A General Canonical Form for Sliding Mode Control of Nonlinear Systems [1]

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Abstract

A new canonical form of the Generalized Hamiltonian type including "dissipation" terms is proposed for single input nonlinear dynamical systems whose state trajectories are required to slide on a given submanifold of the state space.

1. Introduction

In this article, a general canonical form is derived for systems undergoing sliding motions on a submanifold of the state space. The canonical form is based on the use of *projection operators* associated with the sliding surface function and the control input vector field. A natural decomposition is obtained for the systems drift forces which ranks them as : *workless* or conservative forces, i.e., those yielding invariance of the switching surface coordinate; the *attracting* forces, which are those naturally making the sliding manifold attractive and try to drive the surface coordinate function to lower absolute values, and the sliding surface *rejecting* forces, which locally drive the system to achieve higher absolute values of the switching surface coordinate. These two last forces change their nature depending on the local sign of the surface coordinate function i.e., attracting forces *above* the surface become repelling forces *below* the surface and viceversa. By suitably respecting the local beneficial nonlinearities, on each side of the sliding surface, an autonomous non-divergence from the sliding surface is guaranteed and thus, the variable structure feedback controller yielding convergence towards the surface can be designed in a more efficient manner. The controller design simply consists in injecting "damping", or attractivity terms, which suitably complement the local beneficial non-linearities of the system while, at the same time, neutralizes those forces which locally destabilize the sliding surface coordinate.

The results in this article constitue an extension of those found in [4] and [5], for feedback passivity of nonlinear systems. In fact, the results here presented give further insight into the long suspected connections and parallelisms between passivity based control (see Ortega *et al* [3]), Generalized Hamiltonian systems (see [2], [9]) and Sliding Mode control [8].

In Section 2, a canonical form for sliding mode control is derived. In Section 3 a sliding mode controller is proposed which exploits the natural structure of the system with respect to the sliding surface. Section 4 is devoted to some conclusions and suggestions for further research.

2. A Canonical Form for Sliding Mode Control

2.1. Fundamental assumptions, definitions and results

Consider the class of nonlinear single-input single-output systems described by

$$\begin{aligned} \dot{x}(t) &= f(x) + g(x)u \; ; \; x \in \mathcal{X} \subset \mathbf{R}^n \; ; \; u \in \mathcal{U} \subset \mathbf{R} \\ y &= \sigma(x) \; ; \; y \in \mathcal{Y} \subset \mathbf{R} \end{aligned} \quad (1)$$

where $\mathcal{X}$ denotes the *operating region* of the system, constituted by a sufficiently large open set containing a continuum of equilibrium points, possibly parametrized by a constant control input value $u = U \in \mathcal{U}$, of the form $x = \overline{x}(U)$ and given by the solution of $f(\overline{x}) + g(\overline{x})U = 0$. In particular, for $u = 0$, we assume $f(\overline{x}) = 0$ implies $\overline{x} = 0$. However, motivated by a large class of real life systems, we are specifically interested in *nonzero* constant state equilibrium points $x = \overline{x}$, obtained by nonzero constant control inputs $u = U$. The output function $y = \sigma(x)$ is assumed to be zero at the equilibrium point, i.e., $\sigma(\overline{x}) = 0$.

We assume that $\sigma(x)$ is a C^1 scalar function, called the *sliding surface function* $\sigma : \mathbf{R}^n \rightarrow \mathbf{R}$ such that

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when the state trajectories are confined to its zero level set $\mathcal{S}_0 = \{x \in \mathcal{X} \ : \ \sigma(x) = 0\}$, the behaviour of the system is as desired (for instance, asymptotically stable towards a given equilibrium).

