Non-minimum Phase Output Reference Trajectory Tracking for a PVTOL Aircraft[1]

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Abstract

An approximate solution to the general reference trajectory tracking problem, specified in terms of a desired behavior for the non-minimum phase aircraft center of mass position coordinates, is proposed by indirectly defining a suitable trajectory tracking task for the (minimum phase) flat outputs of the system represented by the equivalent to the Huygen's center of oscillation of the aircraft. The approach uses an approximating sequence which appears to be related to an *infinite order differential flatness* of the system with respect to the non-minimum phase output variables.

1 Introduction

The regulation of non-minimum phase outputs represents an interesting problem which has received sustained attention in the past. An indirect approach to such problem was proposed in Benvenuti *et al* in [1] using a judiciously chosen minimum phase output. A second approach approximates the non-minimum phase system by means of a minimum phase system (see Hauser *et al* [2]).

Differential flatness, introduced in [3], is a far reaching ***structural*** system property which can be related to many feedback controller design techniques (backstepping, passivity, exact feedback linearization, etc). Roughly speaking, a multivariable nonlinear system is ***flat*** if there exists a certain vector of independent functions, called the ***flat outputs***, of the same dimension as the vector of control inputs, which are ***differential functions*** of the ***state*** of the system (i.e., these outputs are a function of the state variables and also of a finite number of their time derivatives), with the additional property that, every system variable, i.e., states, original outputs and also the inputs, can, in turn, be expressed as differential functions of the flat outputs.

PVTOL aircraft systems have been the object of study by many researchers. An exact linearization solution to the VTOL position transfer problem has been given by Hauser *et al* in [2] using an approximation of the non-minimum phase system by regarding it to be a ***slightly non-minimum phase*** system. The regulation aspects of the non-minimum phase outputs of the PVTOL aircraft system have been studied in Martin, Devasia and Paden [4] where flatness is exploited in a scheme using inverse trajectory feedforward in combination with a state tracker while guaranteeing a bounded zero dynamics. Recent articles, [5], [6], have proposed an indirect solution to the rest-to-rest stabilization problem via the tracking of a given set of trajectories for the flat outputs, computed on the basis of the non-minimum phase output variables equilibria. The differential parameterization is of little, or no, help on how to proceed with more general kinds of maneuvers which do not necessarily involve stabilization around an equilibrium point. Typically, a trajectory tracking problem is specified in terms of desired trajectories for the non-minimum phase aircraft's center of gravity. Typical maneuvers involve the following a circle or of a "figure eight" and so on.

In this article, we show that the differential parameterization provided by flatness contains enough information as to allow the determination of suitable open loop, non-stationary, relationships between the desired trajectories for the non-minimum phase output variables and the corresponding required flat outputs trajectories. We establish an iterative procedure for generating a sequence of ***finite order differential parameterization*** of the system variables, including the flat outputs, in terms of the non-minimum phase aircraft center of gravity position coordinates. Only the first few elements of the approxi-

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mating sequence need to be evaluated in order to obtain reasonable trajectory candidates that serve as reference trajectories for the flat outputs. The corresponding dynamic feedback controller is then obtained by exact flat output trajectory tracking error linearization.

In section 2 we present the model of the PVTOL aircraft and proceed to obtain the dynamic feedback controller which regulates the system around a set of given flat outputs trajectories. An iterative off-line computational algorithm is then proposed to approximately generate the flat output reference trajectories in terms of the desired non-minimum phase outputs trajectories. Section 3 presents the simulation results. Section 4 is devoted to present some conclusions and suggestions for further research.

