Dynamic second order sliding mode control of the hovercraft vessel *

Hebertt Sira-Ramírez †
Centro de Investigación y Estudios Avanzados
del Instituto Politécnico Nacional (CINVESTAV-IPN)
Departamento Ingeniería Eléctrica
Avenida I.P.N. # 2508
Col. San Pedro Zacatenco, A.P. 14-740
07300 México D.F., México
email hsira@mail. cinvestav.mx
Phone: 52-5-747-7000 Fax: 52-5-7477102

May 24, 2000

Abstract

In this article, a suitable combination of the *differential flatness* property and the *second order sliding mode* controller design technique is proposed for the specification of a robust dynamic feedback multivariable controller accomplishing prescribed trajectory tracking tasks for the earth coordinate position variables of a hovercraft vessel model.

Keywords: hovercraft, flat systems, trajectory planning, second order sliding

*This research was supported by the Centro de Investigación y Estudios Avanzados del Instituto Politécnico Nacional (CINVESTAV-IPN), México and by the Consejo Nacional de Ciencia y Tecnologia (CONACYT) under Research Contract No. 32681-A

†On leave of absence from the Departamento de Sistemas de Control of the Universidad de Los Andes, Mérida-Venezuela

1 Introduction

Differential flatness is a structural property which makes a dynamic system (whether linear, non-linear, monovariable or multivariable) equivalent to a linear controllable system under *endogenous feedback* i.e. one which only requires internal variables to be synthesized. Differential flatness was first introduced from the viewpoint of differential algebra in the work of Fliess *et al* [6]. It soon became apparent, however, that a more general approach was possible from the viewpoint of differential geometry of infinite jet spaces. The geometric approach using *differential varieties*, or "*diffieties*", naturally views a dynamic controlled system as a controlled Cartan field (a diffeity) and system transformations, leading to equivalence, are then cast into the framework of Lie Bäcklund transformations (see the work of Fliess *et al*[7]).

Higher order sliding modes appear as a natural generalization of first order sliding modes, thoroughly studied in the work of Utkin [21], and of a vast, and still growing, list of authors (see the survey prepared by Professor Emely'anov [4]). The basic idea, first proposed by Emely'anov and his coworkers, [3], in the context of "second order sliding modes", is to impose, on a certain auxiliary stabilizing output differential function -or suitably defined tracking error- of arbitrary, but well defined, relative degree, the dynamic behavior of a higher order discontinuous (sliding) dynamics. The chosen sliding dynamics, usually of order higher or equal than two, is such that its trajectories globally, and robustly, converge towards the origin of phase coordinates *in finite time*. This important robustness characteristic of the chosen "higer order sliding algorithm" is usually guaranteed to be preserved even in the presence of, unmodelled, absolutely bounded disturbances. The reader is referred to the works of Levant [11], [12], Fridman and Levant [9], for interesting details, extensions, and generalizations of the second order sliding mode control idea. In the context of uncertain systems, the reader may also benefit from the contents in the recent articles by Bartolini and his coworkers [1], [2]. The article by Sira Ramírez *et al* [20] contains an application example of second order sliding modes to the stabilization of the popular "TORA" system.

In this article, we present an application of the differential flatness property in the controller synthesis of a nonlinear multivariable model of a hovercraft vessel. We propose a robust dynamic feedback control scheme for the hovercraft system based on off-line trajectory planning and dynamic feedback auxiliary trajectory tracking error stabilization to the origin of its phase space coordinates based on second order sliding mode contro. For both, the trajectory planning and the feedback controller design aspects, use is made of the fact that, contrary to the general surface vessel model [8], the hovercraft system model is indeed *differentially flat.* The *flat outputs* are represented by the hovercraft position coordinates with respect to the fixed earth frame. The system is shown to be equivalent, under *endogenous* dynamic feedback, to two fourth order, independent, controllable linear systems in Brunovsky's form. The flatness of the hovercraft model was established in [19].

Section 2 revisits the hovercraft vessel model derivation performed in [5], taking as the starting point the fully actuated, though simplified, ship model found in [8] and also in [13]. In section 2, it is shown that the obtained hovercraft system model is differentially flat. In Section 3, we pose the trajectory tracking problem and derive a robust dynamic feedback controller based on flatness and second order sliding modes. These modes are induced on a set of independent auxiliary polynomial differential functions of the flat outputs tracking errors. Section 4 contains the simulation results for a typical trajectory tracking maneuver and Section 5 is devoted to some conclusions and suggestions for further research.