By $\partial\sigma/\partial x$ we denote the *column* vector field with components $\partial\sigma/\partial x_i \; i = 1, \ldots, n$. The transpose of this gradient field, $(\partial\sigma/\partial x)^T$, is denoted by the *row* vector $\partial\sigma/\partial x^T$. Let $L_g\sigma(x)$ denote the directional derivative of the scalar function $\sigma(x)$ with respect to the control input vector field $g(x)$ at the point $x \in \mathcal{X}$. We assume throughout the entire article that the following assumption holds valid:

$$L_g\sigma(x) = \frac{\partial\sigma}{\partial x^T} g(x) \neq 0 \quad \forall \, x \in \; \mathcal{X} \tag{2}$$

This last condition is usually known as the *transversality condition* and simply establishes that the vector field $g(x)$ is not orthogonal to the gradient of $\sigma(x)$ at any point x in $\mathcal{X}$. In other words, the control vector field $g(x)$ is not tangential, at each x, to the sliding surface function level sets, defined in the state space of the system as, $\mathcal{S}_k = \{x \in \mathcal{X} \ : \ \sigma(x) = \textit{constant} = k\}$. This condition is quite familiar in sliding mode control of nonlinear systems (see [6]) and it amounts to having a sliding surface function which is locally *relative degree* one in $\mathcal{X}$. The *zero dynamics* corresponding to the ideal sliding condition $y = \sigma(x) = 0$ is assumed to be asymptotically stable towards the isolated equilibrium point $\overline{x} \in \mathcal{S}$. In other words the system is *minimum phase* with respect to the output $y = \sigma(x)$. According to the results in [1], the sliding surface function is a *passive* output.

For each $x \in \mathcal{X}$, we define a *projection operator, along the span of the control vector field $g(x)$ onto the tangent space to the constant level sets of the sliding surface function $\sigma(x)$*, as the matrix $M(x)$ given by

$$M(x) = \left[I - \frac{1}{L_g\sigma(x)} g(x) \frac{\partial\sigma}{\partial x^T} \right] \tag{3}$$

The following proposition points out some properties of the matrix $M(x)$ which further justify the given name of "projection operator"

Proposition 2.1 *The matrix $M(x)$ enjoys the following properties:*

$$\begin{aligned} g(x) &\in \mathcal{K}er\, M(x) \\ \frac{\partial\sigma}{\partial x} &\in \mathcal{K}er\, M^T(x) \\ M(x)(I - M(x)) &= 0 \end{aligned} \tag{4}$$

Proof

The first property establishes that, locally, $M(x)g(x) = 0$. Indeed, using the definition of $M(x)$ one has

$$\begin{aligned} & \left[I - \frac{1}{L_g\sigma(x)} g(x) \frac{\partial\sigma}{\partial x^T} \right] g(x) \\ = \;& g(x) - \frac{1}{L_g\sigma(x)} g(x) \frac{\partial\sigma}{\partial x^T} g(x) \\ = \;& g(x) - \frac{1}{L_g\sigma(x)} g(x) L_g\sigma(x) = g(x) - g(x) \\ = \;& 0 \end{aligned} \tag{5}$$

The second property is equivalent to $\partial\sigma/\partial x^T M(x) = 0$.

$$\begin{aligned} & \frac{\partial\sigma}{\partial x^T} \left[I - \frac{1}{L_g\sigma(x)} g(x) \frac{\partial\sigma}{\partial x^T} \right] \\ = \;& \frac{\partial\sigma}{\partial x^T} - \frac{1}{L_g\sigma(x)} \frac{\partial\sigma}{\partial x^T} g(x) \frac{\partial\sigma}{\partial x^T} \\ = \;& \frac{\partial\sigma}{\partial x^T} - \frac{\partial\sigma}{\partial x^T} \\ = \;& 0 \end{aligned} \tag{6}$$

The last property follows immediately from the fact that the columns of the matrix $(I - M(x))$ are all in the subspace *span* $\{g(x)\}$. Indeed,

$$\begin{aligned} I - M(x) &= \frac{1}{L_g\sigma(x)} g(x) \frac{\partial\sigma}{\partial x^T} \\ &= \frac{1}{L_g\sigma(x)} \left[g(x) \frac{\partial\sigma}{\partial x_1}; \cdots ; g(x) \frac{\partial\sigma}{\partial x_n} \right] \end{aligned} \tag{7}$$

This last fact and the use of the first property yields the result.