2 Trajectory Tracking for the PVTOL Aircraft Example

2.1 The PVTOL aircraft model

The simplified description of the dynamics of a planar vertical take-off and landing (PVTOL) aircraft is given by the following magnitude and time normalized model (see Figure 1)

$$\begin{aligned} \ddot{x} &= -u_1 \sin\theta + \epsilon u_2 \cos\theta \\ \ddot{z} &= u_1 \cos\theta + \epsilon u_2 \sin\theta - g \\ \ddot{\theta} &= u_2 \end{aligned} \tag{2.1}$$

where x and z are the horizontal and vertical coordinates of the center of gravity of the aircraft, respectively measured along an orthonormal set of fixed horizontal and vertical coordinates. The angle θ is the aircraft's longitudinal axis angular rotation as measured with respect to the fixed horizontal coordinate axis. The controls u_1 and u_2 represent normalized quantities related to the vertical thrust and the angular rolling torque applied around the longitudinal axis of the aircraft respectively. The constant g is the gravity acceleration and ϵ is a fixed constant related to the geometry of the aircraft.

The system outputs x and z are known to be *non-minimum phase*. Indeed, if x and z are held constant by means of a suitable control action, then, in particular, $\ddot{x} = 0$ and $\ddot{z} = 0$. Using the system dynamics (2.1) one readily obtains the required control inputs as

$$u_1 = g\cos\theta \quad ; \quad u_2 = \frac{g}{\epsilon}\sin\theta$$

The corresponding *remaining dynamics* is then represented by the following autonomous differential equation for the angular position of the aircraft,

$$\ddot{\theta} = \frac{g}{\epsilon}\sin\theta \tag{2.2}$$

The dynamics (2.2) exhibits an unstable (saddle) equilibrium point at the origin $\theta = 0$, $\dot{\theta} = 0$ and a center around $\theta = \pi$, $\dot{\theta} = 0$. For initial conditions with zero angular velocity, the periodic nature of the solutions of 2.2, imply a "rocking" motion of the aircraft around its longitudinal axis. For zero initial conditions of the roll angle and nonzero initial angular velocity, the system (2.2) is unstable and hence, as time increases, the aircraft rotates about its longitudinal axis while its center of gravity remains fixed at a constant position in the x-z plane (see Figure 2).

2.2 Differential flatness of the PVTOL model

It has been shown in [4] that the PVTOL model is differentially flat, with flat output given by the horizontal and vertical coordinates (F, L) of the ***Huygens center of oscillation*** when the aircraft dynamics is re-interpreted as the dynamics of a pendulum of length ϵ. Such outputs are given by,

$$F = x - \epsilon\sin\theta \quad ; \quad L = z + \epsilon\cos\theta \tag{2.3}$$

The PVTOL aircraft system model requires a second order dynamic extension on the control input u_1. Instead of taking u_1 and $\dot{u}_1$ as additional state variables, the following auxiliary variable $\varsigma = u_1 - \epsilon\left(\dot{\theta}\right)^2$ is introduced as a new state variable.

It can be shown, after some algebraic manipulations, that all the system variables x, z, $\dot{z}$, θ, and ς, are expressible as *differential functions* of F and L, i.e.,

$$\begin{aligned} x &= F - \epsilon\frac{\ddot{F}}{\sqrt{\left(\ddot{F}\right)^2 + \left(\ddot{L}+g\right)^2}} \\ z &= L - \epsilon\frac{\left(\ddot{L}+g\right)}{\sqrt{\left(\ddot{F}\right)^2 + \left(\ddot{L}+g\right)^2}} \\ \theta &= -\arctan\left(\frac{\ddot{F}}{\ddot{L}+g}\right) \; ; \; \varsigma = \sqrt{\left(\ddot{F}\right)^2 + \left(\ddot{L}+g\right)^2} \end{aligned} \tag{2.4}$$

Letting

$$F^{(4)} = v_1 \quad ; \quad L^{(4)} = v_2$$

we obtain the following expressions for the original control inputs u_1 and u_2 as well as a second order differential equation describing the states $(\varsigma, \dot{\varsigma})$, corresponding to the second order extension of the control input variable u_1.

$$\begin{aligned} u_1 &= \varsigma + \epsilon\left(\dot{\theta}\right)^2 \\ u_2 &= \frac{1}{\varsigma}\left(-v_1\cos\theta - v_2\sin\theta - 2\dot{\varsigma}\dot{\theta}\right) \end{aligned}$$

$$\ddot{\varsigma} = -v_1 \sin\theta + v_2 \cos\theta + \varsigma \left(\dot{\theta}\right)^2 \qquad (2.5)$$

2.3 A state feedback controller for trajectory tracking in the PVTOL aircraft system

Suppose we are given a set of open loop trajectories $F^*(t)$ and $L^*(t)$ which the flat outputs F and L are required to follow over an indefinite period of time in the real line.

