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REGULATION OF THE LONGITUDINAL DYNAMICS OF AN HELICOPTER : A LIOUVILLIAN SYSTEMS APPROACH [1]

Hebertt Sira-Ramírez[2] and Rafael Castro-Linares
CINVESTAV-IPN
Departamento de Ingeniería Eléctrica
Apartado Postal 14-740
C.P. 07000 México, D.F., México
email (hsira@mail,rcastro@ctrl).cinvestav.mx
Phone: 52-5-7477000 Fax: 52-5-747-7089

Eduardo Licéaga-Castro
Glasgow Caledonian University
Department of Mathematics
Glasgow 64 OBA, Scotland, U.K.
email E.L.Castro@gcal.ac.uk

Abstract

A feedback regulation scheme is presented for a wide range of horizontal and vertical displacement maneuvers on a simplified longitudinal model of a helicopter system. The approach is based on the "Liouvillian" character of the helicopter model, rather than its differential flatness, thus requiring only a static, though time-varying, state feedback controller which tracks an *off-line* planned trajectory planning for the attitude angle reference signal computed in terms of the desired horizontal and vertical displacements. The controller performance is evaluated through digital computer simulations which include initial setting errors and an unmodelled mid course wind gust perturbation of significant magnitude.

1 Introduction

In this article, a feedback regulation scheme is proposed which accomplishes a wide range of horizontal and vertical displacement maneuvers for a six dimensional longitudinal model of a helicopter system, as developed in [1]. The studied model turns out to be a *Liouvillian system*, i.e. it contains a flat subsystem of dimension smaller than the dimension of the overall system. The flat outputs completely determine the rest of the system variables, which we address as the *remaining variables*, up to elementary quadratures (see [2]).

The Liouvillian character of the system allows for an off-line trajectory planning for the attitude angle, computed on the basis of the the desired horizontal and vertical displacements. The feedback controller is then specified on the basis of the ideal (nominal) open loop control complemented with an approximate linearization based controller.

The longitudinal dynamics model of the helicopter adopted here is also *differentially flat* (see the work of Fliess [3]), and it can be exactly linearized by means of *dynamical state feedback* as already done in [5]. The main difference between our Liouvillian based approach and one entirely based on the differential flatness of the suitably extended model, is that the off-line computation needed for the ideal open loop control involves the off-line solution of a second order nonlinear differential equation. The flatness based approach, on the other hand, would not require such an off-line calculation, but its associated burden is instead transfered to the on-line solution of a second order differential equation representing the dynamical feedback controller. Our controller, on the other hand is linear and static, though time-varying, and perhaps simpler in nature than the one we would have obtained based on the exact linearization approach.

Section 2 presents a longitudinal model for the helicopter dynamics as presented in [4] and some pertinent physical assumptions. Section 3 contains a brief introduction to Liouvillian systems and shows that the adopted helicopter model is Liouvillian. A linear time-varying controller is derived guaranteeing robustness to the off-line computed nominal control input. In Section 4, a simulation test is performed for the closed loop system. The conclusions and proposals for further research are presented in the last section.

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[2]On leave of absence from the Departamento de Sistemas de Control, Universidad de Los Andes, Mérida-Venezuela

2 A Simplified Model for a Helicopter System

We consider the following set of simplified second order differential equations for the longitudinal dynamics of the helicopter, shown in Figure 1.

$$\begin{aligned} \ddot{x} &= -\frac{1}{M}\sin\theta u_1 - \frac{1}{M}\cos\theta u_2 \\ \ddot{y} &= g - \frac{1}{M}\cos\theta u_1 + \frac{1}{M}\sin\theta u_2 \\ \ddot{\theta} &= Lu_2 \end{aligned} \quad (1)$$

where x denotes the forward position of the rotorcraft, y its vertical heigth and θ is the attitude angle. M is the helicopter mass while $L = l_h/i_{yy}$ with l_h being the distance between the rotor hub and the fusselage center of mass and i_{yy} is a moment of inertia.

