Sliding Mode Control of a Differentially Flat Vibrational Mechanical System: Experimental Results*

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Abstract

In this paper the differential flatness property and a sliding mode controller are applied to a vibrating mechanical system in order to achieve asymptotic output tracking and disturbance attenuation. The mechanical system consists of two masses connected with springs. The output to be controlled is the position of the underactuated mass, which is directly affected by an undesirable vibration (harmonic force with variable excitation frequency). The active vibration control scheme exploits the differential flatness property during the control design, employing only position measurements and approximate time differentiation, and is dynamically able to track an off-line planned trajectory in spite of small disturbances. The overall system performance is validated by some numerical simulations and experimental results in a physical platform.

Keywords: Asymptotic tracking, Differential flatness, Sliding modes, Vibration control.

1 Introduction

Many engineering systems undergo undesirable vibrations. Vibration control in mechanical systems is an important problem, by means of which vibrations are suppressed or at least attenuated. There are three fundamental methodologies described as passive, semi-active and active vibration control. Passive vibration control relies on the addition of stiffness and damping to the system to reduce the primary response, and serves for an specific excitation frequency and stable operating conditions, but is not recommended for variable frequencies and uncertain system parameters. Semi-active vibration control deals with adaptive spring or damper characteristics which are tuned according to the operating conditions. Active vibration control achieves better performance by adding degrees of freedom to the system and controlling actuator forces depending on the feedback and feedforward information of the system obtained from sensors. In general, the addition of extra degrees of freedom causes complex dynamics, with couplings which can be linear or nonlinear. For more details we refer to [3, 6, 1, 4].

On the other hand, many dynamical system exhibit a structural property called differential flatness. This property is equivalent to the existence of a set of independent outputs, called flat outputs and equal in number to the control inputs, which completely parameterizes every state variable and control input (see [2]). By means of differential flatness the analysis and design of a controller is greatly simplified. In particular, the combination of differential flatness with sliding-mode control techniques (extensively used when a robust control scheme is required, e.g., uncertainty due to modelling errors and dynamic disturbances [7]) qualifies as a valuable control scheme to achieve robust asymptotic output tracking.

This paper considers the problem of controlling a two degree of freedom mechanical platform consisting of mass carriages and springs. The control objective consists of trajectory planning of an underactuated output and disturbance attenuation of external vibrations affecting directly the system. The control law is synthesized applying the differential flatness approach as well as sliding mode control techniques to achieve the asymptotic output tracking of off-line planned trajectories in presence of harmonic vibrations (perturbation forces). The overall closed-loop system comprises an active vibration control scheme, using only position measurements and estimated velocities and accelerations, which is able to compensate harmonic vibrations with variable excitation frequency and, simultaneously, get the asymptotic output tracking. Some numerical simulations and experimental results on a physical system illustrate the robust performance of the active vibration control system.

2 Vibrating mechanical system

An schematic diagram of the vibrating mechanical system is shown in Fig. 1. The mechanical system con-

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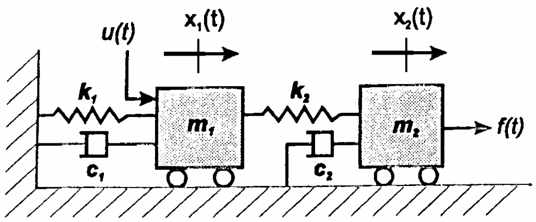

Figure 1: Schematic diagram of the mechanical system.

sists of two mass carriages (m_1,m_2) interconnected by bi-directional springs (k_1,k_2). Each mass carriage suspension has an anti friction ball bearing system, therefore, the linear dashpots (c_1,c_2) shown in Fig. 1 are included only to describe the small viscous dampings. The control force is represented by $u(t)$, which can directly push the mass m_1 to satisfy the desired control objectives (trajectory tracking and disturbance attenuation). This control force is obtained from a brushless servo motor connected to a pinion-rack mechanism. The underactuated mass carriage m_2 is affected by an exogenous (harmonic) force $f(t) = F\sin(\omega t)$ generated from a shaker mounted on the carriage (eccentric connected to a dc-motor shaft with velocity regulation). Optical encoders can measure the mass carriage positions via a cable-pulley system (velocities and accelerations are numerically approximated by software).

