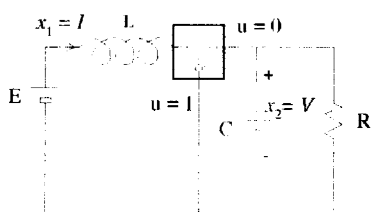

Fig. 1. The "Boost" Converter Circuit

Consider the switch–regulated "boost" converter circuit of Fig. 1. The differential equations describing the circuit are given by

$$
\begin{aligned}
\dot{x}_1 &= -(1-u)\,\frac{1}{L}x_2 + \frac{E}{L} \\
\dot{x}_2 &= (1-u)\,\frac{1}{C}x_1 - \frac{1}{RC}x_2
\end{aligned}
\qquad (1)
$$

where x_1 and x_2 represent, respectively, the input inductor current and the output capacitor voltage variables. The positive quantity E represents the constant voltage value of the external voltage source. The variable u denotes the switch position function, acting as a control input. Such a control input takes values in the discrete set $\{0, 1\}$.

A Pulse Width Modulation (PWM) based regulating policy for the switch position function may be specified as follows,

$$
u = \begin{cases} 1 \text{ for } & t_k \le t < t_k + \mu(t_k)T \\ 0 \text{ for } & t_k + \mu(t_k)T \le t < t_k + T \end{cases}
$$
$$
t_{k+1} = t_k + T \qquad k = 0, 1, \ldots \qquad (2)
$$

where t_k represents a sampling instant; the parameter T is the fixed sampling period, also called the ***duty cycle***; the sampled values of the state vector $x(t)$ of the converter are denoted by $x(t_k)$. The function, $\mu(\cdot)$, is the ***duty ratio function*** acting as a truly feedback policy. The value of the duty ratio function, $\mu(t_k)$, determines, at every sampling instant, t_k, the width of the upcoming "pulse" (switch at the position $u = 1$) as $\mu[t_k]T$. The actual duty ratio function, $\mu(\cdot)$, is evidently a function limited to the closed interval $[0, 1]$ on the real line.

Well known average models of PWM controlled dc–to–dc power converters are constitued by the, so called, ***state average*** models derived in (Ćuk, 1976) and (Middlebrook and Ćuk, 1976). Based on the geometric averaging theory, Sira–Ramírez and coworkers (Sira-Ramírez, 1989; Sira-Ramírez *et al.*, 1992) obtained the same state average models as in (Middlebrook and Ćuk, 1976) and found an interpretation in terms of the ***ideal sliding dynamics***, and its associated ***equivalent control*** (see (Utkin, 1978)).

To obtain the average model of the open loop converter (1), (2), one simply replaces the switch position function, u, by the duty ratio function μ and the actual state variables x_1, x_2 by their averaged values, z_1, z_2 (see (Sira-Ramírez, 1989)).

$$
\begin{aligned}
\dot{z}_1 &= -(1-\mu)\,\frac{1}{L}z_2 + \frac{E}{L} \\
\dot{z}_2 &= (1-\mu)\,\frac{1}{C}z_1 - \frac{1}{RC}z_2
\end{aligned}
\qquad (3)
$$

where we denote by z_1 and z_2 the ***average input current*** and the ***average output capacitor voltage***, respectively, of the PWM regulated "boost" converter.

Note that the average state equilibrium values Z_1 and Z_2 are related by

$$
Z_1 = \frac{Z_2^2}{ER} \qquad (4)
$$

Let $y = H$ denote the total energy of the system, given by

$$
y = H = \frac{1}{2}(Lz_1^2 + Cz_2^2) \qquad (5)
$$

The first order time derivative of y, along the trajectories of (1), is given by:

$$\dot{y} = z_1 E - \frac{1}{R} z_2^2 \qquad (6)$$

The second order time derivative of y is obtained as,

$$\ddot{y} = -(1-\mu)\left(\frac{E}{L} + \frac{2}{RC} z_1\right) z_2 + \frac{2}{R^2 C} z_2^2 + \frac{E^2}{L} \qquad (7)$$