2 The Hovercraft Model

The regulation of a ship vessel, by means of two independent thrusters located at the aft, has received sustained attention in the last few years. Reyhanoglu [17] uses a discontinuous feedback control law for exponential stabilization towards a desired equilibrium. A feedback linearization approach was proposed by Godhavn [10] for the regulation of the position variables. The scheme, however, did not allow for orientation control. In an article by Pettersen and Egeland [13], a time-varying feedback control law is proposed which exponentially stabilizes the vessel state towards a given equilibrium point. Time-varying quasi-periodic feedback control, developed in Pettersen and Egeland [14], has been proposed taking advantage of the homogeneity properties of a suitably transformed model achieving simultaneous exponential stabilization of the position and orientation variables. An interesting experimental set-up has been built which is described in the work of Pettersen and Fossen [15]. In their work, the time-varying feedback control, used by [13], is extended to include integral control actions, including excellent experimiental results. High frequency feedback control signals, in combination with averaging theory and backstepping, have also been proposed by Pettersen and Nijmeijer [16], to obtain practical stabilization of the ship towards a desired equilibrium and also for trajectory tracking tasks. In [18] the ship trajectory tracking control problem was examined from the perspective of Liouvillian systems (a special class of non-differentially flat systems, i.e. systems which are not equivalent to linear controllable systems by means of endogenous feedback).

The hovercract model we use is based on the recent work of Fantoni *et al* [5] where the vessel's dynamics is derived on the basis of the underactuated ship model extensively studied by Fossen [8]. In [5], a series of interesting Lyapunov-based feedback controllers are derived for the stabilization and trajectory tracking of the hovercraft system.

In the book by Fossen [8] the following model is proposed for a rather general surface vessel dynamics,

$$\begin{aligned} M\dot{\nu} + C(\nu)\nu + D\nu &= \tau \\ \dot{\eta} &= J(\eta)\nu \end{aligned} \tag{2.1}$$

where

$$C(\nu) = \begin{bmatrix} 0 & 0 & -m_{22}v \\ 0 & 0 & m_{11}u \\ m_{22}v & -m_{11}u & 0 \end{bmatrix} \quad J(\eta) = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.2}$$

with

$$M = \text{diag}\,\{m_{11}, m_{22}, m_{33}\}, \quad D = \text{diag}\,\{d_{11}, d_{22}, d_{33}\} \tag{2.3}$$

The vector $\nu = [u, v, r]^T$ denotes the linear velocities in surge, sway, and angular velocity in yaw. The vector $\eta = [x, y, \psi]$ denotes the position and orientation in earth fixed coordinates. The vector $\tau = [\tau_1, \tau_2, \tau_3]$ denotes the control forces in surge and sway and the control torque in yaw. The matrices $C(\nu)$ and D represent, respectively, the Coriolis and centripetal forces and the hydrodynamic damping.

Consider the simplified version of the underactuated hovercraft shown in Figure 1. A model for such symmetric vessel can be directly derived, as already done in Fantoni *et al* [5], from equations (2.1)-(2.3) by enforcing the following simplifying assumptions:

$$m_{11} = m_{22}, \quad , \quad \tau_1 = m_{11}\tau_u, \quad \tau_2 = 0, \quad \tau_3 = m_{33}\tau_r, \quad d_{11} = d_{33} = 0, \quad \beta = \frac{d_{22}}{m_{22}} \tag{2.4}$$

We thus obtain the following model of the underactuated hovercraft vessel system,

$$\begin{aligned} \dot{x} &= u\cos\psi - v\sin\psi \\ \dot{y} &= u\sin\psi + v\cos\psi \\ \dot{\psi} &= r \\ \dot{u} &= vr + \tau_u \\ \dot{v} &= -ur - \beta v \\ \dot{r} &= \tau_r \end{aligned} \tag{2.5}$$

2.1 Differential flatness of the hovercraft system

We have the following proposition

Proposition 2.1 *The model (2.5) is differentially flat, with flat outputs given by x and y i.e.,* all *system variables in (2.5) can be* differentially parameterized *solely in terms of x and y, as*

$$\begin{aligned} \psi &= \arctan\left(\frac{\ddot{y} + \beta\dot{y}}{\ddot{x} + \beta\dot{x}}\right) \\ u &= \frac{\dot{x}(\ddot{x} + \beta\dot{x}) + \dot{y}(\ddot{y} + \beta\dot{y})}{\sqrt{(\ddot{x} + \beta\dot{x})^2 + (\ddot{y} + \beta\dot{y})^2}} \end{aligned}$$

