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Sliding Mode-Based Adaptive Learning in Dynamical Adalines [1]

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Abstract

A sliding mode control strategy is proposed for the synthesis of adaptive learning algorithms in perceptron-based feedforward neural networks whose weigths are constituted by first order, time-varying, dynamical systems with adjustable parameters. The approach is shown to exhibit remarkable robustness and fast convergence properties. A simulation example, dealing with an analog signal tracking task, is provided which illustrates the feasibility of the approach.

Keywords : Variable Structure Systems, Sliding Regimes, Adaptive Learning

1 Introduction

The discrete-time context has dominated all proposed adaptive learning strategies in perceptron-based feedforward Neural Networks. The celebrated Widrow-Hoff *Delta Rule* (see Widrow and Lehr, [9]) constitutes a least mean square learning error minimization algorithm by which an asymptotically stable linear convergence dynamics is imposed on the underlying discrete-time error dynamics. Using *quasi-sliding mode control* ideas (see Sira-Ramírez [4]) a modification of the Delta Rule was proposed by Sira-Ramírez and Żak in [6], and in [10], whereby a switching weight adaptation strategy is shown to also impose a discrete time asymptotically stable linear learning error dynamics. This algorithm is at the basis of recently proposed dynamical systems identification and control schemes, based on feedforward neural networks, (see Colina-Morles and Mort [1], and Kuschewsky *et al*, [2]). An entirely different viewpoint in perceptron-based adaptive learning has been recently proposed by considering a class of problems defined on analog (i.e., continuous time) adalines, or perceptrons. In correspondance with such a setting, continuous time - rather than discrete time - adaptive weight adjustment needs to be tackled. From such a continous time viewpoint, the design of learning strategies in adaptive analog perceptrons, from the perspective of sliding mode control, has been recently addressed in the work of Sira-Ramírez and Colina-Morles [5]. The relevance of ordinary differential equations with discontinuous right hand sides, or Variable Structure Systems (see Utkin, [7]), was analyzed in the work of Li *et al* , in [3], also in the context of Analog Neural Networks of the Hopfield type with infinite gain nonlinearities. In that work, it is established under what circumstances sliding mode trajectories do not appear in the behavior of such a class of neurons.

In this article the *continuous time* sliding mode control approach for adaptive "static" weight adjustment problem in adalines is briefly revisited closely following the exposition in [5]. Motivated by the dynamical character of the resulting sliding mode control solution, we proceed to propose a new type of perceptron, refered to as: *dynamical adaline*, where all weights are substituted by first order, linear, time-varying dynamical systems. The weight adjustment maneuvers, from a sliding mode perspective, are now to be carried upon the time-varying "gains" and the time-varying "time constants" of the proposed dynamical "weights". On such a fixed dynamical structure for the perceptron weights, the sliding mode control strategy results in a more versatile, simpler, and easier to implement adaptive learning algorithm. The basic features of the proposed approach are not only fast convergence but, also, enhanced robustness with respect to unknown external perturbation inputs and measurement noises. Such advantageous features are, in general, characteristic of sliding mode control adaptive schemes.

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Section 2 contains the fundamental definitions, assumptions and derivations of the main characteristics of a sliding mode control approach to static and dynamical weight adaptation in adalines. Section 3 contains an illustrative example exploring the behaviour of the proposed dynamical weight adjustment algorithm in an output signal tracking problem. Section 4 contains the conclusions and suggestions for further research.

2 Dynamical Weight Adaptation in Adalines

2.1 Background Results

Consider the perceptron model depicted in Figure 1 where $x(t) = (x_1(t), \ldots, x_n(t))$ represents a vector of bounded time-varying *inputs*, assumed also to exhibit bounded time derivatives, i.e.

$$\begin{aligned} \| x(t) \| &= \sqrt{x_1^2(t) + \ldots + x_n^2(t)} \leq V_x \;\; \forall t \\ \| \dot{x}(t) \| &= \sqrt{\dot{x}_1^2(t) + \ldots + \dot{x}_n^2(t)} \leq V_{\dot{x}} \;\; \forall t \end{aligned} \tag{2.1}$$

where V_x and $V_{\dot{x}}$ are known positive constants.