The following proposition depicts further properties of the projection matrix $M(x)$

Proposition 2.2 *Let $f(x)$ be a smooth vector field, then the vector $M(x)f(x)$ can be written as*

$$M(x)f(x) = \tilde{\mathcal{J}}(x) \frac{\partial\sigma}{\partial x}$$

where $\tilde{\mathcal{J}}(x)$ is a skew-symmetric matrix, i.e., $\tilde{\mathcal{J}}(x) + \tilde{\mathcal{J}}^T(x) = 0$.

On the other hand, the vector field $(I - M(x))f(x)$ can be written as

$$(I - M(x))f(x) = -\frac{1}{2} \tilde{\mathcal{J}}(x) \frac{\partial\sigma}{\partial x} + \mathcal{S}(x) \frac{\partial\sigma}{\partial x}$$

where $\mathcal{S}(x)$ is a symmetric matrix, i.e., $\mathcal{S}(x) = \mathcal{S}^T(x)$

Proof

The first part of the proposition easily follows from the following string of algebraic manipulations

$$\begin{aligned} M(x)f(x) &= \left[I - \frac{1}{L_g\sigma(x)} g(x)\frac{\partial \sigma}{\partial x^T}\right] f(x) \\ &= \frac{1}{L_g\sigma(x)}\left[(L_g\sigma(x))\, f(x) - g(x) L_f\sigma(x)\right] \\ &= \frac{1}{L_g\sigma(x)}\left[\frac{\partial \sigma}{\partial x^T} g(x) f(x) - g(x)\frac{\partial \sigma}{\partial x^T} f(x)\right] \\ &= \frac{1}{L_g\sigma}\left[f(x)g^T(x)\frac{\partial \sigma}{\partial x} - g(x)f^T(x)\frac{\partial \sigma}{\partial x}\right] \\ &= \frac{1}{L_g\sigma}\left[f(x)g^T(x) - g(x)f^T(x)\right]\frac{\partial \sigma}{\partial x} \\ &= \tilde{\mathcal{J}}(x)\frac{\partial \sigma}{\partial x} \end{aligned} \quad (8)$$

For the proof of the second part of the proposition note that,

$$\begin{aligned} (I - M(x))f(x) &= \frac{1}{L_g\sigma} g(x)\frac{\partial \sigma}{\partial x^T} f(x) \\ &= \frac{1}{L_g\sigma}\left[g(x)f^T(x)\right]\frac{\partial \sigma}{\partial x} \end{aligned} \quad (9)$$

The result follows from the fact that *any* square matrix $N(x)$ and, in particular,

$$N(x) = (1/L_g\sigma(x))\left[g(x)f^T(x)\right]$$

can always be written as

$$N(x) = (1/2)(N(x) - N^T(x)) + 1/2(N(x) + N^T(x))$$

The first summand, which is written as,

$$\begin{aligned} &\frac{1}{2}(N(x) - N^T(x)) = \\ &\frac{1}{2L_g\sigma}\left[g(x)f^T(x) - f(x)g^T(x)\right] \\ &= -\frac{1}{2}\tilde{\mathcal{J}}(x) \end{aligned} \quad (10)$$

is clearly skew-symmetric, while the second summand $(1/2)(N(x) + N^T(x))$ is symmetric. For the purposes of further reference we define the matrix $\mathcal{S}(x)$ as follows

$$\begin{aligned} \mathcal{S}(x) &= \frac{1}{2}\left[N(x) + N^T(x)\right] \\ &= \frac{1}{2L_g\sigma}\left[g(x)f^T(x) + f(x)g^T(x)\right] \end{aligned}$$

and the matrix $\mathcal{J}(x)$ as

$$\mathcal{J}(x) = \frac{1}{2}\tilde{\mathcal{J}}(x)$$

2.2. Vector field decompositions through projection operators

As a consequence of the above propositions and definitions we have the following result.