$$
\begin{aligned}
v &= \frac{yx - xy}{\sqrt{(\ddot{x} + \beta \dot{x})^2 + (\ddot{y} + \beta \dot{y})^2}} \\
r &= \frac{y^{(3)}(\ddot{x} + \beta \dot{x}) - x^{(3)}(\ddot{y} + \beta \dot{y}) + \beta^2 (\ddot{x}\dot{y} - \ddot{y}\dot{x})}{(\ddot{x} + \beta \dot{x})^2 + (\ddot{y} + \beta \dot{y})^2} \\
\tau_u &= \frac{\ddot{x}(\ddot{x} + \beta \dot{x}) + \ddot{y}(\ddot{y} + \beta \dot{y})}{\sqrt{(\ddot{x} + \beta \dot{x})^2 + (\ddot{y} + \beta \dot{y})^2}} \\
\tau_r &= \frac{y^{(4)}(\ddot{x} + \beta \dot{x}) - x^{(4)}(\ddot{y} + \beta \dot{y}) + \beta \left(y^{(3)}\ddot{x} - x^{(3)}\ddot{y}\right) - \beta^2 \left(x^{(3)}\dot{y} - y^{(3)}\dot{x}\right)}{(\ddot{x} + \beta \dot{x})^2 + (\ddot{y} + \beta \dot{y})^2} \\
&\quad -2\frac{\left[y^{(3)}(\ddot{x} + \beta \dot{x}) - x^{(3)}(\ddot{y} + \beta \dot{y}) - \beta^2(\ddot{x}\dot{y} - \ddot{y}\dot{x})\right]\left[(\ddot{x} + \beta \dot{x})(x^{(3)} + \beta \ddot{x}) + (\ddot{y} + \beta \dot{y})(y^{(3)} + \beta \ddot{y})\right]}{\left[(\ddot{x} + \beta \dot{x})^2 + (\ddot{y} + \beta \dot{y})^2\right]^2}
\end{aligned}
\tag{2.6}
$$

Proof

From the first two equations in (2.5) we readily obtain

$$
\begin{aligned}
v &= \dot{y}\cos\psi - \dot{x}\sin\psi \\
u &= \dot{x}\cos\psi + \dot{y}\sin\psi
\end{aligned}
\tag{2.7}
$$

Differentiating the first two equations in (2.5) with respect to time, yields, after use of (2.5) and (2.7),

$$
\begin{aligned}
\ddot{x} &= \dot{u}\cos\psi - u\dot{\psi}\sin\psi - \dot{v}\sin\psi - v\dot{\psi}\cos\psi \\
&= \tau_u \cos\psi + \beta v \sin\psi \\
\ddot{y} &= \dot{u}\sin\psi + u\dot{\psi}\cos\psi + \dot{v}\cos\psi - v\dot{\psi}\sin\psi \\
&= \tau_u \sin\psi - \beta v \cos\psi
\end{aligned}
\tag{2.8}
$$

Multiplying the first equation in (2.8) by $\sin\psi$ and the second equation by $\cos\psi$ and then subtracting the obtained expressions we obtain, after use of (2.5),

$$
\ddot{x}\sin\psi - \ddot{y}\cos\psi = \beta v \tag{2.9}
$$

Similarly, multiplying the first equation in (2.8) by $\cos\psi$ and the second by $\sin\psi$ and adding, we obtain

$$
\tau_u = \ddot{x}\cos\psi + \ddot{y}\sin\psi \tag{2.10}
$$

Substituting the first of (2.7) into (2.9) one obtains, after some further algebraic manipulations

$$
\tan\psi = \frac{\ddot{y} + \beta \dot{y}}{\ddot{x} + \beta \dot{x}} \longrightarrow \psi = \arctan\left(\frac{\ddot{y} + \beta \dot{y}}{\ddot{x} + \beta \dot{x}}\right) \tag{2.11}
$$

Using (2.11) in (2.7) we obtain,

$$
v = \frac{\dot{y}\,(\ddot{x} + \beta \dot{x}) - \dot{x}\,(\ddot{y} + \beta \dot{y})}{\sqrt{(\ddot{x} + \beta \dot{x})^2 + (\ddot{y} + \beta \dot{y})^2}} = \frac{\dot{y}\ddot{x} - \dot{x}\ddot{y}}{\sqrt{(\ddot{x} + \beta \dot{x})^2 + (\ddot{y} + \beta \dot{y})^2}} \tag{2.12}
$$

and

$$
u = \frac{\dot{x}(\ddot{x} + \beta \dot{x}) + \dot{y}(\ddot{y} + \beta \dot{y})}{\sqrt{(\ddot{x} + \beta \dot{x})^2 + (\ddot{y} + \beta \dot{y})^2}} \tag{2.13}
$$

Substituting in (2.10) the value of ψ, obtained in (2.11), leads to the expression for the force input, τ_u, given in the proposition. Finally, we make use of the fact that $r = \dot{\psi}$ and $\tau_r = \ddot{\psi}$.