We denote by $\tilde{x}(t)$ the vector of *augmented inputs*, which includes a constant input of value $B \neq 0$, affecting the *bias*, or *threshold* weight w_{n+1} in the perceptron model, i.e

$$\tilde{x}(t) = \text{col}(x_1(t), \ldots, x_n(t), B) = \text{col}(x(t), B) \tag{2.2}$$

The vector $\omega(t) = \text{col}(\omega_1(t), \ldots, \omega_n(t))$ represents the set of time-varying *weights*. It will be assumed that, due to physical constraints, the magnitude of the vector $\omega(t)$ is bounded $\| \omega(t) \| \leq W \; \forall t$, for some constant W. We also define the vector of *augmented weights* by including the bias weight component

$$\begin{aligned} \tilde{\omega}(t) &= \text{col}(\omega_1(t), \ldots, \omega_n(t), \omega_{n+1}(t)) \\ &= \text{col}(\omega(t), \omega_{n+1}(t)) \end{aligned} \tag{2.3}$$

Similarly, $\tilde{\omega}(t)$ is assumed to be bounded at each instant of time t by means of

$$\| \tilde{\omega}(t) \| = \sqrt{\omega_1^2(t) + \cdots + \omega_n^2(t) + \omega_{n+1}^2(t)} \leq \tilde{W} \;\; \forall t \tag{2.4}$$

for some constant $\tilde{W}$.

The scalar signal $y_d(t)$ represents the time-varying *desired output* of the perceptron. It will be assumed that $y_d(t)$ and $\dot{y}_d(t)$ are also bounded signals, i.e.

$$\begin{aligned} | y_d(t) | &\leq V_y \;\; \forall t \\ | \dot{y}_d(t) | &\leq V_{\dot{y}} \;\; \forall t \end{aligned} \tag{2.5}$$

The output signal $y(t)$ is a scalar quantity defined as:

$$\begin{aligned} y(t) &= \sum_{i=1}^{n} \omega_i(t) x_i(t) + \omega_{n+1}(t) \\ &= \omega^T(t) x(t) + \omega_{n+1}(t) \; B = \tilde{\omega}^T(t) \tilde{x}(t) \end{aligned} \tag{2.6}$$

We define the *learning error* $e(t)$ as the scalar quantity obtained from

$$e(t) = y(t) - y_d(t) \tag{2.7}$$

The nonlinear function $\Gamma(y)$ is generally assumed to be an *odd* function of y, i.e. $\Gamma(y) = -\Gamma(-y)$, known as the *activation* function. The activation functions are relevant to the analysis of networks involving several layers of neurons, which we will not be considering here.

Using the theory of *Sliding Mode Control* of *Variable Structure Systems* (see [8]) we propose to consider the zero value of the learning error coordinate $e(t)$ as a time–varying *sliding surface*, i.e.

$$s(e(t)) = e(t) = 0 \tag{2.8}$$

Condition (2.8) is, therefore, deemed as a *desired* condition for the learning error signal $e(t)$ and one which guarantees that the perceptron output $y(t)$ coincides with the desired output signal $y_d(t)$ for all time $t > t_h$ where t_h is addressed as the *hitting time.*

Definition 2.1 *A* sliding motion *is said to exist on a sliding surface $s(e(t)) = e(t) = 0$, after time t_h, if the condition $s(t)\dot{s}(t) = e(t)\dot{e}(t) < 0$ is satisfied for all t in some nontrivial semi-open subinterval of time of the form $[t, t_h) \subset (-\infty, t_h)$.*

It is desired to devise a dynamical feedback *adaptation mechanism*, or *adaptation law*, for the augmented vector of variable weights $\tilde{\omega}(t)$ such that the sliding mode condition of definition 2.1 is enforced.