Proposition 2.3 *A drift vector field $f(x(t))$ can be naturally decomposed in the following sum,*

$$\begin{aligned} f(x) &= M(x)f(x) + (I - M(x))f(x) \\ &= \mathcal{J}(x)\frac{\partial \sigma}{\partial x} + \mathcal{S}(x)\frac{\partial \sigma}{\partial x} \end{aligned} \quad (11)$$

Proof

Indeed,

$$M(x)f(x) = \tilde{\mathcal{J}}(x)\frac{\partial \sigma}{\partial x}$$

and

$$(I - M(x))f(x) = -\frac{1}{2}\tilde{\mathcal{J}}(x)\frac{\partial \sigma}{\partial x} + \mathcal{S}(x)\frac{\partial \sigma}{\partial x}$$

then,

$$\begin{aligned} f(x) &= M(x)f(x) + (I - M(x))f(x) \\ &= \tilde{\mathcal{J}}(x)\frac{\partial \sigma}{\partial x} - \frac{1}{2}\tilde{\mathcal{J}}(x)\frac{\partial \sigma}{\partial x} + \mathcal{S}(x)\frac{\partial \sigma}{\partial x} \\ &= \frac{1}{2}\tilde{\mathcal{J}}(x)\frac{\partial \sigma}{\partial x} + \mathcal{S}(x)\frac{\partial \sigma}{\partial x} \\ &= \mathcal{J}(x)\frac{\partial \sigma}{\partial x} + \mathcal{S}(x)\frac{\partial \sigma}{\partial x} \end{aligned} \quad (12)$$

The following lemma is well known,

Lemma 2.4 *Let $\mathcal{S}(x)$ be a symmetric matrix, then $\mathcal{S}(x)$ can always be decomposed (nonuniquely) as the sum of a positive semi-definite matrix $\mathcal{S}_p(x)$ and a negative semi-definite matrix $\mathcal{S}_n(x)$. If the matrix is already positive (semi) definite or, else, it is negative (semi) definite then the decomposition is trivial.*

2.3. A canonical form for sliding mode controlled nonlinear systems

As a corollary to the above results, a nonlinear system of the form (1), with a sliding surface function $\sigma(x)$, satisfying the transversality condition $L_g\sigma(x) \neq 0$, can always be rewritten as

$$\dot{x}(t) = \mathcal{J}(x)\frac{\partial \sigma}{\partial x} + \mathcal{S}_p(x)\frac{\partial \sigma}{\partial x} + \mathcal{S}_n(x)\frac{\partial \sigma}{\partial x} + g(x)u \quad (13)$$

with $\mathcal{J}(x)$ being skew-symmetric, and $\mathcal{S}_p(x)$ being positive semi-definite and $\mathcal{S}_n(x)$ negative semi-definite. However, if $\mathcal{S}_p(x)$ is positive definite, then $\mathcal{S}_n(x)$ is zero and conversely if $\mathcal{S}_n(x)$ is negative definite then $\mathcal{S}_p(x)$ is zero.

2.4. Feedback sliding mode control for systems in canonical form

Consider a nonlinear system, given in the following form,

$$\begin{aligned} \dot{x} &= \mathcal{J}(x)\frac{\partial \sigma}{\partial x} + \mathcal{S}(x)\frac{\partial \sigma}{\partial x} + g(x)u \\ y &= \sigma(x) \end{aligned} \tag{14}$$

where the symmetric matrix $\mathcal{S}(x)$ is assumed to be decomposed as the sum of two symmetric matrices $\mathcal{S}_p(x) + \mathcal{S}_n(x)$, as explained above.