The *primary* and *secondary* subsystems are composed by the elements (m_2, c_2, k_2) and (m_1, c_1, k_1), respectively. The undesirable vibrations on the mass carriage m_2 are produced by the action of the harmonic force $f(t)$. The secondary subsystem (m_1, c_1, k_1) may be considered as a passive vibration absorber for the primary subsystem (m_2, c_2, k_2); it is typically designed to satisfy the resonance conditions, where the harmonic force $f(t)$ is actually attenuated from $x_2(t)$ only for a specific excitation frequency ω and stable operation conditions. The main disadvantage of this open-loop control scheme is the absence of robustness with respect to parameter uncertainties or varying excitation frequencies.

The introduction of the control force $u(t)$ result in an active vibration control scheme. This will be properly designed using differential flatness, sliding mode techniques and trajectory planning.

2.1 Mathematical model

The mathematical model of the two degree of freedom mechanical system shown in Fig. 1 is given by

$$m_1\ddot{x}_1 + c_1\dot{x}_1 + k_1x_1 + k_2(x_1 - x_2) = u(t) \quad (1)$$

$$m_2\ddot{x}_2 + c_2\dot{x}_2 - k_2(x_1 - x_2) = f(t) \quad (2)$$

where x_1 and x_2 denote the horizontal positions of the mass carriages m_1 and m_2 with respect to the resting point, respectively. The control input $u(t)$ and the exogenous signal $f(t)$ are forces able to push or pull the mass carriages m_1 and m_2, respectively.

By defining the state variables $z_1 = x_1$, $z_2 = \dot{x}_1$, $z_3 = x_2$, $z_4 = \dot{x}_2$ the model in (1)-(2) is represented in the state space form

$$\dot{z} = Az + Bu + Ef,\ z \in R^4,\ f \in R^1 \quad (3)$$

$$y = z_3,\ y \in R^1 \quad (4)$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{(k_1+k_2)}{m_1} & -\frac{c_1}{m_1} & \frac{k_2}{m_1} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k_2}{m_2} & 0 & -\frac{k_2}{m_2} & -\frac{c_2}{m_2} \end{bmatrix},$$

$$B = \begin{bmatrix} 0 \\ \frac{1}{m_1} \\ 0 \\ 0 \end{bmatrix},\ E = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{m_2} \end{bmatrix}$$

It can be easily proved that the free vibrations of system (3)-(4) are only stable for the undamped case (i.e., when $c_1 = 0$ and $c_2 = 0$ the matrix A has all its eigenvalues over the imaginary axis) and asymptotically stable for the damped case (i.e., when $c_1 > 0$ and $c_2 > 0$ all the eigenvalues of A have negative real part). Moreover, this system is completely controllable.

The transfer function of the system (3)-(4) relating the disturbance $f(t)$ with the output $y = z_3$ is easily obtained as

$$G(s) = \frac{Y(s)}{F(s)} = \frac{m_1s^2 + c_1s + (k_1+k_2)}{d_4s^4 + d_3s^3 + d_2s^2 + d_1s + d_0}$$

where $d_4 = m_1m_2$, $d_3 = m_1c_2 + m_2c_1$, $d_2 = m_1k_2 + m_2(k_1+k_2) + c_1c_2$, $d_1 = k_2c_1 + c_2(k_1+k_2)$ and $d_0 = k_1k_2$. When $f(t) \equiv 0$ the equilibrium state is given by $x^e = \mathrm{col}(u^e/k_1, 0, u^e/k_1, 0)$, where u^e is a constant input (force), hence, $u^e = 0$ implies $x^e = 0$. Otherwise, the disturbance f is in conflict with the transfer of the equilibrium state to any desired trajectory.