Let " sign $e(t)$ " stand for the *signum* function, defined as:

$$\text{sign } e = \begin{cases} +1 & \text{for } e(t) > 0 \\ 0 & \text{for } e(t) = 0 \\ -1 & \text{for } e(t) < 0 \end{cases} \tag{2.9}$$

We then have the following result,

Theorem 2.2 *If the adaptation law for the augmented weight vector $\tilde{\omega}(t)$ is chosen as*

$$\begin{aligned}\dot{\tilde{\omega}}(t) &= -\left(\frac{\tilde{x}(t)}{\tilde{x}^T(t)\tilde{x}(t)}\right) k \text{ sign } e(t) \\ &= -\left(\frac{\begin{bmatrix} x(t) \\ B \end{bmatrix}}{B^2 + x^T(t)x(t)}\right) k \text{ sign } e(t)\end{aligned} \tag{2.10}$$

with k being a sufficiently large positive design constant satisfying

$$k > \tilde{W}V_{\dot{x}} + V_{\dot{y}} \tag{2.11}$$

then, given an arbitrary initial condition $e(0)$, the learning error $e(t)$ converges to zero in finite time, t_h, estimated by

$$t_h \leq \frac{| e(0) |}{k - \tilde{W}V_{\dot{x}} - V_{\dot{y}}} \tag{2.12}$$

and a sliding motion is sustained on $e = 0$ for all $t > t_h$.

Proof see [5]

Note that the proposed dynamical feedback adaptation law for the vector of weights in (2.10) results in a *continuous regulated evolution* of the vector of variable weights $\tilde{\omega}(t)$.

Note also that if the quantity $\dot{\tilde{x}}(t)$ is measurable, one can obtain a more relaxed variable structure feedback control strategy than the one obtained in (2.10). Generally speaking, such an adaptive feedback strategy for the variable weights requires smaller design gains k to obtain a corresponding sliding motion.

Since such a case is of some practical importance, we summarize its details in the following theorem, whose proof can be found in [5]

Theorem 2.3 *If the adaptation law for the augmented weight vector $\tilde{\omega}(t)$ is chosen as*

$$\begin{aligned}\dot{\tilde{\omega}}(t) &= -\frac{\tilde{x}(t)}{\tilde{x}^T(t)\tilde{x}(t)} \left(k \text{ sign } e(t) + \tilde{\omega}^T(t)\dot{\tilde{x}}(t)\right) \\ &- \left(\frac{\tilde{x}(t)\dot{\tilde{x}}^T(t)}{\tilde{x}^T(t)\tilde{x}(t)}\right)\tilde{\omega}(t) - \left(\frac{\tilde{x}(t)}{\tilde{x}^T(t)\tilde{x}(t)}\right) k \text{ sign } e(t)\end{aligned} \tag{2.13}$$

with k being a positive design constant satisfying

$$k > V_{\dot{y}} \tag{2.14}$$

then, given an arbitrary initial condition $e(0)$, the learning error $e(t)$ converges to zero in finite time t_h satisfying

$$t_h \leq \frac{| e(0) |}{k - V_{\dot{y}}} \tag{2.15}$$

and a sliding motion is sustained on $e = 0$ for all $t > t_h$.

The proposed solution for $\dot{\tilde{\omega}}(t)$ in (2.13) is, necessarily, aligned with the augmented vector of inputs $\tilde{x}(t)$. The total disregard for the effect of the scalar signal $\dot{y}_d(t)$ in the above adaptation schemes, (2.10) and (2.13), arises from the implicit assumption that such a signal is not, generally speaking, measurable in practise, nor can it be estimated with sufficient precision. The previous theorem shows that as long as $\dot{y}_d(t)$ is bounded, the adaptation policy always manages to bring the learning error to zero in finite time. A similar remark can be made about the control law in (2.10). Figure 2 depicts the (instantaneous) geometric features lying at the basis of the proposed algorithm.