Along the solutions of the system, the time derivative of the "energy function"

$$V(x) = \frac{1}{2}\sigma^2(x) \tag{15}$$

is evaluated as $\dot{V} = \sigma(x)\dot{\sigma}(x)$.

For $\sigma(x) > 0$ we have,

$$\begin{aligned} \sigma\dot{\sigma} &= \sigma\left[\frac{\partial \sigma}{\partial x^T}\mathcal{J}(x)\frac{\partial \sigma}{\partial x} + \frac{\partial \sigma}{\partial x^T}\mathcal{S}_p(x)\frac{\partial \sigma}{\partial x} \right. \\ &\quad \left. + \frac{\partial \sigma}{\partial x^T}\mathcal{S}_n(x)\frac{\partial \sigma}{\partial x} + L_g\sigma(x)u\right] \\ &= \sigma\left[\frac{\partial \sigma}{\partial x^T}\mathcal{S}_p(x)\frac{\partial \sigma}{\partial x} + \frac{\partial \sigma}{\partial x^T}\mathcal{S}_n(x)\frac{\partial \sigma}{\partial x} + L_g\sigma(x)u\right] \\ &\leq \sigma\left[\frac{\partial \sigma}{\partial x^T}\mathcal{S}_p(x)\frac{\partial \sigma}{\partial x} + L_g\sigma(x)u\right] \end{aligned} \tag{16}$$

while for $\sigma(x) < 0$ we obtain

$$\begin{aligned} \sigma\dot{\sigma} &= \sigma\left[\frac{\partial \sigma}{\partial x^T}\mathcal{J}(x)\frac{\partial \sigma}{\partial x} + \frac{\partial \sigma}{\partial x^T}\mathcal{S}_p(x)\frac{\partial \sigma}{\partial x} \right. \\ &\quad \left. + \frac{\partial \sigma}{\partial x^T}\mathcal{S}_n(x)\frac{\partial \sigma}{\partial x} + L_g\sigma(x)u\right] \\ &= \sigma\left[\frac{\partial \sigma}{\partial x^T}\mathcal{S}_p(x)\frac{\partial \sigma}{\partial x} + \frac{\partial \sigma}{\partial x^T}\mathcal{S}_n(x)\frac{\partial \sigma}{\partial x} + L_g\sigma(x)u\right] \\ &\leq \sigma\left[\frac{\partial \sigma}{\partial x^T}\mathcal{S}_n(x)\frac{\partial \sigma}{\partial x} + L_g\sigma(x)u\right] \end{aligned} \tag{17}$$

Consider then the following variable structure input coordinate transformation, with v denoting a new external independent control input,

For $\sigma(x) > 0$,

$$u = \frac{1}{L_g\sigma}\left[v - \frac{\partial \sigma}{\partial x^T}\mathcal{S}_p(x)\frac{\partial \sigma}{\partial x}\right] \tag{18}$$

For $\sigma(x) < 0$.

$$u = \frac{1}{L_g\sigma}\left[v - \frac{\partial \sigma}{\partial x^T}\mathcal{S}_n(x)\frac{\partial \sigma}{\partial x}\right] \tag{19}$$

It is clear that the transformed system is given by the following variable structure system:

For $\sigma(x) > 0$

$$\begin{aligned} \dot{x} &= \mathcal{J}(x)\frac{\partial \sigma}{\partial x} + \mathcal{S}_n(x)\frac{\partial \sigma}{\partial x} \\ &+ \left(I - \frac{1}{L_g\sigma(x)}g(x)\frac{\partial \sigma}{\partial x^T}\right)\mathcal{S}_p(x)\frac{\partial \sigma}{\partial x} + \frac{1}{L_g\sigma}g(x)v \\ y &= \sigma(x) \end{aligned} \tag{20}$$

while, for $\sigma(x) < 0$

$$\begin{aligned} \dot{x} &= \mathcal{J}(x)\frac{\partial \sigma}{\partial x} + \mathcal{S}_p(x)\frac{\partial \sigma}{\partial x} \\ &+ \left(I - \frac{1}{L_g\sigma(x)}g(x)\frac{\partial \sigma}{\partial x^T}\right)\mathcal{S}_n(x)\frac{\partial \sigma}{\partial x} + \frac{1}{L_g\sigma}g(x)v \\ y &= \sigma(x) \end{aligned} \tag{21}$$