The implicit function theorem guarantees that z_1, z_2 and μ are functions of $y, \dot{y}, \ddot{y}$, i.e., differential functions of the total energy y, in the region where the determinant of the Jacobian Matrix $\frac{\partial\{y,\dot{y},\ddot{y}\}}{\partial\{z_1,z_2,\mu\}}$ is non-zero, i.e., whenever,

$$\det \frac{\partial\{y,\dot{y},\ddot{y}\}}{\partial\{z_1,z_2,\mu\}} = -LC\left(\frac{E}{L} + \frac{2}{RC} z_1\right)^2 z_2^2 \neq 0 \qquad (8)$$

The total energy y of the average boost converter, thus, locally qualifies as a *linearizing output* for the average system. Differential flatness, i.e., linearizability, of the average model (3) is hence locally valid in the region of the state space where expression (8) holds true.

2.2 *The "Buck–Boost" Converter*

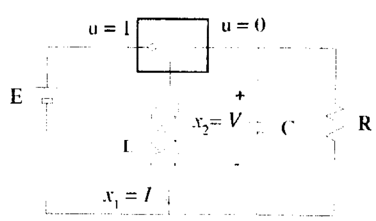

Fig. 2. The "Buck–Boost" Converter Circuit

The switch–regulated "buck–boost" converter circuit is shown in Fig. 2. The state average model of the PWM regulated buck–boost converter is given by

$$\begin{aligned} \dot{z}_1 &= (1-\mu)\frac{1}{L} z_2 + \mu \frac{E}{L} \\ \dot{z}_2 &= -(1-\mu)\frac{1}{C} z_1 - \frac{1}{RC} z_2 \end{aligned} \qquad (9)$$

where z_1 and z_2 denote the average input current and the average output capacitor voltage, respectively. μ denotes, as before, the duty ratio function acting as an external input to the average system.

The components $Z_1 > 0$, $Z_2 < 0$ of the state equilibrium vector are related by

$$Z_1 = \frac{Z_2}{R}\left(\frac{Z_2}{E} - 1\right) \qquad (10)$$

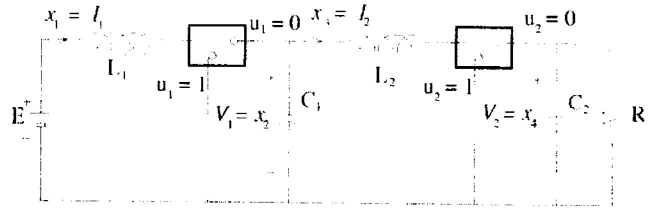

Fig. 3. The "Boost–Boost" Converter Circuit

Let y denote the following energy-like function of the system, given by

$$y = \frac{1}{2}(L z_1^2 + C(z_2 - E)^2) \qquad (11)$$

It is straightfroward to see that (11) is, by the Jacobian matrix test, a differentially flat output for the buck boost averaged PWM model, whenever,

$$\frac{1}{LC}\left[\frac{L}{R} z_1 (2z_2 - E) z_1 + EC(z_2 - E)\right]^2 \neq 0 \qquad (12)$$

In this region, the energy-like function (11) qualifies as a linearizing output.

2.3 *The "Boost boost" Converter*

Consider now a cascade connection of two "boost" dc-to-dc power converters, as shown in Fig. 3. For simplicity, we assume a multivariable PWM feedback regulation policy for the switch position functions u_1 and u_2, characterized by ***synchronous samplings***,

$$\begin{aligned} u_i &= \begin{cases} 1 \text{ for } & t_k \le t < t_k + \mu_i(t_k)T \\ 0 \text{ for } & t_k + \mu_i(t_k)T \le t < t_k + T \end{cases} \\ i &= 1, 2. \\ t_{k+1} &= t_k + T \ ; \qquad k = 0, 1, \ldots \end{aligned} \qquad (13)$$

where μ_i ; $i = 1, 2$, are the duty ratios. T is the common sampling interval and t_k is the sampling instant. The duty ratio functions take values in the closed interval $[0, 1]$ of the real line.