2.1 Some physically plausible assumptions

Motivated by simple equilibrium considerations for a null horizontal displacement, we assume that the following conditions are valid for any given maneuver

$$(g-\ddot{y})\cos\theta - \ddot{x}\sin\theta > \mu > 0 \ ; \ -\frac{\pi}{2} < \theta < \frac{\pi}{2} \quad (2)$$

where μ is a strictly positive constant.

We also assume that u_2 is a bounded control input and that

$$|\,(g-\ddot{y})\sin\theta + \ddot{x}\cos\theta\,| < \kappa \quad (3)$$

for a strictly positive constant κ.

A reasonable practical assumption on the magnitude of the rate of change of the control input u_1 is given by the condition

$$|\,\dot{u}_1\,| < \epsilon \quad (4)$$

3 Regulation of the Helicopter Model via Trajectory Planning

3.1 The helicopter model as a Liouvillian system

The helicopter model (1), although being differentially flat may also be regarded as a Liouvillian system, with the flat subsystem being represented by the state variables $(\theta, \dot{\theta}, y, \dot{y})$. The "flat" outputs for this subsystem are given by the attitude angle θ and by the vertical displacement y, which we denote by F and R, respectively. The following partial *differential parameterization* of the system variables allows for some elementary equilibrium analysis and also establishes the main features of the system to be controlled

$$\begin{aligned} \theta &= F \ ; \ \dot{\theta} = \dot{F} \ ; \ u_2 = \frac{\ddot{F}}{L} \ ; \ y = R \ ; \ \dot{y} = \dot{R} \\ u_1 &= \frac{M}{\cos F}\left(g + \frac{\ddot{F}}{LM}\sin F - \ddot{R}\right) \end{aligned} \quad (5)$$

The "remaining" system variables, represented by the horizontal displacement variables $(x, \dot{x})$, are expressible in terms of quadratures of the proposed flat outputs F and R and its second order time derivatives $\ddot{F}$, $\ddot{R}$. Indeed, from (1) and the previous considerations, we obtain, modulo initial conditions and specific integration limits,

$$\begin{aligned} x &= -\int\int \tan F\left(g + \frac{\ddot{F}}{LM}\sin F - \ddot{R}\right) d\sigma dt \\ & \quad -\frac{1}{LM}\int\int \ddot{F}\cos F d\sigma dt \end{aligned} \quad (6)$$

Thus, system (1) qualifies as a Liouvillian system with flat subsystem outputs given by F and R. The *zero dynamics* corresponding to a resting hovering position, characterized by $x = constant$, $y = constant$, is given, according to (1), by the following dynamics

$$\ddot{F} = -MgL\sin F \quad (7)$$

The zero dynamics (7) represents a locally stable oscillatory system with equilibria located at the origin and, also, at attitude angles of the form, $F = \pm k\pi$; $k = 1, 2, \ldots$. The system is hence *weakly minimum phase* with respect to the horizontal and vertical coordinates x, y, taken as the system outputs.

3.2 Off-line trajectory planning

Suppose that a desired displacement maneuver is specified by sufficiently smooth trajectories $x^*(t)$ and $y^*(t)$ for the horizontal and vertical position variables x and y. The desired maneuver is to take place in given, finite, but independent amounts of time T_{hf} and T_{vf}. The specified trajectories are supposed to take the helicopter from the initial equilibrium hovering position, located at $(x(t_{hi}), y(t_{vi}))$, on an arbitrary vertical plane, towards a final hovering horizontal and vertical coordinate values, specified as $(x(T_{hf}), y(T_{vf}))$. The displacement maneuveres may include an arbitrary number of intermediate resting equilibria or hovering positions as well as "backing-ups", advancements, and ascents and descents of arbitrary lengths and durations. The time evolution of the "off-line" planned trajectories $x^*(t)$, $y^*(t)$ are also assumed to start with a sufficient number of zero initial and final time derivatives (with similar features for intermediate resting positions). This last requirement guarantees smooth departures from initial or intermediate points as well as smooth arrivals at intermediate equilibria, or at final resting positions.