□

2.2 Invertibility of the control parameterization

The differential parameterization of the input torque τ_r depends on the flat outputs, and their time derivatives, up to the fourth order. Note, however, that the corresponding parameterization of the control input τ_u only depends up to the second order time derivatives of x and y . This simple fact clearly reveals an " obstacle" to achieve *static* feedback linearization and points to the need for a second order *dynamic* extension of the control input τ_u in order to exactly linearize the system.

Use of (2.5) allows the following (simpler) expressions for the control inputs τ_r and $\ddot{\tau}_u$, in terms of the system's state variables, the highest order derivatives of the flat outputs x and y, and a first order extension of the control input τ_u.

$$\tau_r = \frac{y^{(4)}\cos\psi - x^{(4)}\sin\psi - \beta r\tau_u - 2r\dot{\tau}_u - 2\beta r^2 v - \beta^2 ur + \beta^3 v}{\beta u + \tau_u} \tag{2.14}$$

$$\ddot{\tau}_u = x^{(4)}\cos\psi + y^{(4)}\sin\psi + 2\beta ur^2 + 2\beta^2 rv - \beta v\tau_r + r^2\tau_u \tag{2.15}$$

(2.16)

Using (2.6) we have that, $\beta u + \tau_u = \sqrt{(\ddot{x}+\beta\dot{x})^2 + (\ddot{y}+\beta\dot{y})^2}$. Clearly, we are interested in maneuvers for which this quantity is bounded (which is physically reasonable and natural) and it is also bounded away from zero (which somehow limits the class of desired trajectories).

Assumption 2.2 *We assume that the positive quantity, $\sqrt{(\ddot{x}+\beta\dot{x})^2 + (\ddot{y}+\beta\dot{y})^2}$, is uniformly bounded by a constant for all times, and it is nowhere identically zero along the evolution of the system.*

The previous assumption specifically precludes us from considering trajectories that either lead, or contain, a resting point for the earth position coordinates, x, and y, of the hovercraft vessel. These may be treated by time reparameterizations, or "control of the clock" techniques (see [7]) which are outside the scope of this paper. On the other hand, straight lines, circles, and many other types of trajectories, which are to be followed at constant speeds, can be handled by the method here proposed.

Let ξ and ϕ denote two independent auxiliary control inputs. Under the above assumption, the, locally defined, input coordinate transformation

$$\begin{aligned} \tau_r &= \frac{\phi\cos\psi - \xi\sin\psi - \beta r\tau_u - 2r\dot{\tau}_u - 2\beta r^2 v - \beta^2 ur + \beta^3 v}{\beta u + \tau_u} \\ \ddot{\tau}_u &= \xi\cos\psi + \phi\sin\psi + 2\beta ur^2 + 2\beta^2 rv - \beta v\tau_r + r^2\tau_u \end{aligned} \tag{2.17}$$

yields the following transformed system

$$x^{(4)} = \xi, \quad y^{(4)} = \phi$$

The hovercraft system is thus equivalent, under endogenous feedback, to a set of two independent linear systems in Brunovsky's controllable canonical form.

3 Trajectory Tracking for the Hovercraft System

3.1 The unperturbed case

Suppose a desired trajectory is given for the position coordinates x and y in the form of specified time functions: $x^*(t)$ and $y^*(t)$, respectively. The following proposition gives a dynamic feedback solution to the trajectory tracking problem based on flatness and exact tracking error linearization through imposition of second order sliding modes on stable differential polynomials of the tracking errors.

Proposition 3.1 *Let the set of constant real coefficients $\{\alpha_2, \alpha_1, \alpha_0\}$ and $\{\gamma_2, \gamma_1, \gamma_0\}$ represent independent sets of* Hurwitz coefficients, *so that the polynomials in the complex variable s,*

$$p(s) = s^2 + \alpha_2 s + \alpha_1, \qquad q(s) = s^2 + \gamma_2 s + \gamma_1$$

have symmetric roots located in the left portion of the complex plane. Let $x^*(t)$ and $y^*(t)$ de a given a set of desired trajectories for the position coordinates x and y which satisfies assumption 2.2. Associated with the polynomials, $p(s)$ and $q(s)$, define the following two auxiliary differential functions of the position tracking errors:

$$\begin{aligned}\rho &= (\ddot{x} - \ddot{x}^*(t)) + \alpha_2(\dot{x} - \dot{x}^*(t)) + \alpha_1(x - x^*(t)) \\ \eta &= (\ddot{y} - \ddot{y}^*(t)) + \gamma_2(\dot{y} - \dot{y}^*(t)) + \gamma_1(y - y^*(t))\end{aligned}$$