One proceeds to impose on the flat output tracking errors $e_F(t) = F - F^*(t)$ and $e_L(t) = L - L^*(t)$ the following asymptotically stable behaviors,

$$\begin{aligned} e_F^{(4)}(t) + a_3 e_F^{(3)}(t) + a_2 \ddot{e}_F(t) + a_1 \dot{e}_F(t) + a_0 e_F(t) &= 0 \\ e_L^{(4)}(t) + b_3 e_L^{(3)}(t) + b_2 \ddot{e}_L(t) + b_1 \dot{e}_L(t) + b_0 e_L(t) &= 0 \end{aligned} \qquad (2.6)$$

where the sets of coefficients $\{a_3, a_2, a_1, a_0\}$ and $\{b_3, b_2, b_1, b_0\}$ are chosen so that the corresponding characteristic polynomials are both *Hurwitz* polynomials, i.e., with all their roots having strictly negative real parts.

The specification of the tracking errors dynamics (2.6) results in the following feedback controller explicitly based on the specification of the desired flat outputs trajectories,

$$\begin{aligned} v_1 &= F^{*(4)}(t) - a_3 \left(-\dot{\varsigma}\sin\theta - \varsigma\dot{\theta}\cos\theta - F^{*(3)}(t)\right) \\ &\quad - a_2 \left(-\varsigma\sin(\theta) - \ddot{F}^*(t)\right) \\ &\quad - a_1 \left(\dot{x} - \epsilon\dot{\theta}\cos\theta - \dot{F}^*(t)\right) \\ &\quad - a_0 \left(x - \epsilon\sin\theta - F^*(t)\right) \\ v_2 &= L^{*(4)}(t) - b_3 \left(\dot{\varsigma}\cos(\theta) - \varsigma\dot{\theta}\sin(\theta) - L^{*(3)}(t)\right) \\ &\quad - b_2 \left(\varsigma\cos(\theta) - g - \ddot{L}^*(t)\right) \\ &\quad - b_1 \left(\dot{z} + \epsilon\dot{\theta}\sin\theta - \dot{L}^*(t)\right) \\ &\quad - b_0 \left(z + \epsilon\cos\theta - L^*(t)\right) \end{aligned} \qquad (2.7)$$

2.4 A suitable reference trajectory for the flat outputs in terms of desired center of gravity displacements

Sometimes, maneuvers are required which do not involve a rest-to-rest equilibrium transfer. Moreover, the most common specification of a desired trajectory is made in terms of time-varying functions $x^*(t)$, $z^*(t)$ for the displacement variables x and z and not in terms of the flat outputs. Notice that in order to determine suitable trajectories $F^*(t)$, $L^*(t)$ for the flat outputs in terms of the desired displacement variables trajectories, $x^*(t)$ and $z^*(t)$, a complete specification of the corresponding angular position trajectory, $\theta^*(t)$, of the aircraft is also necessary. It is not intuitively clear how to specify it and neither is it possible to obtain it, in an exact way, from knowledge of $x^*(t)$ and $z^*(t)$ alone.

Consider the set of relations linking the involved variables $x^*(t)$, $z^*(t)$, and $\theta^*(t)$ through the flat outputs $F^*(t)$, $L^*(t)$, as obtained from the definitions of the flat outputs (2.3) and the differential parameterization (2.4).

$$\begin{aligned} F^*(t) &= x^*(t) - \epsilon\sin\theta^*(t) \\ L^*(t) &= z^*(t) + \epsilon\cos\theta^*(t) \\ \theta^*(t) &= -\arctan\left(\frac{\ddot{F}^*(t)}{\ddot{L}^*(t) + g}\right) \end{aligned} \qquad (2.8)$$