3.3 A trajectory tracking feedback controller

The partial differential flatness properties of the variables $\theta = F$ and $y = R$, allows one to express the control inputs u_1 and u_2 as the open loop control laws

$$u_1 = \frac{M}{\cos F}\left(g - \ddot{R} + \frac{\ddot{F}}{LM}\sin F\right) \quad \text{and} \quad u_2 = \frac{\ddot{F}}{L}, \quad (8)$$

respectively. Therefore, if a desired displacement maneuver is given by $x^*(t)$, $y*(t)$ then, the corresponding attitude angular trajectory may be computed by finding the solution $F^*(t)$ of the following nonlinear second order differential equation, obtained from the horizontal displacement dynamics,

$$\ddot{F} = -LM\left[\ddot{x}^*(t)\cos F^* + (g - \ddot{y}\ (t))\sin F^*\right] \quad (9)$$

with initial conditions in complete accordance with the desired maneuver.

The "off-line" computed attitude angle trajectory $F\ (t)$, generated by the differential equation (9), is to be used in an "on-line" feedback control scheme obtained from the following approximate linearization scheme.

For doing this, let us define the state variables tracking errors and control inputs errors

$$\begin{aligned}
x_{1\delta} &= x - x^*(t)\ \ ;\ \ x_{2\delta} = \dot{x} - \dot{x}^*(t) \\
x_{3\delta} &= y - R^*(t)\ \ ;\ \ x_{4\delta} = \dot{y} - \dot{R}^*(t) \\
x_{5\delta} &= \theta - F^*(t)\ \ ;\ \ x_{6\delta} = \dot{\theta} - \dot{F}^*(t) \\
u_{1\delta} &= u_1 \\
&\quad - \frac{M}{\cos F^*(t)}\left(g - \ddot{R}^*(t) + \frac{\ddot{F}^*(t)}{LM}\sin F^*(t)\right) \\
u_{2\delta} &= u_2 - \frac{\ddot{F}^*(t)}{L} \qquad (10)
\end{aligned}$$

The linearized dynamics, around the ideally regulated open loop trajectories, is given by

$$\begin{aligned}
\dot{x}_{1\delta} &= x_{2\delta} \\
\dot{x}_{2\delta} &= (\ddot{R}^* - g)x_{5\delta} - \frac{1}{M}\sin(F^*)u_{1\delta} - \frac{1}{M}\cos(F^*)u_{2\delta} \\
\dot{x}_{3\delta} &= x_{4\delta} \\
\dot{x}_{4\delta} &= -\ddot{x}^* x_{5\delta} - \frac{1}{M}\cos(F^*)u_{1\delta} + \frac{1}{M}\sin(F^*)u_{2\delta} \\
\dot{x}_{5\delta} &= x_{6\delta} \\
\dot{x}_{6\delta} &= Lu_{2\delta} \qquad (11)
\end{aligned}$$

A linear time-varying state feedback controller, of the "proportional plus derivative" (PD) type, including time-varying compensation terms results in the incremental correction inputs

$$\begin{bmatrix} u_{1\delta} \\ u_{2\delta} \end{bmatrix} = \begin{bmatrix} M\sin F^* & M\cos F^* \\ M\cos F^* & -M\sin F^* \end{bmatrix} \times \begin{bmatrix} \left(\ddot{R}^* - g\right)x_{5\delta} + k_{xp}x_{1\delta} + k_{xd}x_{2\delta} \\ -\ddot{x}^* x_{5\delta} + k_{yp}x_{3\delta} + k_{yd}x_{4\delta} \end{bmatrix} - \begin{bmatrix} 0 \\ \frac{1}{L}\left(k_{\theta p}x_{5\delta} + k_{\theta d}x_{6\delta}\right) \end{bmatrix}$$

where k_{xp}, k_{xd}, k_{yp}, k_{yd}, $k_{\theta p}$ and $k_{\theta d}$, are strictly positive design constants.