Then, for any set of real parameters A_ρ, B_ρ and A_η, B_η such that $A_\rho > B_\rho$ and $A_\eta > B_\eta$, the following dynamic second order sliding feedback controller

$$\tau_r = \frac{\phi\cos\psi - \xi\sin\psi - \beta r\tau_u - 2r\dot{\tau}_u - 2\beta r^2 v - \beta^2 ur + \beta^3 v}{\beta u + \tau_u} \tag{3.1}$$

$$\ddot{\tau}_u - r^2\tau_u = \xi\cos\psi + \phi\sin\psi + 2\beta ur^2 + 2\beta^2 rv - \beta v\tau_r \tag{3.2}$$

$$\xi = x^{*(4)}(t) - \alpha_2(x^{(3)} - x^{*(3)}(t)) - \alpha_1(\ddot{x} - \ddot{x}^*(t)) - \frac{1}{2}sign(\rho)[A_\rho + B_\rho + (A_\rho - B_\rho)sign(\rho\,\dot{\rho})]$$

$$\phi = y^{*(4)}(t) - \gamma_2(y^{(3)} - y^{*(3)}(t)) - \gamma_1(\ddot{y} - \ddot{y}^*(t)) - \frac{1}{2}sign(\eta)[A_\eta + B_\eta + (A_\eta - B_\eta)sign(\eta\,\dot{\eta})]$$

with

$$\begin{aligned}\dot{x} &= u\cos\psi - v\sin\psi \\ \dot{y} &= u\sin\psi + v\cos\psi \\ \ddot{x} &= \beta v\sin\psi + \tau_u\cos\psi \\ \ddot{y} &= \tau_u\sin\psi - \beta v\cos\psi \\ x^{(3)} &= -\left[r(\beta u + \tau_u) + \beta^2 v\right]\sin\psi + (\beta rv + \dot{\tau}_u)\cos\psi \\ y^{(3)} &= \left[r(\beta u + \tau_u) + \beta^2 v\right]\cos\psi + (\beta rv + \dot{\tau}_u)\sin\psi\end{aligned} \tag{3.3}$$

semi-globally stabilizes the auxiliary tracking errors ρ and η and their first order time derivatives $\dot{\rho}$, $\dot{\eta}$ to zero, in finite time. As a consequence, the trajectory tracking errors, $e_x = x - x^*(t)$ and $e_y = y - y^*(t)$ exponentially asymptotically converge to zero.

Proof

Subtracting the controller expression, for $\ddot{\tau}_u$ in (3.1), from the open loop expression in (2.15) we obtain, after some algebraic manipulations,

$$\begin{aligned}&\left[\ddot{\rho} + \frac{1}{2}sign(\rho)[A_\rho + B_\rho + (A_\rho - B_\rho)sign(\rho\,\dot{\rho})]\right]\cos\psi \\ +&\quad \left[\ddot{\eta} + \frac{1}{2}sign(\eta)[A_\eta + B_\eta + (A_\eta - B_\eta)sign(\eta\,\dot{\eta})]\right]\sin\psi = 0\end{aligned} \tag{3.4}$$

Proceeding in a similar fashion with respect to the corresponding closed and open loop expressions for τ_r, one finds:

$$\begin{aligned}-&\left[\ddot{\rho} + \frac{1}{2}sign(\rho)[A_\rho + B_\rho + (A_\rho - B_\rho)sign(\rho\,\dot{\rho})]\right]\sin\psi \\ &+ \left[\ddot{\eta} + \frac{1}{2}sign(\eta)[A_\eta + B_\eta + (A_\eta - B_\eta)sign(\eta\,\dot{\eta})]\right]\cos\psi = 0\end{aligned} \tag{3.5}$$

Then, clearly, the tracking errors functions ρ and η satisfy the ideal second order sliding dynamics

$$\begin{aligned}\ddot{\rho} + \frac{1}{2}sign(\rho)[A_\rho + B_\rho + (A_\rho - B_\rho)sign(\rho\,\dot{\rho})] &= 0 \\ \ddot{\eta} + \frac{1}{2}sign(\eta)[A_\eta + B_\eta + (A_\eta - B_\eta)sign(\eta\,\dot{\eta})] &= 0\end{aligned} \tag{3.6}$$