Notice that, as shown in the previous section, the projected vector field given by either

$$\left(I - \frac{1}{L_g\sigma(x)}g(x)\frac{\partial \sigma}{\partial x^T}\right)\mathcal{S}_p(x)\frac{\partial \sigma}{\partial x}$$

or

$$\left(I - \frac{1}{L_g\sigma(x)}g(x)\frac{\partial \sigma}{\partial x^T}\right)\mathcal{S}_n(x)\frac{\partial \sigma}{\partial x}$$

can be rewritten, respectively, as,

$$\mathcal{K}_p(x)\frac{\partial \sigma}{\partial x} \quad \text{and} \quad \mathcal{K}_n(x)\frac{\partial \sigma}{\partial x}$$

with $\mathcal{K}_p(x)$ and $\mathcal{K}_n(x)$ being skew-symmetric matrices. In other words, the transformed system is of the form,

For $\sigma(x) > 0$,

$$\begin{aligned} \dot{x} &= \mathcal{I}_p(x)\frac{\partial \sigma}{\partial x} + \mathcal{S}_n(x)\frac{\partial \sigma}{\partial x} + \frac{1}{L_g\sigma}g(x)v \\ y &= \sigma(x) \end{aligned} \tag{22}$$

with $\mathcal{I}_p(x) = \mathcal{J}(x) + \mathcal{K}_p(x)$ being skew symmetric and,

For $\sigma(x) < 0$,

$$\begin{aligned} \dot{x} &= \mathcal{I}_n(x)\frac{\partial \sigma}{\partial x} + \mathcal{S}_p(x)\frac{\partial \sigma}{\partial x} + \frac{1}{L_g\sigma}g(x)v \\ y &= \sigma(x) \end{aligned} \tag{23}$$

with $\mathcal{I}_n(x) = \mathcal{J}(x) + \mathcal{K}_n(x)$ being skew-symmetric.

The input coordinate transformations, viewed as a partial variable structure feedback, has achieved *neutralization* of the non-beneficial nonlinearities in the system. Notice that this is far less demanding than the usual practise of *elimination* of the non-beneficial nonlinearities. The partial variable sturcture feedback has also achieved passivity for the variable structure system with respect to the sliding surface function viewed as a "degenerate" storage fucntion, as the following proposition establishes,

Proposition 2.5 *The system (14) is* passive *with respect to the storage function $V(x) = 1/2\sigma^2(x)$, viewed as a positive semidefinite (i.e., degenerate) storage function, whenever $S_n(x)$, (respectively $S_p(x)$) is negative semidefinite (resp. positive semidefinite) and it is strictly passive if $S_n(x)$ is strictly negative definite (resp. strictly positive definite).*

Proof

Taking the time derivatives of $V(x)$, along the solutions of the transformed system, away from the sliding surface $\mathcal{S}_0$ one obtains:

For $\sigma(x) > 0$

$$\begin{aligned} \dot{V}(x) &= \sigma(x)\left[\frac{\partial\sigma}{\partial x^T}\mathcal{I}_p(x)\frac{\partial\sigma}{\partial x}+\frac{\partial\sigma}{\partial x^T}\mathcal{S}_n(x)\frac{\partial\sigma}{\partial x}\right. \\ &\quad \left.+\frac{\partial\sigma}{\partial x^T}\frac{1}{L_g\sigma}g(x)v\right] \\ &= \sigma(x)\left[\frac{\partial\sigma}{\partial x^T}\mathcal{S}_n(x)\frac{\partial\sigma}{\partial x}\right]+\sigma(x)v \\ &\leq \sigma(x)v = yv \end{aligned} \tag{24}$$