The closed loop linearized system is given by

$$\begin{aligned}
\dot{x}_{1\delta} &= x_{2\delta} \\
\dot{x}_{2\delta} &= -k_{xp}x_{1\delta} - k_{xd}x_{2\delta} \\
&\quad + \frac{1}{LM}\cos F^*\left[k_{\theta p}x_{5\delta} + k_{\theta d}x_{6\delta}\right] \\
\dot{x}_{3\delta} &= x_{4\delta} \\
\dot{x}_{4\delta} &= -k_{yp}x_{3\delta} - k_{yd}x_{4\delta} \\
&\quad - \frac{1}{LM}\sin F^*\left[k_{\theta p}x_{5\delta} + k_{\theta d}x_{6\delta}\right] \\
\dot{x}_{5\delta} &= x_{6\delta} \\
\dot{x}_{6\delta} &= -\left[k_{\theta p} + LM\left((g - \ddot{R}^*)\cos F^* - \ddot{x}^*\sin F^*\right)\right]x_{5\delta} \\
&\quad - k_{\theta d}x_{6\delta} + LM\cos F^*\left[k_{xp}x_{1\delta} + k_{xd}x_{2\delta}\right] \\
&\quad - LM\sin F^*\left[k_{yp}x_{3\delta} + k_{yd}x_{4\delta}\right] \qquad (12)
\end{aligned}$$

It is not difficult to show that the closed loop system (12) may be rendered exponentially stable to zero under the established physically meanigful assumptions and for a set of suitably chosen controller design constants and displacement reference trajectories.

A full feedback controller for the helicopter model, based on the above considerations, is thus given by

$$\begin{aligned}
u_1 &= \frac{M}{\cos F^*(t)}\left(g - \ddot{R}^*(t) + \frac{\ddot{F}^*(t)}{LM}\sin F^*(t)\right) \\
&\quad + u_{1\delta} \\
u_2 &= \frac{\ddot{F}\ (t)}{L} + u_{2\delta} \qquad (13)
\end{aligned}$$

4 Simulation Results

For the simulations presented in this section, the following values were assigned to the helicopter system parameters: $M = 4313$ Kg, $g = 9.8$ m/s^2, $L = 1.0456 \times 10^{-4}$ rad/N-s^2.

4.1 Off line computations example

We first consider a trajectory planning example corresponding to the off-line computations represented by equation (9) for a given desired displacements $x^*(t)$ and $y^*(t)$, starting and ending with ideal hovering conditions while requiring a position transfer between two known equlibrium values in the x-y plane. The desired horizontal and vertical displacement maneuvers were specified as polynoimal splines. For this, we used a polynomial function $\eta(t; t_1, t_2)$, $t_1 \le t \le t_2$, satisfying $\eta(t_1; t_1, t_2) = 0$ and $\eta(t_2; t_1, t_2) = 1$, with a suitable number of time derivatives being zero at times, $t = t_1$ and $t = t_2$. For the simulations we have used the following polynomial spline interpolating between 0 and 1 with five time derivatives being zero at $t = t_1$ and five time derivatives also being zero at $t = t_2$.

$$\eta(t; t_1, t_2) = \left[\frac{t - t_1}{t_2 - t_1}\right]^5 \left\{r_1 - r_2\frac{t - t_1}{t_2 - t_1} + r_3\left(\frac{t - t_1}{t_2 - t_1}\right)^2\right.$$

$$-r_4\left(\frac{t-t_1}{t_2-t_1}\right)^3+r_5\left(\frac{t-t_1}{t_2-t_1}\right)^4-r_6\left(\frac{t-t_1}{t_2-t_1}\right)^5\Bigg\}$$

with $r_1 = 252$, $r_2 = 1050$, $r_3 = 1800$, $r_4 = 1575$, $r_5 = 700$, $r_6 = 126$. Thus, $x^*(t)$ and $y^*(t)$ are given by