We proceed to embed the above set of relations into an off-line iterative computational algorithm of the form

$$\begin{aligned} F_k(t) &= x^*(t) - \epsilon\sin\theta_k(t) \\ L_k(t) &= z^*(t) + \epsilon\cos\theta_k(t) \\ \theta_{k+1}(t) &= -\arctan\left(\frac{\ddot{F}_k(t)}{\ddot{L}_k(t) + g}\right) \end{aligned} \qquad (2.9)$$

whose terms coincide with the original relations only after convergence. [1] In these relations $x^*(t)$ and $z^*(t)$ represent the desired trajectories for the displacement variables and they are supposed to be known. We initialize the above algorithm with a reasonable value of the function θ, say $\theta_0 = 0$. The first two iterations of the algorithm yield,

$$\left\{ \begin{aligned} \theta_0 &= 0 \\ F_0(t) &= x^*(t) \\ L_0(t) &= z^*(t) + \epsilon \end{aligned} \right.$$

[1] In fact one may view (2.8) as a set of relations implicitly defining a nonlinear (unstable) differential equation. One may view such an equation also as a static *unbounded nonlinear differential operator*, as already suggested, years ago, by Chaplygin (see the book by Kurpel' [7]). The proposed embedding, (2.9), stands as the iterative computational algorithm usually devised to approximately solve the underlying operator equation. Since the main purpose of this algorithm is *off-line trajectory planning*, convergence issues may be safely side-stepped, provided that the candidate elements, generated by the iterative process, produce reasonable approximations to the desired trajectory, which, as shown below, is precisely the case in this example, after $k = 1$.

$$\begin{cases} \theta_1(t) & = & -\arctan\left(\dfrac{\ddot{x}^*(t)}{\ddot{z}^*(t)+g}\right) \\ F_1(t) & = & x^*(t)+\epsilon\left(\dfrac{\ddot{x}^*(t)}{\sqrt{(\ddot{x}^*(t))^2+(\ddot{z}^*(t)+g)^2}}\right) \\ L_1(t) & = & z^*(t)+\epsilon\left(\dfrac{\ddot{z}^*(t)+g}{\sqrt{(\ddot{x}^*(t))^2+(\ddot{z}^*(t)+g)^2}}\right) \end{cases}$$

The proposed algorithm produces a sequence of *finite order differential parameterizations* of the angular position θ in terms of the desired reference trajectories, $x^*(t)$ and $z^*(t)$, which constitutes an approximation to an eventual *infinite order differential parameterization* of θ, of the form

$$\theta_\infty(t) = \psi\left(x^*(t), z^*(t), ..., \frac{d^k}{dt^k}x^*(t), \frac{d^k}{dt^k}z^*(t), ...\right)$$

Stopping the algorithm at some desired level of approximation (say, when $k = K$), yields a set of reference trajectories candidates $F_K^*(t)$ and $L_K^*(t)$ for the flat outputs F and L which can be directly used in the previously proposed tracking error state feedback controller (2.7). The reference trajectories for F and L are, thus, given purely in terms of the desired horizontal and vertical displacements $x^*(t)$ and $z^*(t)$ and a finite number of their time derivatives as

$$\begin{aligned} F_K^*(t) &= x^*(t) \\ &\quad -\epsilon\sin\theta_K(x^*(t), z^*(t), ..., \frac{d^{2K}}{dt^{2K}}x^*(t), \frac{d^{2K}}{dt^{2K}}z^*(t)) \\ L_K^*(t) &= z^*(t) \\ &\quad +\epsilon\cos\theta_K(x^*(t), z^*(t), ..., \frac{d^{2K}}{dt^{2K}}x^*(t), \frac{d^{2K}}{dt^{2K}}z^*(t)) \end{aligned}$$

2.5 Simulation Results

We tested the performance of the feedback controlled PVTOL using only the first two outcomes of the off-line approximation algorithm, i.e., $k = 0, 1$. The desired horizontal and vertical displacements were given by the time functions

$$x^*(t) = A\sin\omega t \quad ; \quad z^*(t) = A(1-\cos\omega t)$$

i.e., it was desired to follow a circle of radius A, centered at the point $x = 0, z = A/2$ of the x-z plane, with an angular velocity of revolution around this center given by ω.