2.2 Adalines with Dynamical Weights

Consider an adaline in which the traditional adjustable weights have been substituted by first order, linear, time-varying, dynamical systems described by

$$\dot{y}_i = a_i(t)y_i + K_i(t)x_i(t) \;\; ; \;\; i = 1, \ldots, n \tag{2.16}$$

where the time-varying scalar functions $a_i(t)$; $i = 1, \ldots, n$ and $K_i(t)$; $i = 1, \ldots, n$ play the role of adjustable parameters. For the lack of better names, we sometimes improperly will refer tosuch parameters as, "gains" and "time constants", respectively, in plain reminiscense of the traditional terms associated with the time-invariant counterparts (see figure 3).

As in traditional adalines, $x(t) = (x_1(t), \ldots, x_n(t))$ represents a vector of bounded time-varying inputs, also assumed to possess bounded time derivatives. We define the vectors $a(t)$ and $K(t)$ as n-dimensional vectors constituted by the "time constants" and "gains", i.e.,

$$\begin{aligned} a(t) &= \text{col}\,(a_1(t), \ldots, a_n(t)) \\ K(t) &= \text{col}\,(K_1(t), \ldots, K_n(t)) \end{aligned} \tag{2.17}$$

The vector $y(t)$ is constituted by the outputs of the dynamical systems acting as weights, $y(t) = (y_1(t), \ldots, y_n(t))$. Each one of the outputs $y_i(t)$; $i = 1, \ldots, n$, qualifies as the *state* of the corresponding dynamical weight. The scalar signal $y_d(t)$ represents the desired output of the perceptron and it constitutes the signal to be tracked by the adaline scalar output $y_0(t)$. It will be assumed that $y_d(t)$ and $\dot{y}_d(t)$ are also bounded signals. The output of the perceptron, $y_0(t)$, is given by

$$y_0(t) = \sum_{i=1}^{n} y_i(t) \tag{2.18}$$

It is assumed that the set of dynamical weights, characterized by the state vector $y(t)$, has an initial state vector given by $y(t_0)$ The learning error, denoted by

$e(t)$, is the scalar quantity defined by,

$$e(t) = y_0(t) - y_d(t) \qquad (2.19)$$

As in the traditional case, it is desired to derive a feedback adaptation law for the adjustable parameter vectors $a(t)$ and $K(t)$, such that the learning error $e(t)$ reaches the value zero, for any set initial conditions - represented by the vector $y(t_0)$ - of the dynamical weights. Moreover it is desired that once the learning error reaches the value zero, such a value is sustained for the remining time horizon.

In the following theorem we assume that the external signal $y_d(t)$ and $\dot{y}_d(t)$ are bounded as in (2.5).

Theorem 2.4 *If the adaptation laws for the adjustble parameters of the dynamical weights is chosen as*

$$\begin{bmatrix} a(t) \\ K(t) \end{bmatrix} = -\left(\frac{W \text{signe}(t)}{\|y(t)\|^2 + \|x(t)\|^2}\right) \begin{bmatrix} y(t) \\ x(t) \end{bmatrix} \qquad (2.20)$$

with W being a sufficiently large positive design constant, satisfying $W > V_{\dot{y}}$ then, given an arbitrary set of initial conditions $y(t_0)$, the learning error $e(t)$ converges to zero in finite time t_h, estimated as,

$$t_h \leq \frac{|y(t_0) - y_d(t_0)|}{W - V_{\dot{y}}}$$

and a sliding motion is sustained on $e = 0$ for all $t > t_h$.

Proof

Compute the time derivative of the learning error as

$$\begin{aligned} \dot{e}(t) &= \sum_{i=1}^{n} \dot{y}_i(t) - \dot{y}_d(t) \\ &= \sum_{i=1}^{n} (a_i(t)y_i(t) + K_i(t)x_i(t)) - \dot{y}_d(t) \\ &= a^T(t)y(t) + K^T(t)x(t) - \dot{y}_d(t) \\ &= [a(t) \;\; K(t)] \begin{bmatrix} y(t) \\ x(t) \end{bmatrix} - \dot{y}_d(t) \\ &= -\dot{y}_d - W \text{sign}\, e(t) \end{aligned}$$

where the last equality is obtained after substitution of the proposed parameter adaptation laws (2.20). Evidently, for all $e(t) \neq 0$,

$$\begin{aligned} e(t)\dot{e}(t) &= -e(t)\dot{y}_d(t) - W|e(t)| \\ &\leq |e(t)|V_{\dot{y}} - W|e(t)| \\ &= -|e(t)|\,(W - V_{\dot{y}}) < 0 \end{aligned}$$

The learning error $e(t)$, thus, satisfies a differential equation with discontinuous right hand side whose solution exhibits a sliding regime in finite time t_h [8].