The average PWM model of the multivariable switch-regulated "boost-boost" converter is given as

$$\begin{aligned} \dot{z}_1 &= -(1-\mu_1)\frac{1}{L_1} z_2 + \frac{E}{L_1} \\ \dot{z}_2 &= (1-\mu_1)\frac{1}{C_1} z_1 - \frac{z_3}{C_1} \\ \dot{z}_3 &= -(1-\mu_2)\frac{1}{L_2} z_4 + \frac{z_2}{L_2} \\ \dot{z}_4 &= (1-\mu_2)\frac{1}{C_2} z_3 - \frac{1}{RC_2} z_4 \end{aligned} \qquad (14)$$

For the average state model (14), we take as the linearizing outputs y_1, y_2, the total energy functions H_1, H_2 associated with each one of the constitutive converters:

$$y_1 = H_1 = \frac{1}{2}\left(L_1 z_1^2 + C_1 z_2^2\right)$$
$$y_2 = H_2 = \frac{1}{2}\left(L_2 z_3^2 + C_2 z_4^2\right) \qquad (15)$$

Some simple calculations show that the energy functions y_1 and y_2 constitute linearizing outputs for the average model of the cascaded converter system. The "boost–boost" converter model is, therefore, locally differentially flat and linearizable by multivariable static state feedback [2].

3. LINEARIZING AVERAGE PWM CONTROL POLICIES FOR DC-TO-DC POWER CONVERTERS

3.1 *The "Boost" Converter*

Consider the average PWM model of the "boost" converter (3). Let H^* denote the total energy associated to the state equilibrium $Z = (Z_1, Z_2)$, i.e.,

$$H^* = \frac{1}{2}\left(LZ_1^2 + CZ_2^2\right) = \frac{Z_2^2}{2}\left(C + \frac{L}{R^2E^2}Z_2^2\right) \quad (16)$$

Note that, given a desired Z_2, then both Z_1 and H^* are *uniquely* determined. Let $\tilde{H}$ denote the energy error $\tilde{H} = H - H^*$.

Proposition 1 *Let $Z_2 > 0$ be a desired equilibrium point for model (3). Then, the closed loop system represented by the average "boost" converter (3) and the following feedback duty ratio synthesizer,*

$$\mu = 1 - \frac{1}{\left(\frac{E}{L} + \frac{2}{RC}z_1\right)z_2}\left[\left(\frac{2}{R^2C} - \frac{2\zeta\omega_n}{R} + \frac{\omega_n^2 C}{2}\right)z_2^2 + \left(2\zeta\omega_n E + \frac{\omega_n^2 L}{2}\right)z_1 + \frac{E^2}{L} - H^*\right] \quad (17)$$

where $\zeta > 0$ and ω_n are design constants, result in the asymptotically stable linear dynamics for the energy error $\tilde{H}$.

$$\frac{d^2}{dt^2}\tilde{H} + 2\zeta\omega_n\frac{d}{dt}\tilde{H} + \omega_n^2\tilde{H} = 0 \qquad (18)$$

[2] For a cascade combination of the boost and the buck-boost converter, the flat outputs are constitued by energy-related functions of its components but the system is linearizable by *dynamical* state feedback.

PROOF. The proof is immediate upon substituting the expressions (5), (6), (7) and (16) onto the linear error dynamics (18), and then solving for the required duty ratio function μ. □

3.2 *The "Buck-Boost" Converter*

Consider the average PWM model of the "buck-boost" converter (9). Let H^* denote the energy-like function associated to the equilibrium $Z = (Z_1, Z_2)$, i.e.,

$$H^* = \frac{1}{2}\left[LZ_1^2 - C(Z_2 - E)^2\right] \qquad (19)$$

Given a desired $Z_2 < 0$ then both Z_1 and H^* are *uniquely* determined. Let $\tilde{H}$ denote the energy error $\tilde{H} = H - H^*$. As before, it is possible to obtain a feedback duty ratio synthesizer from the following asymptotically stable linear dynamics on the energy function error $\tilde{H}$.

$$\frac{d^2}{dt^2}\tilde{H} + 2\zeta\omega_n\frac{d}{dt}\tilde{H} + \omega_n^2\tilde{H} = 0 \qquad (20)$$

where $\zeta > 0, \omega_n$ are design parameters, substituting the expressions (11) and (19) onto the linear error dynamics (20), and then solving for the duty ratio function μ.

