On the Control of the Variable Length Pendulum [1]

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Abstract

A new notion of *differential flatness of infinite order* is used to approximately regulate, along an off-line prescribed oscillatory trajectory, a mechanical system, which is: nonlinear in the control input, it is *not* feedack linearizable by means of dynamical, or static, state feedback, and it is also non-minimum phase.

1 The variable length pendulum

Consider the normalized model of the so called "variable length pendulum" studied in by Bressan and Rampazzo [1], given by

$$\begin{aligned} \dot{q} &= p \\ \dot{p} &= -\cos\theta + qu^2 \\ \dot{\theta} &= u \end{aligned} \qquad (1)$$

where q denotes the distance from the center of the sliding ball to the origin of coordinates around which the rod rotates. The variable p is the velocity of the ball, as measured along the rod, and θ is the angle formed by the rod and the vertical line passing through the origin.

The system is known to be *non differentially* flat (see Fliess *et al* [3]). Notice that the *defect* variable, q, satisfies the *linear* time varying differential equation

$$\ddot{q} = q\left(\dot{F}^2\right) - \cos F \qquad (2)$$

Hence, the system is *Liouvillian* since the variable q can be expressed in terms of quadratures of differential functions of the flat output F (see Chelouah [2]). When $F = constant$, the zero dynamics $\ddot{q} = -\cos F$ is unstable around the equilibrium position $F = \pi/2$. Thus $\theta = F$ is a *nonminimum phase* output.

Regard equation (2) as a *nonlinear differential operator equation*, in the unknown F, of the form, $F = T(F, \xi)$, where ξ is a *known* desired trajectory for q, with $T(F,\xi) = \arccos(-\ddot{\xi} + \xi\dot{F}^2)$. The following functional iterative procedure would be aimed at approximately solving the proposed operator equation [1]

$$F_{k+1} = \arccos\left(-\ddot{\xi} + \xi\dot{F}_k^2\right) \qquad (3)$$

We can thus obtain a sequence of *finite order differential parametrization* of the flat output $F = \theta$ in terms of the defect variable $q = \xi$, by considering the embedding of the equation (2) into the *functional iterative process* (3).

Indeed, the following *sequence* of differential functions, of finite increasing order, for the flat output F, in terms of ξ, is immediately generated by (3),

$$\begin{aligned} F_0 &= constant \\ F_1 &= \arccos\left(-\ddot{\xi}\right) \\ F_2 &= \arccos\left(-\ddot{\xi} + \frac{\xi\left(\xi^{(3)}\right)^2}{1-\left(\ddot{\xi}\right)^2}\right) \\ &\vdots \\ F_\infty &= \psi\left(\xi, \dot{\xi}, \ddot{\xi}, \ldots, \xi^{(k)}, \ldots\right) \end{aligned} \qquad (4)$$

All the system variables are, thus, differentially parametrizable in terms of ξ as follows.

$$\begin{aligned} q = \xi,\ p &= \dot{\xi},\ \theta = \psi\left(\xi, \dot{\xi}, \ddot{\xi}, \ldots, \xi^{(k)}, \ldots\right) \\ u &= \sum_{k=0}^{\infty} \frac{\partial \psi}{\partial \xi^{(k)}} \xi^{(k+1)} \end{aligned} \qquad (5)$$

The output $\xi = q$ is a flat output of infinite order.

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[1] There exists an extensive literature on the topic of approximately solving differential operator equations, through iterative processes, as evidenced by the book of Kurpel' [4] and the references therein.

2 A trajectory trakcing problem

Suppose it is desired to perform the following maneuver for the ball position on the rod

$$q^*(t) = \begin{cases} X, & t \leq T_1 \\ X + R\,\psi(t, T_1, T_2)\,\sin(\omega t) & T_1 < t \leq T_2 \\ X + R\,\sin(\omega t) & T_2 < t < \infty \end{cases} \quad (6)$$

with $\psi(t, T_1, T_2)$ being a Bezier polynomial, smoothly interpolating between 0 and 1 satisfying the conditions $\psi(T_1, T_2, T_2) = 0$, $\psi(T_2, T_1, T_2) = 1$.

We use the first term of the approximation algorithm (4) in order to off-line generate a set of *nominal* reference trajectories for the state variables $p = p^*(t)$, $\theta = \theta^*(t)$ and the control input, $u = u^*(t)$, in terms of a desired nominal trajectory for the ball position variable $q^*(t)$ (see Figure 1). The closed loop controller was designed using approximate linearization and linear time-varying feedback compensation by constant pole placement techniques, as developed in [5] in the context of nonlinear observer design.

Figure 2 shows the closed loop performance of the system for a significant deviation of the initial position with respect to the prescribed nominal trajectory.

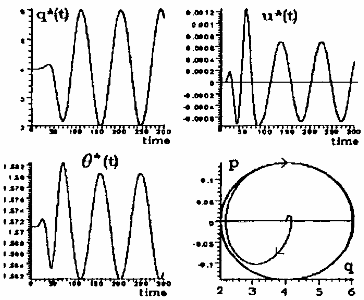

Figure 1: Nominal (open loop) state and control input trajectories.

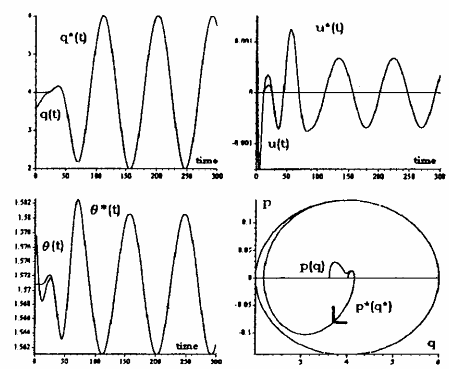

Figure 2: Closed loop system responses for trajectory tracking task.

3 Conclusions

In this article we have introduced the rudimentary steps towards the uncovering of a differential parametrization of infinite order for non-differentially flat systems. It was shown that the required trajectories of the largest flat subsystem linearizing output, which effectivley complies with a desired trajectory for the defect variable, can be approximated using the first few terms of the sequence of parametrizations leading to infinite order flatness. This task usually required the off-line solution of nonlinear differential equations which, in the case of nonminimum phase systems, could lead to unfeasible, i.e. unstable, solutions. The class of non flat systems where this technique can be naturally applied seems to be the class of *Liouvillian* systems.

References

[1] A. Bressand and F. Rampazzo, "On Differential Systems with Quadratic Impulses and their Applications to Lagrangian Dynamics", SIAM J. Control Optimization Vol 31, pp. 1205-1220, 1993.

[2] A. Chelouah, "Extensions of differential flat fields and Liouvillian systems" in *Proceedings of the 36th Conference on Decision and Control*, San Diego, California, USA. December 1997, pp. 4268-4273.

[3] M. Fliess, J. Lévine, Ph. Martín and P. Rouchon, "Sur les systèmes nonlinéaires différentiellement plats", *C. R. Acad. Sci. Paris*, I-315, pp. 619-624, 1992.

[4] N. S. Kurpel', *Projection-Iterative Methods for Solution of Operator Equations*, Vol. 46, Translations of Mathematical Monographs. American Mathematical Society, 1976.

[5] J. Rudolph and M. Fliess, "Corps de Hardy et Observateurs Asymptotiques Locaux pour Systèmes Différentiellement Plats" *C. R. Acad. Sci. Paris*, Serie Automatique, t. 324, Serie II-b, pp. 513-519, 1997.

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