Remark 1 *The output $y = z_3$ has relative degree 4 with respect to u and relative degree 2 with respect to f, hence, the disturbance decoupling problem is not solvable. Although disturbance attenuation is still possible by application of a robust controller.*

3 Differential flatness of the vibrating system

A dynamical system is *differentially flat* if there exists a set of differentiable functions of the states, the so-called *flat outputs*, by means of which all the state variables and control inputs can be expressed in terms of the flat outputs and its time derivatives. This structural property and many illustrative applications have been widely explained in several papers by Fliess and co-workers (see [2] and references therein). With this property the full linearization problem can be established and extended to different classes of nonlinear systems. Differential flatness is specially useful for asymptotic stabilization, exact linearization, non-minimum phase outputs, trajectory planning, etc.

For simplicity in the analysis of the differential flatness for the mechanical system (3)-(4) assume that $f(t) = 0$. Because this system is completely controllable then is differentially flat. In fact, the output to be controlled $y = z_3$ is the simplest flat output. In order to show the parameterization of all the state variables and control input, we firstly compute the time derivatives up to fourth order for $y = z_3$, resulting

$$
\begin{aligned}
y &= z_3 \\
\dot{y} &= \dot{z}_3 = z_4 \\
\ddot{y} &= \dot{z}_4 = \frac{k_2}{m_2}(z_1 - z_3) - \frac{c_2}{m_2} z_4 \\
\dddot{y} &= -\frac{k_2 c_2}{m_2^2} z_1 + \frac{k_2}{m_2} z_2 + \frac{k_2 c_2}{m_2^2} z_3 + \left[\left(\frac{c_2}{m_2}\right)^2 - \frac{k_2}{m_2}\right] z_4 \\
\overset{(4)}{y} &= \frac{k_2}{m_2}\left[\left(\frac{c_2}{m_2}\right)^2 - \frac{k_2}{m_2} - \frac{(k_1+k_2)}{m_1}\right] z_1 \\
&\quad - \frac{k_2}{m_2}\left[\frac{c_1}{m_1} + \frac{c_2}{m_2}\right] z_2 \\
&\quad + \frac{k_2}{m_2}\left[\frac{k_2}{m_1} + \frac{k_2}{m_2} - \left(\frac{c_2}{m_2}\right)^2\right] z_3 \\
&\quad + \left[\frac{2k_2 c_2}{m_2^2} - \left(\frac{c_2}{m_2}\right)^3\right] z_4 + \frac{k_2}{m_1 m_2} u
\end{aligned}
$$

Then, the state variables and control input are parameterized in terms of the flat output as follows:

$$
\begin{aligned}
z_1 &= \frac{1}{k_2}(m_2 \ddot{y} + c_2 \dot{y} + k_2 y) \\
z_2 &= \frac{m_2}{k_2}\dddot{y} + \frac{c_2}{k_2}\ddot{y} + \dot{y} \\
z_3 &= y \\
z_4 &= \dot{y} \qquad (5) \\
u &= \frac{1}{b}\left[\overset{(4)}{y} - (a_1 z_1 + a_2 z_2 + a_3 z_3 + a_4 z_4)\right] \\
&= \frac{1}{b}\{\overset{(4)}{y} - [\frac{a_2 m_2}{k_2}\dddot{y} + \left(\frac{a_1 m_2 + a_2 c_2}{k_2}\right)\ddot{y} + \\
&\quad + \left(\frac{a_1 c_2}{k_2} + a_2 + a_4\right)\dot{y} + (a_1 + a_3) y]\} \qquad (6)
\end{aligned}
$$