$$x^*(t) = \begin{cases} x(t_{hi}) \text{ for } t < t_{hi} \\ x(t_{hi}) + (x(T_{hf}) - x(t_{hi}))\eta(t; t_{hi}, T_{hf}) \\ \quad \text{for } t_{hi} \le t \le T_{hf} \\ x(T_{hf}) \text{ for } t > T_{hf} \end{cases} \tag{14}$$

$$y^*(t) = \begin{cases} y(t_{vi}) \text{ for } t < t_{vi} \\ y(t_{vi}) + (y(T_{vf}) - y(t_{vi}))\eta(t; t_{vi}, T_{vf}) \\ \quad \text{for } t_{vi} \le t \le T_{vf} \\ y(T_{vf}) \text{ for } t > T_{vf} \end{cases} \tag{15}$$

with $x(t_{hi}) = 100$ m, $x(T_{hi}) = 300$ m $t_{hi} = 30$ sec, $T_{hf} = 60$ sec, $y(t_{vi}) = 30$ m, $y(T_{vi}) = 200$ m, $t_{vi} = 50$ sec, $T_{vf} = 90$ sec.

Figure 2 shows the computed attitude angle trajectory corresponding to the prescribed horizontal and vertical motions.

The corresponding open loop control inputs, which would be given to the helicopter under ideal conditions impling no external perturbations and no initial setting errors, are specified, by virtue of the partial differential flatness of the system, as the open loop signals,

$$\begin{aligned} u_1^*(t) &= \frac{M}{\cos F^*(t)}\left(g - \ddot{R}^*(t) + \frac{\ddot{F}^*(t)}{LM}\sin F^*(t)\right) \\ u_2^*(t) &= \frac{\ddot{F}^*(t)}{L} \end{aligned}$$

The simulated responses of the helicopter dynamics to such an open loop control is precisely represented by the same curves in Figure 2 as long as the initial conditions for the kinematical and dynamical variables are taken to exactly coincide with the ideal hovering conditions, $F^*(t_0) = 0$, $\dot{F}^*(t_0) = 0$ and the flat output trajectory is given by the off-line data computed from (9).

4.2 Feedback controller performance

The performance of the proposed multivariable feedback controller (13) was tested in a combination of desired horizontal and vertical displacement maneuvers, involving an intermediate rest point with a subsequent "backing up" while ascending, or descending, requirements, was also prescribed as indicated below, with a suitable polynomial spline function interpolating between 0 and 1, specified as before by a function now denoted by $\nu(t;\tau,\sigma)$, for $\sigma \ge t \ge \tau$. A possible realization of the described trajectory is then

$$x^*(t) = \begin{cases} x(t_{hi}) \text{ for } t < t_{hi} \\ x(t_{hi}) + (x(t_{h1}) - x(t_{hi}))\nu(t; t_{hi}, t_{h1}) \\ \quad \text{for } t_{hi} \le t \le t_{h1} \\ x(t_{h1}) \text{ for } t_{h1} \le t \le t_{h2} \\ x(t_{h3}) + (x(t_{h3}) - x(t_{h2}))\nu(t; t_{h2}, t_{h3}) \\ \quad \text{for } t_{h2} \le t \le t_{h3} \\ x(t_{h3}) \text{ for } t_{h3} \le t \le t_{h4} \\ x(T_{hf}) + (x(T_{hf}) - x(t_{h4}))\nu(t; t_{h4}, T_{hf}) \\ \quad \text{for } t_{h4} \le t \le T_{hf} \\ x(T_{hf}) \text{ for } t > T_{hf} \end{cases} \tag{16}$$

$$y^*(t) = \begin{cases} y(t_{vi}) \text{ for } t < t_{vi} \\ y(t_{vi}) + (y(t_{v1}) - y(t_{vi}))\nu(t; t_{vi}, t_{h1}) \\ \quad \text{for } t_{vi} \le t \le t_{v1} \\ y(t_{v1}) \text{ for } t_{v1} \le t \le t_{v2} \\ y(t_{v3}) + (y(t_{v3}) - y(t_{v2}))\nu(t; t_{v2}, t_{v3}) \\ \quad \text{for } t_{v2} \le t \le t_{v3} \\ y(t_{v3}) \text{ for } t_{v3} \le t \le t_{v4} \\ y(T_{vf}) + (y(T_{vf}) - y(t_{v4}))\nu(t; t_{v4}, T_{vf}) \\ \quad \text{for } t_{v4} \le t \le T_{vf} \\ y(T_{vf}) \text{ for } t > T_{vf} \end{cases} \tag{17}$$