Under ideal sliding mode conditions (i.e., $e(t) = 0$ and $\dot{e}(t) = 0$), the vectors of "equivalent adjustable parameters", denoted by, $a_{eq}(t)$ and $K_{eq}(t)$, are given by

$$\begin{bmatrix} a_{eq}(t) \\ K_{eq}(t) \end{bmatrix} = \frac{\dot{y}_d(t)}{\|y(t)\|^2 + \|x(t)\|^2} \begin{bmatrix} y(t) \\ x(t) \end{bmatrix} \qquad (2.21)$$

As it easily verified, this choice renders

$$\dot{y}_0(t) = \dot{y}_d(t) \;\; \forall\, t > t_h \qquad (2.22)$$

This fact, added to the ideal condition $e(t) = 0 \; \forall t > t_h$, implies that $y_0(t) = y_d(t) \; \forall \; t > t_h$. Ideally, perfect tracking is thus guaranteed for all bounded external output signals which have bounded time derivatives.

A relaxed version of Theorem 2.4 is obtained if one assumes that the signal $\dot{y}_d(t)$ is measurable.

Theorem 2.5 *If the adaptation laws for the adjustble parameters of the dynamical weights is chosen as*

$$\begin{bmatrix} a(t) \\ K(t) \end{bmatrix} = \left(\frac{\dot{y}_d(t) - W \text{signe}(t)}{\|y(t)\|^2 + \|x(t)\|^2}\right) \begin{bmatrix} y(t) \\ x(t) \end{bmatrix} \qquad (2.23)$$

with W being a sufficiently large positive design constant, then, given an arbitrary set of initial conditions $y(t_0)$, the learning error $e(t)$ converges to zero in finite time t_h,given by,

$$t_h = \frac{|y(t_0) - y_d(t_0)|}{W} \qquad (2.24)$$

and a sliding motion is sustained on $e = 0$ for all $t > t_h$.

Proof

The proof is immediate upon realizing that the controlled learning error satisfies the following differential equation with discontinuous right hand side

$$\dot{e}(t) = -W \text{ sign } e(t) \qquad (2.25)$$

and hence a sliding regime exists on $e(t) = 0$, since $e(t)\dot{e}(t) = -W|e(t)| < 0$ for all nonzero $e(t)$. The sliding regime is reachable in finite time t_h given by $|e(0)|/W$, as it can be inferred from the explicity solution of (2.25).

3 An Illustrative Simulation Example

Consider a dynamic adaline consisting of three first order, linear, time-varying systems acting as adjustable weights through their time-varying parameters. Suppose, furthermore, that the input signals x_1, x_2 and x_3 to the adaline are known constants. It is desired to track the scalar signal

$$y_d(t) = A \sin \omega t \cos 2\omega t$$

with $A = 0.4$, $\omega = 10$ [rad/s], by means of the output, $y(t) = y_1(t) + y_2(t) + y_3(t)$, of the adaline, specified by the sum of the states y_1,y_2,y_3 of the three independent linear systems constituting the dynamical adaline. The adaptive algorithm used to adjust the "gains" and the 'time constants" was not fed with any information regarding the time derivative of the signal $y_d(t)$. Figure 4 depicts the computer simulation results for this example. The state components of the adaline are seen to be bounded signals as well as the adaptation parameters constituting the three dimensional vectors $a(t)$ and $K(t)$. The tracking error $e(t)$ is seen to rapidly converge to zero in spite of lack of knowledge of $\dot{y}_d(t)$. In order to smooth out the natural chattering generated by the discontinuity present in the "sign" function, we substituted such a function, as it is customarily done, in sliding mode control practise by the approximating function