where

$$
\begin{aligned}
a_1 &= \frac{k_2}{m_2}\left[\left(\frac{c_2}{m_2}\right)^2 - \frac{k_2}{m_2} - \frac{(k_1+k_2)}{m_1}\right], \\
a_2 &= -\frac{k_2}{m_2}\left[\frac{c_1}{m_1} + \frac{c_2}{m_2}\right], \\
a_3 &= \frac{k_2}{m_2}\left[\frac{k_2}{m_1} + \frac{k_2}{m_2} - \left(\frac{c_2}{m_2}\right)^2\right] \\
a_4 &= \left[\frac{2k_2 c_2}{m_2^2} - \left(\frac{c_2}{m_2}\right)^3\right], \; b = \frac{k_2}{m_1 m_2}
\end{aligned}
$$

If the control objective is the asymptotic output tracking to a desired reference trajectory $y^*(t)$ then, we can compute the state variables and control input required to get perfect tracking of the given nominal trajectory. Therefore, in an ideal setting the parameterization in (5)-(6) can be used to determine $z^*(t)$ and $u^*(t)$ leading to $y(t) = y^*(t)$. As a matter of fact, the differential flatness approach itself does not provide asymptotic tracking and robustness in presence of the exogenous signal or parameter uncertainties into the system. In case that $f(t) \neq 0$, the above parameterization must explicitly include the effect of f and time derivatives up to second order.

A sliding mode controller based on differential flatness will result in a more realistic control scheme and better system behavior.

4 Sliding mode control based on differential flatness

Specifying a desired trajectory $y^*(t)$ for the flat output $y(t) = z_3(t)$ and defining the tracking error by $e(t) := y(t) - y^*(t)$ a state feedback controller, based on exact tracking error, is obtained as follows (see eq. (6)):

$$
\begin{aligned}
u &= \frac{1}{b}\{-[\frac{a_2 m_2}{k_2}\dddot{y} + \left(\frac{a_1 m_2 + a_2 c_2}{k_2}\right)\ddot{y} \\
&\quad + \left(\frac{a_1 c_2}{k_2} + a_2 + a_4\right)\dot{y} + (a_1 + a_3) y] + v(t)\} \qquad (7)
\end{aligned}
$$

Application of (7) to the system (3)-(4), but in coordinates $(y, \dot{y}, \ddot{y}, \dddot{y})$, leads to a closed-loop system in the Brunovsky canonical form. The new control input $v(t)$ is designed to stabilize the system and obtain the desired tracking. For instance, the linear error feedback

$$v(t) = \alpha_3 \dddot{e}(t) + \alpha_2 \ddot{e}(t) + \alpha_1 \dot{e}(t) + \alpha_0 e(t) + \overset{(4)}{y}{}^*(t) \quad (8)$$

where α_i, $i = 0, ..., 3$, are positive real constants selected in such a way that the polynomial $s^4 + \alpha_3 s^3 + \alpha_2 s^2 + \alpha_1 s + \alpha_0$ be Hurwitz according to some prescribed linear dynamics (pole placement). Thus, the tracking error dynamics is described by the linear ordinary differential equation

$$\overset{(4)}{e}(t) + \alpha_3 \dddot{e}(t) + \alpha_2 \ddot{e}(t) + \alpha_1 \dot{e}(t) + \alpha_0 e(t) = 0 \quad (9)$$

which is globally asymptotically stable (i.e., $\lim_{t\to\infty} e(t) = 0$).

Note that this linear controller does not consider the perturbation $f(t)$ and parameter uncertainties. To overcome this kind of difficulties we shall exploit the well-known robustness properties of the sliding mode control techniques.