3.3 *The "Boost-Boost" Converter*

A feedback policy arises in an similar way for the "boost-boost" converter. The feedback duty ratio functions sinthesized for $\mu_1(.)$ and $\mu_2(.)$ in (13) can be calculated directly from the linearizing outputs y_1 and y_2 in (15) and their derivatives up to order 2. Thus, for the averaged system (14) these feedback laws provide asymptotically stable linear dynamics for the energy function errors, $\tilde{H}_1 = H_1 - H_1^*$ and $\tilde{H}_2 = H_2 - H_2^*$, given by

$$\frac{d^2}{dt^2}\tilde{H}_1 + 2\zeta\omega_n\frac{d}{dt}\tilde{H}_1 + \omega_n^2\tilde{H}_1 = 0 \qquad (21)$$
$$\frac{d^2}{dt^2}\tilde{H}_2 + 2\zeta\omega_n\frac{d}{dt}\tilde{H}_2 + \omega_n^2\tilde{H}_2 = 0 \qquad (22)$$

where H_1^* and H_2^* are the energy funtions associated with the desired average state equilibrium points.

Remark 2 *The actual duty ratio functions used in the PWM strategies described above are to be, necessarily, limited to the closed interval* $[0, 1]$. *Thus, the feedback policies only guarantee local stabilization to equilibrium values provided the average closed loop state responses do not abandon the region bounded by the relations $0 \leq \mu(\cdot), \mu_i(\cdot) \leq 1$. As a consequence, the set of initial states that satisfies such a restriction is not arbitrary.*

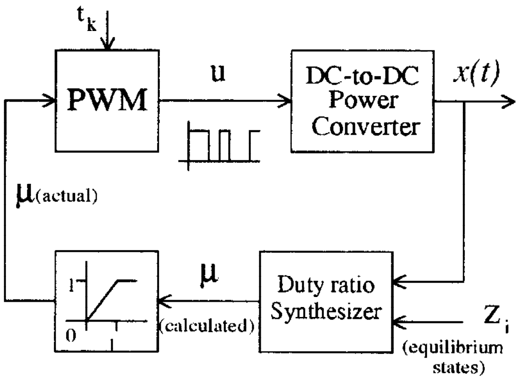

Fig. 4. A Linearizing PWM Feedback Regulation Scheme for DC-to-DC Power Converters

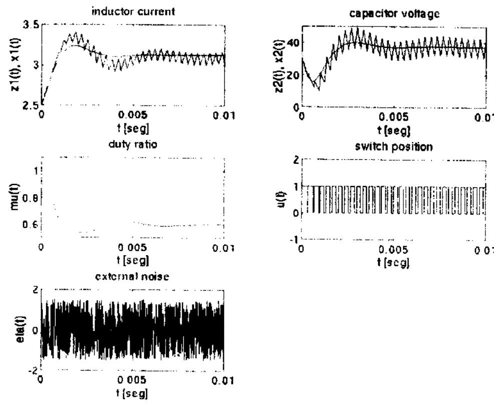

Fig. 5. Performance Evaluation of Linearizing PWM controller for Perturbed "Boost" Converter.