The calculation is similar for $\sigma(x) < 0$

Notice that if we let $\tilde{g}(x)$ denote the transformed control input vector field $\frac{\sigma(x)}{L_g\sigma}g(x)$ then the variable structure system may, in fact, be written as:

$$\begin{aligned} \dot{x} &= \begin{cases} \mathcal{I}_p(x)\frac{\partial\sigma}{\partial x}+\mathcal{S}_n(x)\frac{\partial\sigma}{\partial x}+\tilde{g}(x)\vartheta \text{ for } \sigma > 0 \\ \mathcal{I}_n(x)\frac{\partial\sigma}{\partial x}+\mathcal{S}_p(x)\frac{\partial\sigma}{\partial x}+\tilde{g}(x)\vartheta \text{ for } \sigma < 0 \end{cases} \\ y &= \tilde{g}^T(x)\frac{\partial\sigma}{\partial x} \end{aligned} \tag{25}$$

which, except for the "damping" terms $\mathcal{S}_n(x)\partial\sigma/\partial x$ and $\mathcal{S}_p(x)\partial\sigma/\partial x$ are, each one, in the same form as the *Generalized Hamiltonian systems*, widely studied in the literature (see [2],[9]). Notice that the state-dependent input coordinate transformation $\vartheta = v/\sigma(x)$ is defined away from the sliding surface $\mathcal{S}_0$.

3. A Sliding Mode Controller

The sliding mode controller for the input v may now be obtained by simply injecting complementary "damping" to the natural beneficial nonlinearities acting on each side of the sliding manifold. Let $\mathcal{S}_{nI}(x)$ be a symmetric negative semidefinite matrix such that $\mathcal{S}_n(x) + \mathcal{S}_{nI}(x)$ is negative definite. Similarly, let $\mathcal{S}_{pI}(x)$ be a symmetric positive semidefinite matrix such that $\mathcal{S}_p(x) + \mathcal{S}_{pI}(x)$ is positive definite. The following variable structure controller achieves the reaching of the sliding surface and the creation of a local sliding regime on such a surface

$$v = \begin{cases} \frac{\partial\sigma}{\partial x^T}S_{nI}(x)\frac{\partial\sigma}{\partial x} & \text{for } \sigma(x) > 0 \\ \frac{\partial\sigma}{\partial x^T}S_{pI}(x)\frac{\partial\sigma}{\partial x} & \text{for } \sigma(x) < 0 \end{cases} \tag{26}$$

The *ideal sliding dynamics* is readily obtained from the invariance condition of the sliding surface coordinate $\dot{\sigma} = 0$. Consider the system canonical form before any state feedback precompensation

$$\begin{aligned} \dot{x} &= \mathcal{J}(x)\frac{\partial\sigma}{\partial x}+\mathcal{S}(x)\frac{\partial\sigma}{\partial x}+g(x)u \\ y &= \sigma(x) \end{aligned} \tag{27}$$

The ideal control input, or equivalent control, achieving surface coordinate invariance for all motions starting on the sliding surface $\mathcal{S}_0$ is obtained as

$$u = -\frac{1}{L_g\sigma}\frac{\partial\sigma}{\partial x^T}\mathcal{S}(x)\frac{\partial\sigma}{\partial x}$$

The *ideal sliding dynamics* is governed by a projected vector field of the form

$$\hat{\mathcal{J}}(x)\frac{\partial\sigma}{\partial x} = \mathcal{J}(x)\frac{\partial\sigma}{\partial x}+\left[I-\frac{1}{L_g\sigma}g(x)\frac{\partial\sigma}{\partial x^T}\right]\mathcal{S}(x)\frac{\partial\sigma}{\partial x}$$