As a consequence, the variables ρ and η , as well as their corresponding time derivatives, $\dot{\rho}$ and $\dot{\eta}$, converge to zero in finite time (see [20] for a proof of the semi-global convergence to the origin, in finite time, of the trajectories generated by the second order sliding dynamics), thus imposing the following asymptotically exponentially stable dynamics on the flat outputs tracking errors,

$$\begin{aligned}(\ddot{x} - \ddot{x}^*(t)) + \alpha_2(\dot{x} - \dot{x}^*(t)) + \alpha_1(x - x^*(t)) &= 0 \\ (\ddot{y} - \ddot{y}^*(t)) + \gamma_2(\dot{y} - \dot{y}^*(t)) + \gamma_1(y - y^*(t)) &= 0\end{aligned}$$

3.2 The perturbed case

A simple tracing of the influence of unmodelled perturbations, in the open loop system (2.5), reveals that a hypothesized perturbation in either the surge velocity equation, or the sway dynamics, affects all the state variables of the system, except the orientation angle ψ. On the contrary, a similar perturbation affecting the yaw rate dynamics, propagates to *all* of the states in the system. We consider first the latter case.

Consider an unmodelled *matched* perturbation input, $\varsigma(t)$, which is absolutely uniformly bounded by a strictly positive constant S, i.e. $\mid \varsigma(t) \mid \leq S \ \forall \ t$. This perturbation affects the ship's yaw rate dynamics in the form:

$$\begin{aligned}\dot{x} &= u\cos\psi - v\sin\psi \\ \dot{y} &= u\sin\psi + v\cos\psi \\ \dot{\psi} &= r \\ \dot{u} &= vr + \tau_u \\ \dot{v} &= -ur - \beta v \\ \dot{r} &= \tau_r + \varsigma(t)\end{aligned} \tag{3.7}$$

The input coordinate transformation (2.14), (2.15) on the perturbed system (3.7) results now in the following set of *perturbed Brunovsky canonical forms*

$$x^{(4)} = \xi - \varsigma(t)\left[(\beta u + \tau_u)\sin\psi + \beta v\cos\psi\right], \quad y^{(4)} = \phi + \varsigma(t)\left[(\beta u + \tau_u)\cos\psi + \beta v\sin\psi\right] \tag{3.8}$$

Evidently, from assumption 2.2, and for absolutely uniformly bounded surge and sway velocities u and v, the perturbation terms affecting the transformed system are also absolutely uniformly bounded. [1] Thus, for some strictly positive constant parameters Q and R, the perturbed transformed system can be assumed to be of the form: $x^{(4)} = \xi + m(t)$, $y^{(4)} = \phi + n(t)$ with $\mid m(t) \mid \leq Q$ and $\mid n(t) \mid \leq R$, for all t.

Following the same steps in the proof of the previous proposition, we find that the perturbed version of the dynamics of the auxiliary tracking error functions ρ and η are now governed by,

$$\begin{aligned}\ddot{\rho} + \frac{1}{2}sign(\rho)[A_\rho + B_\rho + (A_\rho - B_\rho)sign(\rho\,\dot{\rho})] &= m(t) \\ \ddot{\eta} + \frac{1}{2}sign(\eta)[A_\eta + B_\eta + (A_\eta - B_\eta)sign(\eta\,\dot{\eta})] &= n(t)\end{aligned} \tag{3.9}$$

According to the robustness results of second order sliding modes [20], the perturbed evolutions of ρ and η converge to zero *in finite time*, provided, $B_\rho < A_\rho$, with $Q < \min\{B_\rho, (A_\rho - B_\rho)/2\}$ and $B_\eta < A_\eta$, with $R < \min\{B_\eta, (A_\eta - B_\eta)/2\}$. This implies that for a set of suitable controller parameters, $A_{(\cdot)}, B_{(\cdot)}$, and for an absolutely uniformly bounded perturbation input signal, $\varsigma(t)$, the trajectory tracking errors, $x - x^*(t)$, and $y - y^*(t)$, still asymptotically exponentially converge to zero in spite of the influence of the perturbations.

A similar conclusion can be reached for the case of *unmatched perturbations* affecting the non-actuated sway velocity dynamics. For absolutely uniformly bounded perturbations $\chi(t)$ with similarly bounded first order time derivatives, acting on the non-actuated sway velocity equation, as $\dot{v} = -ur - \beta v + \lambda(t)$, the input coordinate transformation (2.14), (2.15) yields the following perturbed Brunovsky canonical forms:

[1] Note that the sway velocity dynamics is a *linear* time-invariant dynamics, with a strictly negative eigenvalue, excited by the product of the surge velocity, u, and the yaw rate, r. Since it is physically plausible to assume that these two velocities are absolutely uniformly bounded, then it follows that the absolute value of the sway velocity is also uniformly bounded.