The simulation results of the closed loop system resposes are shown in Figure 3. The simulation test included the action of an unmodeled wind gust disturbance occuring around time $t = 30$ s of the maneuver execution time. The helicopter is commanded to advance from $x = 100$ m, to $x = 300$ m, and from $y = 30$ m to $y = 200$ m, where it rests for some time. After this, it is commanded to back up from $x = 300$ m to $x = 200$ m and, while backing up, it is directed to start an ascent maneuver from $y = 200$ m to $y = 350$ m. The horizontal and vertical displacment reference trajectories $x^*(t)$ and $y^*(t)$ are very closely followed by the closed loop system with little alteration. The tracking of the attitude reference signal (not shown in the figure) is temporarily lost around the location of the wind gust.

5 Conclusions

In this article we have proposed a linear time-varying state feedback controller complementing a nonlinear off-line (i.e. open loop) computed controller ideally solving a trajectory tracking task for a simplified, underactuated, longitudinal model of an helicopter. The approach is based on exploting the fact that the system belongs to the class of "Liouvillian" systems, which generalizes the class of differentially flat systems. This last property allows for an off-line trajectory planning of a chosen subsystem "flat" output, represented by the attitude angular position, in terms of the required horizontal and vertical displacement trajectories. Given such desired horizontal and vertical displacement trajectories, the corresponding attitude angle trajectory and the required control inputs are computed using the partial differential flatness of the model with respect to the vertical displacement and the attitude angle variables. The ideal open loop control is then completed with a linearization based static, though time-varying, state feedback controller bestowing the required robustness to the open loop control scheme. The proposed static feedback controller has been tested through computer simulations, with very encouraging results. A wide range of longitudinal maneuvers, including initial state and mid course unknown perturbations, are efficiently handled by the proposed feedback controller scheme.

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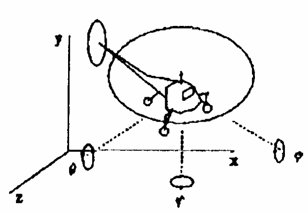

Figure 1: Schematical diagram of a helicopter.

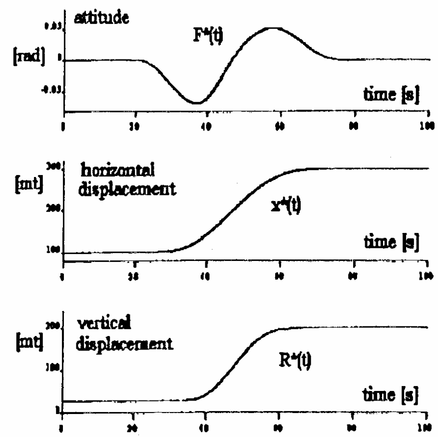

Figure 2: Ideal open loop horizontal and vertical displacement maneuvers via computed attitude trajectory

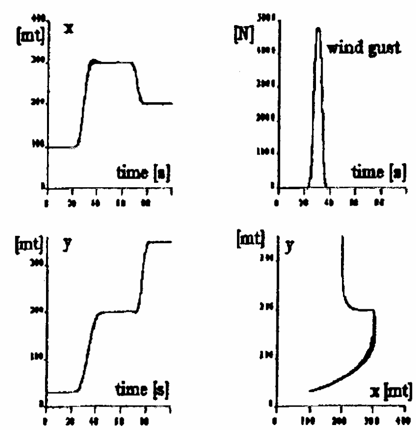

Figure 3: Forward and backing maneuver, while ascending, including unmodelled wind gust perturbation.