This vector field evidently belongs to the tangent subspace to the sliding manifold. The equivalent control neutralizes *all working forces* in the system (beneficial and non-beneficial forces) and it evidently bestows a workless, or conservative, character to closed loop ideal sliding dynamics.

$$\dot{x} = \hat{\mathcal{J}}(x)\frac{\partial\sigma}{\partial x} \;\; ; \;\; \hat{\mathcal{J}}(x)+\hat{\mathcal{J}}^T(x) = 0$$

3.1. Removing the transversality condition limitation

The transversality condition (2) plays an essential role in all our previous developments and provisions should be taken for those cases in which it is not immediately satisfied.

Suppose that, in $\mathcal{X}$, the sliding surface function $\sigma(x)$ is not *relative degree* equals to one with respect to the control input u, i.e., the transversality condition $L_g\sigma(x) \neq 0$ is not satisfied on the operating region $\mathcal{X}$. It is intuitively clear that, in such a case, the sliding surface function $\sigma(x)$ must have *some* relative degree on a subset of $\mathcal{X}$. For if not, then the sliding surface function cannot be modified by any control action whatsoever. In order to avoid needless specifications we make the following assumption:

Assume the sliding surface function $\sigma(x)$ is relative degree $r > 1$, in the operating region $\mathcal{X}$ of the state space, i.e.,

$$\begin{aligned} L_g L_f^j \sigma(x) &= 0 \; ; \; j = 0, 1, \ldots, r-2 \; \forall \, x \in \mathcal{X} \\ L_g L_f^{r-1} \sigma(x) &\neq 0 \; ; \; \forall \, x \in \mathcal{X} \end{aligned} \tag{28}$$

From the above assumption it should also be clear that, in the operating region $\mathcal{X}$, the condition $L_f^{r-1}\sigma(x) \neq 0$ is also trivially satisfied, for, otherwise, the relative degree of $\sigma(x)$ is not r in $\mathcal{X}$, as assumed.

Let α be a nonzero scalar constant. Consider next the following sliding surface function,

$$\omega(x) = \sigma(x) + \alpha L_f^{r-1}\sigma(x)$$

Then it is obviously true that $W(x)$ does satisfy the transversality condition in all of $\mathcal{X}$.

$$\begin{aligned} L_g\omega(x) &= L_g\sigma(x) + \alpha L_g L_f^{r-1}\sigma(x) \\ &= \alpha L_g L_f^{r-1}\sigma(x) \neq 0 \end{aligned} \tag{29}$$

A possibly more suggestive modified sliding surface function $W(x)$ may be taken to be

$$\omega(x) = \alpha_0\sigma(x) + \left[\sum_{k=1}^{r-1} \alpha_k \; L_f^k \sigma(x)\right]$$

with $\alpha_k \neq 0 \; \forall \; k$ being appropriate *Hurwitz coefficients.*

4. Conclusions

We have proposed a natural canonical form for nonlinear systems for which sliding motions are to be created on a given sliding surface. The canonical form is largely motivated from passivity based considerations on the same class of systems. As a result, a clear and natural decomposition of the internal system forces is revealed which definitely helps in designing a more efficient feedback variable structure controller. The approach leads to a controller design characterized by: 1) It respects the useful nonlinearities of the system which help in locally reaching the sliding surface. 2) It does not eliminate but, rather, neutralizes those non-benefiical forces of the system which tend to make the trajectories move away from the sliding manifold and 3) It simply complements the useful nonlinearities of the system, on each side of the sliding surface, so as to locally achieve sliding surface reachability. These last three characteristics make the feedback controller signals more naturally tunned to the system structure and control amplitude limitations while achieving the control objective.

The results can be extended to the sliding mode control of multivariable nonlinear systems as already demonstrated, in the context of feedback passivity, in [7].

References

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