$$x^{(4)} = \xi - r\left[2\dot{\lambda}(t) - \beta\lambda(t)\right]\cos\psi + \left[\beta\dot{\lambda}(t) + (r^2 - \beta^2)\lambda(t)\right]\sin\psi$$
$$y^{(4)} = \phi - \left[\beta\dot{\lambda}(t) + (r^2 - \beta^2)\lambda(t)\right]\cos\psi - r\left[2\dot{\lambda}(t) - \beta\lambda(t)\right]\sin\psi$$

The perturbations affecting the right hand sides of the Brunovsky forms are evidently absolutely bounded for bounded yaw rates, r, and absolutely bounded perturbation inputs, $\lambda(t)$, with an absolutely bounded first order time derivative. Thus, expressions similar to (3.9) are also valid. In the simulations presented below, we test the proposed nominal dynamic second order sliding mode controller of Proposition 3.1 with an unmatched perturbation input signal of the form just discussed.

4 Simulation Results

Simulations were carried out to evaluate the performance of the proposed dynamic feedback controller for a rather common trajectory tracking task: The tracking of a circular trajectory, defined in the earth fixed coordinate frame, of radius R, centered around the origin.

4.1 Tracking a circular trajectory

A circular trajectory, or radius R, is to be followed in a clockwise sense in the plane (y, x), with a given constant angular velocity of value ω. In other words, the flat outputs are nominally specified as,

$$x^*(t) = R\ \cos\omega t, \quad y^*(t) = R\ \sin\omega t \tag{4.1}$$

For this particular choice of x and y, the nominal orientation angle $\psi^*(t)$ is given by

$$\psi^*(t) = \arctan\left(\frac{\omega\sin\omega t - \beta\cos\omega t}{\omega\cos\omega t + \beta\sin\omega t}\right) = \arctan(\tan(\omega t - \theta)) = \omega t - \theta \tag{4.2}$$

with $\theta = \arctan(\beta/\omega)$.

The nominal surge and sway velocities and the nominal yaw angular velocity are given, according to (2.7) and the fact that $r = \dot{\psi}$, by the following constant values

$$u^*(t) = -R\omega\sin\theta, \quad v^*(t) = R\omega\cos\theta, \quad r^*(t) = \omega \tag{4.3}$$

Similarly, using (2.10) and the fact that $\tau_r = \ddot{\psi}$ we obtain that the nominal applied inputs are given by the following constant values

$$\tau_u^*(t) = -R\omega^2\cos\theta, \quad \tau_r^*(t) = 0 \tag{4.4}$$

Note that for the chosen trajectory, the nominal value of the quantity $\beta u + \tau_u$, appearing in the denominator of the controller expression for τ_r, is given by

$$\beta u + \tau_u = R\omega(\omega\cos\theta + \beta\sin\theta) = R\omega\sqrt{\beta^2 + \omega^2} \neq 0$$

The only system parameter β was set to be $\beta = 1.2$. We have chosen the following parameters for the circular reference trajectory

$$R = 10, \quad \omega = 0.1,$$

which result in $\theta = 1.4876$ rad, $\tau_u^* = -8.304 \times 10^{-3}$. The controller parameters were set to be:

$$\alpha_2 = \gamma_2 = 1.4142, \quad \alpha_1 = \gamma_1 = 1, \quad A_\rho = A_\eta = 0.2, \quad B_\rho = B_\eta = 0.05$$

Figure 2 depicts the controlled evolution of the hovercraft position coordinates when the vessel motions are started significantly far away from the desired trajectory. Figure 3 shows the corresponding surge, the sway, and the yaw angular velocities. Figure 4 depicts the applied external inputs, $\tau_u(t)$, $\tau_r(t)$.

4.2 Robustness with respect to unmodelled, unmatched, perturbations

In order to test the robustness of the proposed controller, we introduced in the non-actuated dynamics (i.e., in the sway acceleration equation) an unmodelled external perturbation force, simulating a rather strong "wave field" effect, of the form:

$$\lambda(x(t)) = L\left[\sin(fx(t)) + \frac{1}{5}\cos(\pi f x(t))\right], \quad \left(\dot{v} = -ur - \beta v + \lambda(x(t)) \right)$$

with $L = 0.15$ and $f = 10$. The parameters of the second order sliding dynamics and the auxiliary function ρ were set to be,

$$A = 0.2, \quad B = 0.05, \quad \alpha_1 = \gamma_1 = 1.414, \alpha_0 = \gamma_0 = 1$$

In spite of the unmatched nature of the perturbation signal, the proposed dynamic feedback controller, using the same controller parameters used before, efficiently corrects the undesirable deviations, due to the persistent perturbation, and manages to accomplish the trajectory tracking task with satisfactory precision.