The following set of system (normalized) parameter values and prescribed trajectory specifications,

$$\epsilon = 0.5 \ ; \ g = 1 \ ; \ A = 1 \ ; \ \omega = 0.3$$

were chosen for the digital computer simulations.

The controller design parameters were chosen so that the polynomials $p_F(s)$ and $p_L(s)$ each had four roots located at the point $-2+0j$ in the real axis of the complex plane, i.e,

$$a_3 = 8 \ ; \ a_2 = 24 \ ; \ a_1 = 32 \ ; \ a_0 = 16$$

$$b_3 = 8 \ ; \ b_2 = 24 \ ; \ b_1 = 32 \ ; \ b_0 = 16$$

Figure 3 shows the evolution of the controlled horizontal and vertical positions, as well as the angular displacement corresponding to the first step reference trajectory approximation functions. Notice that, in this case, the flat output trajectories $F^*(t)$ and $L^*(t)$ are being approximated by the non-minimum phase quantities

$$\begin{aligned} F_0^*(t) &= x^*(t) = A\sin\omega t, \\ L_0^*(t) &= z^*(t)+\epsilon = A(1-\cos\omega t)+\epsilon \end{aligned}$$

It is, therefore, expected to obtain a typical non-minimum phase behavior of the controlled responses with an "undershoot" around the initial time. The feedback control inputs will then tend to compensate this momentary drift by exerting a large feedback control action that we try to limit by imposing some saturations to the control amplitudes. For this, the feedback control inputs were restricted by, $|u_1| < u_{1,max}, |u_2| < u_{2,max}$ with $u_{1,max} = u_{2,max} = 10$. As a result of the tracking of a shifted circle by the flat outputs, the corresponding position variables x and z track a nearly circular trajectory. The closed loop performance results, for the first order approximation flat output reference trajectories, is already reasonably good for such a low order approximation, as shown in Figure 3.

We also implemented a set of reference trajectories for the flat outputs using the second step approximation of the proposed off-line trajectory generation algorithm. The expressions used for the corresponding desired maneuver for the flat outputs were given by,

$$\begin{aligned} F_1^*(t) &= A\sin(\omega t) \\ &\quad -\epsilon\left(\frac{A\omega^2\sin(\omega t)}{\sqrt{A^2\omega^4\sin^2(\omega t)+(A\omega^2\cos(\omega t)+g)^2}}\right) \\ L_1^*(t) &= A(1-\cos(\omega t)) \\ &\quad +\epsilon\left(\frac{A\omega^2\cos(\omega t)+g}{\sqrt{A^2\omega^4\sin^2(\omega t)+(A\omega^2\cos(\omega t)+g)^2}}\right) \end{aligned} \quad (2.10)$$

The time derivatives of the reference trajectories (2.10) required by the dynamic feedback controller (2.5),(2.7) were obtained using the Maple symbolic computation package.

Figure 4 shows the computer simulation results corresponding to the closed loop responses of the controlled system. A non-minimum phase behavior is still observed in the system's responses at the beginning of the maneuver. This time, however, the control inputs do not reach a value high enough to exceed the imposed amplitude saturation limits. The accuracy with which the displacement variables x and z track the desired circular trajectory, as the flat outputs track the corresponding off-line computed trajectories (2.10), is remarkably good.

In order to test the robustness of the proposed trajectory tracking scheme, an unmodeled additive wind perturbation force $\zeta(t)$ was assumed to affect both the horizontal and the vertical displacement dynamics. This wind gust perturbation was modeled as a Gauss distribution function on the (x, z) plane, centered around the point $x = -0.5, z = 1.9$, with maximum amplitude B, given by

$$\zeta(t) = B \exp\left(-\frac{(x(t)+0.45)^2 + (z(t)-1.9)^2}{\sigma^2}\right)$$

Figure 5 shows the corresponding closed loop performance of the controlled system in the (x, z) plane, for several values of the wind gust amplitudes, $B = 0.1,\ 0.2,\ 0.4$, with $\sigma = 0.1$. The controlled trajectories are seen to be momentarily disturbed from its prescribed circular course and, as the perturbation fades, when the aircraft moves away from the turbulence point, the controller brings the motion of the aircraft back to the prescribed ideal circular trajectory.