4. SIMULATION RESULTS

The general PWM feedback policy is indicated in Fig. 4. This feedback scheme implies the sampling of the *actual* duty ratio functions at sampling instants t_k ; $k = 0, 1, 2 \ldots$ regularly spaced with period T. The actual switch position function u is specified as in (2), or u_j, $j = 1, 2$, in (13), with the use of the *actual* state variables x_i instead of their averaged values z_i. This procedure has been justified from a theoretical viewpoint and supported by simulations in many references dealing with the average-based PWM feedback control of dc-to-dc power converters (see, for instance, (Sira-Ramírez, 1989; Sira-Ramírez and Lischinsky-Arenas, 1991; Sira-Ramírez *et al.*, 1992) and references therein). For simulation purposes, we only present the boost and boost-boost converter cases.

4.1 *"Boost" converter*

Simulations of the closed loop behaviour of the "boost" converter and the linearizing PWM control policy were performed on the following perturbed version of the "boost" converter circuit,

$$\dot{x}_1 = -(1-u)\,\frac{1}{L}x_2 + \frac{E+\eta}{L}$$
$$\dot{x}_2 = (1-u)\,\frac{1}{C}x_1 - \frac{1}{RC}x_2 \qquad (23)$$

where η represents an external stochastic perturbation input affecting the system behaviour directly through the external voltage source value. This perturbation input is of the "unmatched" type, i.e., it enters the system equations through an input channel vector field, given by $[1/L \quad 0]^T$, which is *not* in the range space of the control input channel, given by the vector field $[1/Lx_2 \quad 1/Cx_1]^T$. The magnitude of this noise was chosen to be, approximately, 20 % of the value of E. The circuit parameter values were taken to be the following "typical" values, $C = 20\ \mu$F; $R = 30\ \Omega$; $L = 20$ mH; $E = 15$ V. The sampling frequency for the PWM policy was set to be 3 KHz. The duty ratio function is obtained from a sampling process carried out on the output $\mu(t)$ of the smooth static duty ratio synthesizer derived above. To avoid the construction of low pass filters, the actual PWM controlled states x_1, x_2 are used on the controllers expressions. The desired average equilibrium output voltage was set to be $Z_2 = 37.5$ Volts, with a corresponding value of the average input current $Z_1 = 3.125$ Amp. The steady state value for the duty ratio is $U = 0.6$.

Fig. 5 shows the closed loop PWM regulated state trajectories and the average state ones corresponding to the linearizing duty ratio synthesizer derived for the "boost" converter. This figure also presents the trajectory of the duty ratio function, the corresponding trajectory of the switch position function, and a realization of the computer generated stochastic perturbation signal η.

4.2 *"Boost-Boost" Converter*

The perturbed switch-regulated "boost-boost" model used in the simulations was:

$$\dot{x}_1 = -(1-u_1)\frac{1}{L_1}x_2 + \frac{E+\eta}{L_1}$$
$$\dot{x}_2 = (1-u_1)\frac{1}{C_1}x_1 - \frac{x_3}{C_1}$$
$$\dot{x}_3 = -(1-u_2)\frac{1}{L_2}x_4 + \frac{x_2}{L_2}$$
$$\dot{x}_4 = (1-u_2)\frac{1}{C_2}x_3 - \frac{1}{RC_2}x_4 \qquad (24)$$

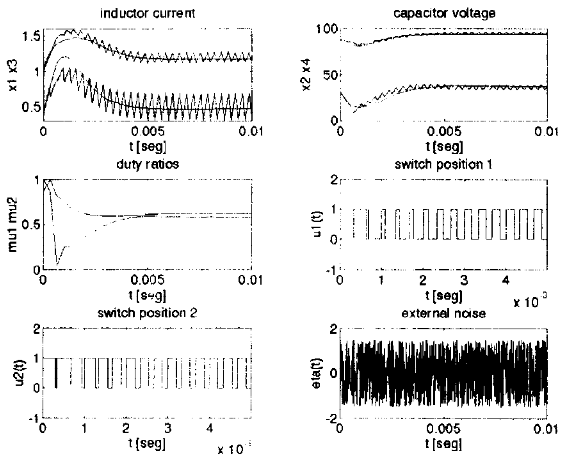

Fig. 6. Performance Evaluation of Linearizing PWM controller for Perturbed "Boost–Boost" Converter.

where $C_1 = C_2 = 20\ \mu$F ; $L_1 = L_2 = 20$ mH ; $R = 500\ \Omega$; $E = 15$ V The desired average output voltages were set to be $Z_4 = 93.75$ Volts and $Z_2 = 37.5$ Volts which correspond to average input currents given by $Z_3 = 0.4687$ Amp. and $Z_1 = 1.1719$ Amp., with steady state duty ratios of $U_1 = U_2 = 0.6$. Fig. 6 shows the closed loop PWM behavior of the regulated "boot–boost" converter.