Consider a linear switching surface defined by

$$\sigma(e, \dot{e}, \ddot{e}, \dddot{e}) = \dddot{e} + \beta_2 \ddot{e} + \beta_1 \dot{e} + \beta_0 e \quad (10)$$

where β_i, $i = 1, ..., 3$, are design parameters incorporating the regulation and stability requirements. The sliding motion of the tracking error $e(t)$, that is, the error dynamics restricted to $\sigma(\cdot) = 0$, is governed by the linear differential equation

$$\dddot{e} + \beta_2 \ddot{e} + \beta_1 \dot{e} + \beta_0 e = 0 \quad (11)$$

Next, the constants $\beta_0, \beta_1, \beta_2$ are selected to verify that the associated polynomial $s^3 + \beta_2 s^2 + \beta_1 s + \beta_0$ be Hurwitz. As a consequence, the error dynamics on the switching surface $\sigma(\cdot) = 0$ is globally asymptotically stable.

The sliding surface $\sigma(\cdot) = 0$ is made globally attractive with the continuous approximation to the discontinuous sliding mode controller as given in [5], i.e., by forcing σ to satisfy the dynamics

$$\dot{\sigma} = -\mu \left[\sigma + \gamma \operatorname{sign}(\sigma)\right] \quad (12)$$

where μ, γ denote positive real constants and $\operatorname{sign}(\cdot)$ is the standard *signum* function. By using the Lyapunov function σ^2 it is easy to prove the asymptotic convergence of σ to the sliding surface $\sigma = 0$ (i.e., $\sigma\dot{\sigma} < 0$).

The sliding mode controller is then synthesized by

$$\begin{aligned} v &= -\left[\beta_2 \dddot{e}(t) + \beta_1 \ddot{e}(t) + \beta_0 \dot{e}(t)\right] + \\ &\quad -\mu \left[\sigma(\cdot) + \gamma \operatorname{sign}(\sigma(\cdot))\right] + \overset{(4)}{y}{}^{*}(t) \end{aligned} \quad (13)$$

Taking into account the physical limitations of the vibrating mechanical system, the degree of 'smoothness' introduced in (12) is quite reasonable to avoid the high switching frequencies. In addition, we shall use the following approximation of the *signum* function

$$sign(\sigma) \approx \frac{\sigma}{|\sigma| + \varepsilon} \quad (14)$$

where ε is an arbitrarily small positive constant.

5 Simulation and experimental results

The closed-loop system performance is illustrated with some numerical simulations. By means of several real time experiments on the vibrating mechanical system previously described.

The experimental platform is comprised of three subsystems: *i*) electromechanical plant with actuator (brushless dc servomotor) and sensors (high resolution encoders of 4000 pulses per revolution with an effective linear resolution of 2266 counts/cm) as described in Fig. 1; *ii*) real time controller based on a DSP installed into a PC; and, *iii*) software installed on a PC to program the control algorithms and collect the information. The maximum sampling rate of the overall system is 1.131 Khz (minimum sampling time is 0.000884 seconds), however, to avoid damages and high frequencies this was fixed to 0.00884 seconds.

Because only position measurements are directly available on the experimental platform (via the encoders), then the time derivatives up to fourth order of the corresponding flat output are numerically approximated into the real time control algorithms. In fact, these computations will be clearly observed on the noisy control input.

The physical parameters of the vibrating mechanical system are given in Table I.

Table I. System parameters	
$m_1 = 1.85$ kg,	$m_2 = 2.55$ kg
$k_1 = 400$ N/m,	$c_1 \approx 10$ N/(m/s)
$k_2 = 175$ N/m,	$c_2 \approx 10$ N/(m/s)

The above parameters were actually validated to fit the open-loop time and frequency response of the mechanical system.

The parameters of the proposed sliding mode controller (7)-(13) were designed to have polynomials

$$s^3 + \beta_2 s^2 + \beta_1 s + \beta_0 = (s + p)\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)$$

hence

$$\beta_0 = p\omega_n^2 \quad \beta_1 = \omega_n^2 + 2p\zeta\omega_n \quad \beta_2 = (p + 2\zeta\omega_n)$$

Thus, we select $p = 20$, $\zeta = 0.7071$, $\omega_n = 12.566$ rad/s, $\mu = 10$, $\gamma = 10$ and $\varepsilon = 0.01$.