3 Conclusions

In this article, initial steps have been taken to exploit differential flatness, in a non-traditional manner, for reference trajectory tracking problems. In many examples of physical nature, as in the PVTOL example, it is difficult, if not impossible, to directly specify a suitable reference trajectory for the flat outputs which, in turn, generate a desired, or pre-specified, set of trajectories form the (non-minimum phase) system outputs. We have shown that the differential parameterization allowed by the flatness of the system contains the key to uncover a natural sequence of finite differential parameterizations approximating a certain flatness property of infinite order (i.e., one involving an infinite number of time derivatives) exhibited by the non-minimum phase center of mass position coordinates variables. Only the first few terms of such an approximating sequence are required in order to obtain an accurate reference solution. The approach consists then in using an element of the described sequence, to off-line generate a suitable reference trajectory for the flat outputs. An exact tracking error linearization controller is then used to provide a feedback control solution, devoid of internal instabilities, to the proposed non-minimum phase outputs trajectory tracking problem.

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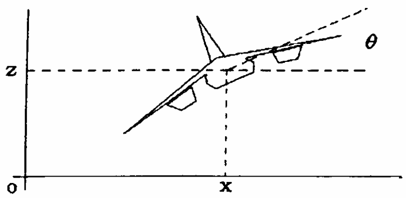

Figure 1: Planar Vertical Take-Off and Landing Aircraft.

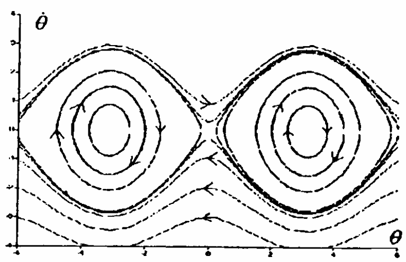

Figure 2: Unstable remaining dynamics ($\epsilon = 0.5$, $g = 1$).

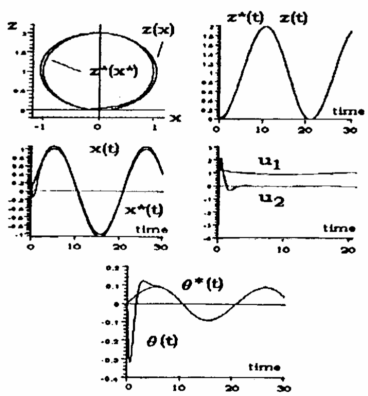

Figure 3: Closed loop responses for first order approximation of flat output reference trajectories in terms of the desired displacement trajectories.

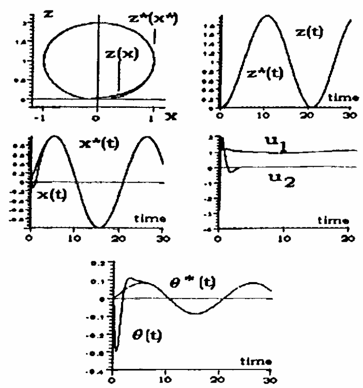

Figure 4: Closed loop responses for second order approximation of flat output reference trajectories in terms of the desired displacement trajectories.

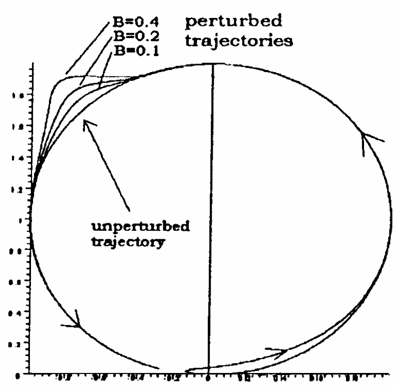

Figure 5: Closed loop responses including an unmodelled wind gust perturbation force centered around the point $x = -0.45$, $z = 1.9$ in the (x, z) plane.