$$\frac{e(t)}{|e(t)| + \epsilon}$$

with ϵ taken to be a small constant of value $\epsilon = 0.01$

4 Conclusions

In this article, a sliding mode feedback adaptative learning algorithm has been proposed for a special class of adalines with first order, linear, time-varying, dynamical systems acting as adjustable "weights". The sliding mode strategy was used here in the context of an output signal tracking problem, but it can be equally utilized in more complex tasks, such as *direct* and *inverse dynamics* identification, commonly used in automatic control applications of perceptron-based neural networks. As in the traditional analog adaline case, the sliding mode learning algorithm robustly drives the learning error to zero in finite time. The approach is also highly insensitive to bounded external perturbation inputs and measurement noises. The assumptions made about the bounded nature of external input signals and desired outputs, as well as of their time derivatives, are quite standard in relation to adaptive neuron elements, but they restrict the considerations to memoryless activation functions of the differentiable type.

Extensions of the results to more general classes of multilayer dynamical neuron arrangements is being pursued at the present time, with highly encouraging results.

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Figures

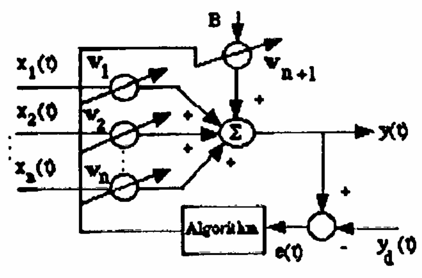

Figure 1: Adaptive linear element

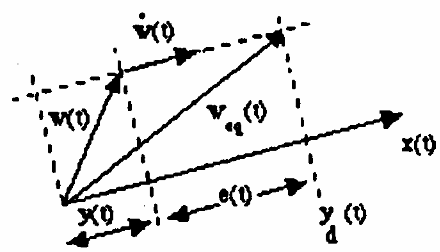

Figure 2: Geometric interpretation of learning algorithm

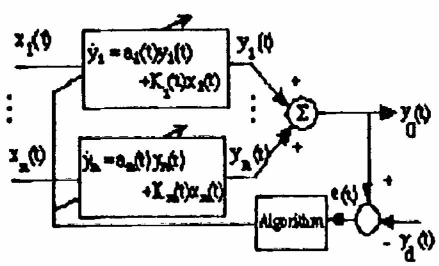

Figure 3: Dynamical Adaline

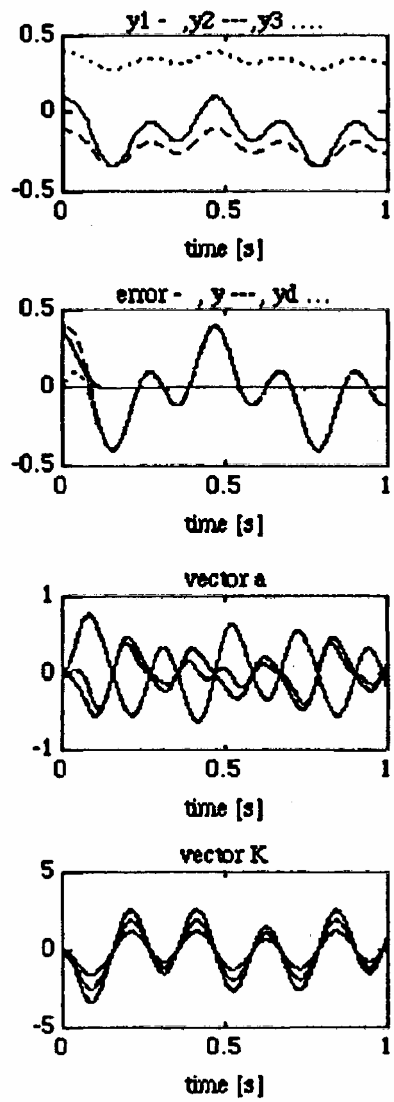

Figure 4: Simulation results for the example