The desired trajectory for the flat output $y(t) = x_2(t)$ is planned off-line according to

$$y^*(t) = \begin{cases} \bar{y} = \text{constant} & \text{if } 0 \le t < T_1 \\ \bar{y}\left[1 - \psi(t, T_1, T_2)\right] & \text{if } T_1 \le t < T_2 \\ 0 & \text{if } t > T_1 \end{cases}$$

where $\bar{y} = 0.010$ m, $T_1 = 6$ s, $T_2 = 8$ s, and $\psi(t, T_1, T_2)$ is a Beziér type polynomial such that $\psi(T_1, T_1, T_2) = 0$ and $\psi(T_2, T_1, T_2) = 1$. The polynomial is given by

$$\psi(t, T_1, T_2) = \left(\frac{t - T_1}{T_2 - T_1}\right)^3 [r_1 - r_2\left(\frac{t - T_1}{T_2 - T_1}\right) + +r_3\left(\frac{t - T_1}{T_2 - T_1}\right)^2 + ... + r_6\left(\frac{t - T_1}{T_2 - T_1}\right)^5]$$

with constants $r_1 = 252$, $r_2 = 1050$, $r_3 = 1800$, $r_4 = 1575$, $r_5 = 700$ and $r_6 = 126$ chosen to guarantee smooth connections of the three piecewise continuous trajectories. Observe that $y^*(t)$ is obtained as the concatenation of non-smooth and sufficiently smooth trajectories. This is considered to compare the closed-loop tracking dynamics in both cases.

Several numerical simulations were performed with and without the presence of the exogenous vibration force given by $f(t) = F\sin(\omega t)$, with $F = 2.295$ N and $\omega = 62.8$ rad/s. Note the small amplitude of the vibration combined with a high excitation frequency for the natural frequencies of the physical mechanical system (6.7 rad/s and 18.3 rad/s). It is evident that such a perturbation force causes highly oscillatory responses on the system dynamics, thus exciting possible unmodelled dynamics.

5.1 Simulation results

In Fig. 2 is shown the numerical simulation using the sliding mode controller and differential flatness for $f(t) = 0$ [N]. There one can see a good dynamic performance with respect to the asymptotic output tracking. In case of $f(t) = 2.295\sin(62.8t)$ [N] (see Fig. 3) the asymptotic output tracking is quite reasonable, because the undesirable vibration is attenuated, but with the compromise of having internal behaviors and control actions which are highly oscillatory.

5.2 Experimental results

In Figs. 4 and 5 is illustrated the experimental results obtained with the application of the sliding mode control scheme based on differential flatness. Specifically, when $f(t) = 0$ [N] the closed-loop response is quite similar to that obtained via numerical simulations (see Fig. 4). The control input present high frequencies close to the abrupt and smooth changes on the desired reference trajectory. This is due to deficient numerical approximations for those time derivatives needed into the real time control algorithm.

When the mechanical system undergoes the vibration force $f(t) = 2.295\sin(62.8t)$ [N] (see Fig. 5) the asymptotic output tracking is quite reasonable and even better than that obtained through numerical simulations. The active vibration control is able to attenuate the disturbance from the output, resulting in a robust asymptotic output tracking. Again, the internal dynamics and the control force show highly oscillatory responses.

6 Conclusions

Differential flatness and sliding mode control techniques were applied to a vibrating mechanical system in order to achieve asymptotic output tracking and disturbance attenuation. The mechanical system consists of two masses connected with springs. The output to be controlled is the position of the underactuated mass, which is directly affected by an undesirable vibration (harmonic force with variable excitation frequency). The active vibration control scheme exploits the differential flatness property during the control design, employing only position measurements and approximate time differentiation, and is dynamically able to track an off-line planned trajectory in spite of small disturbances. The overall system performance is validated via numerical simulations and experimental results in a physical platform. The active vibration controller is quite robust against the action of exogenous (harmonic) vibrations with high excitation frequencies.