As it can be seen from the simulations in the previous cases, the proposed controllers achieved the desired stabilization of the output voltages around the desired equilibrium values while exhibiting a high degree of robustness with respect to the "unmatched" external stochastic perturbation input.

5. CONCLUSIONS

In this article, we have exploited the differential flatness property of average models of PWM regulated dc–to–dc power converters for the design of linearizing duty ratio synthesizers. These were shown to achieve local asymptotic stabilization towards predetermined equilibrium points. The approach is based on a dynamic shaping of energy–like coordinates which act as the linearizing outputs of the converters. For the "boost" converter the linearizing output was shown to be the total energy of the circuit while for the "buck boost" converter, the linearizing output is an energy like function closely related to the total energy. In the case of the "boost-boost" converter, the linearizing outputs correspond to the total energy funtions asociated with each one of the constitutive converters.

Differential flatness can also be conveniently exploited in the *adaptive* feedback regulation of uncertain dynamical systems. In particular, dc–to–dc power converters with unknown resistive loads can be treated in a most elegant manner by a novel technique known as *control of the clock*, as proposed in (Fliess *et al.*, 1993*b*). This topic will be the subject of a forthcomming publication.

6. REFERENCES

Ćuk, S. (1976). Modeling, Analysis and Design of Switching Converters. PhD thesis. California Institute of Technology. Pasadena.

Fliess, M., J. Lévine, P. Martin and P. Rouchon (1992). Sur les systèmes linéaires différentiellement plats. *C.R. Acad. Sci. Paris Serie I* **315**, 619–624.

Fliess, M., J. Lévine, P. Martin and P. Rouchon (1993*a*). Défaut d'un systéme non linéaire et commande haute fréquence. *C.R. Acad. Sci. Paris Serie I* **316**, 513–518.

Fliess, M., J. Lévine, P. Martin and P. Rouchon (1993*b*). Linèarisation par bouclage dynamique et transformations de lie-bäcklund. *C.R. Acad. Sci. Paris Serie I* **317**, 981–986.

Isidori, A. (1989). *Nonlinear Control Systems*. 2nd ed.. Springer–Verlag. New York.

Martin, P. (1992). Contribution á l'étude des systémes différentiellement plats. PhD thesis. École des Mines de Paris. Paris.

Middlebrook, R.D. and S. Ćuk (1976). A general unified approach to modelling switching–converter power stages. In: *IEEE Power Electronics Specialists Conference*. pp. 18–34.

Sira-Ramírez, H. (1989). A geometric approach to pulse-width-modulated control in nonlinear dynamical systems. *IEEE Transactions on Automatic Control* **34**(2), 184–187.

Sira-Ramírez, H. and M. Ilic (1989). Exact linearization in switch mode dc-to-dc power converters. *International Journal of Control* **50**(2), 511–524.

Sira-Ramírez, H. and P. Lischinsky-Arenas (1991). The differential algebraic approach in nonlinear dynamical compensator design for dc-to-dc power converters. *International Journal of Control* **54**, 111–134.

Sira-Ramírez, H., P. Lischinsky-Arenas and O. Llanes-Santiago (1992). Dynamic compensator design in nonlinear aerospace systems. *IEEE Transactions on Aerospace and Electronic Systems* **29**, 374–369.

Utkin, V.I. (1978). *Sliding Regimes and Their Applications in Variable Structure Systems*. MIR Editors. Moscow.