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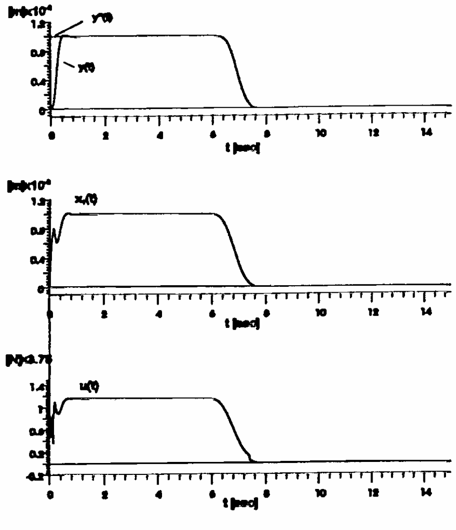

Figure 2: Simulation results using sliding mode control and differential flatness when $f(t) = 0$ [N].

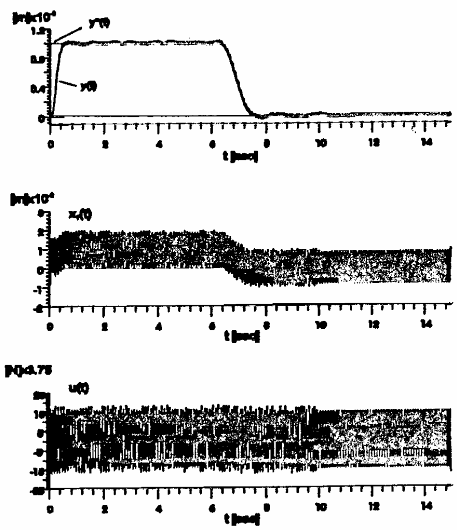

Figure 3: Simulation results using sliding mode control and differential flatness when $f(t) = 2.295\sin(62.8t)$ [N].

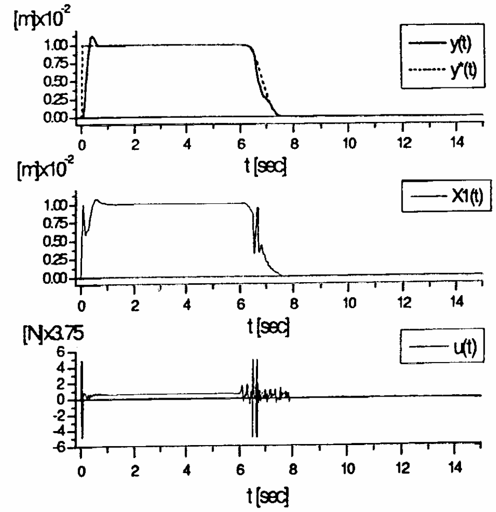

Figure 4: Experimental results using sliding mode control and differential flatness when $f(t) = 0$ [N].

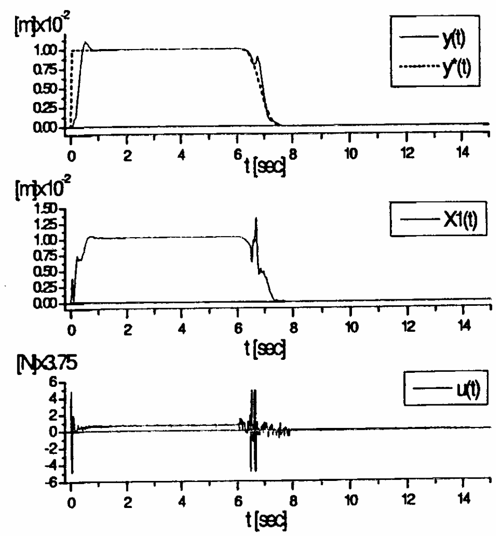

Figure 5: Experimental results using sliding mode control and differential flatness when $f(t) = 2.295\sin(62.8t)$ [N].