5 Conclusions

In this article, we have illustrated how the property of differential flatness can be advantageously combined with the robustness and simplicity of higher order sliding modes. We have carried out this combined controller design option in the context of the trajectory tracking regulation of an underactuated hovercraft system model, derived through some simplifying assumptions from the general surface vessel model. This model is shown to be differentially flat. The flatness property immediately allows to establish the equivalence of the model, by means of dynamic state feedback, to a set of two decoupled controllable linear systems. A trajectory planning, combined with a second order sliding mode trajectory tracking scheme , allows to obtain a direct feedback controller synthesis for arbitrary position trajectory following. The design was shown to be robust with respect to significant perturbation input forces even when they affect the non actuated portion of the hovercraft velocity dynamics.

The characteristics, and simplicity, of higher order sliding mode controllers, beyond those of the second order type treated here, seem to be a natural, and remarkably robust, alternative for the efficient regulation and trajectory tracking tasks of perturbed nonlinear systems which are nominally differentially flat.

The more difficult problem of hovercraft position regulation towards trajectories that include a resting equilibrium point is the object of ongoing research by many authors. The problem certainly deserves attention from the flatness viewpoint using time-reparameterizations, and other suitable techniques.

References

[1] G. Bartolini, P. Pidynowsky, "An improved chattering free VSC scheme for uncertain dynamic systems" *IEEE Trans. on Auto. Contr.*, Vol. 41, pp. 1220-1226, 1996.

[2] G. Bartolini, A. Ferrara, A. Levant, E. Usai, "On second order sliding mode controllers" in *Variable Structure Systems, Sliding Mode and Nonlinear Control*, K.D. Young, and Ü. Özgüner (Eds). Lecture Notes in Control and Information Sciences, Vol. 247, pp. 329-350. Springer-Verlag, London, 1999.

[3] S.V. Emel'yanov, S.K. Korovin and L.V. Levantovsky, "Higher order sliding modes in the binary control systems" *Soviet Physics*, Dokady, vol. 31, No. 4, pp. 291-293, 1986.

[4] S. V. Emelyanov, *Titles in the theory of variable structure control systems*, International Research Institute for Management Sciences, Irmis, Moscow, 1990.

[5] I. Fantoni, R. Lozano, F. Mazenc and K. Y. Pettersen, " Stabilization of a nonlinear underactuated hovercraft" *Int. Journal of Robust and Nonlinear Control* (to appear).

[6] M. Fliess, J. Lévine, Ph. Martin and P. Rouchon, "Flatness and defect of nonlinear systems: Introductory theory and examples ", *International Journal of Control*, Vol. 61 , pp. 1327-1361, 1995.

[7] M. Fliess, J. Lévine, Ph. Martin and P. Rouchon, " A Lie-Bäcklund approach to equivalence and flatness" *IEEE Transactions on Automatic Control*, Vol. 44, No. 5, pp. 922-937, May 1999.

[8] T. I. Fossen, *Guidance and Control of Ocean Vehicles* John Wiley and Sons Ltd. Chichester, 1994.

[9] L. Fridman, A. Levant, "Higher order sliding modes as a natural phenomenon in control theory" in *Robust Control via Variable Structure and Lyapunov Techniques*, F. Garofalo and L. Glielmo (Eds.). Lecture Notes in Control and Information Science, No. 217, pp. 107-133, Springer-Verlag, London, 1996.

[10] J. M. Godhavn, "Nonlinear tracking of underactuated surface vessels" In *Proceedings of the 35th IEEE Conference on Decision and Control.* Kobe, Japan,pp. 987-991.

[11] A. Levant, "Sliding order and sliding accuracy in sliding mode control," *International Journal of Control*, Vol. 58, No. 6, pp. 1247-1263, 1993.

[12] A. Levant, "Controlling output variables via higher order sliding modes" in *Proc. of the European Control Conference ECC'97*, Brussels, 1-4, June 1997. (CD Rom version).

[13] K. Y. Pettersen and O. Egeland, " Exponential stabilization of an underactuated surface vessel" *Proceedings of the 35th IEEE Conference on Decision and Control*, Kobe, Japan, pp. 967-971, 1996.

[14] K. Y. Pettersen and O. Egeland, "Robust control of an underactuated surface vessel with thruster dynamics", In *Proceedings of the 1997 American Control Conference*, Albuquerque, New Mexico, 1997.

[15] K. Y. Pettersen and T. L. Fossen, " Underactuated ship stabilization using integral control: Experimental results with Cybership I" in *Proceedings of the 1998 IFAC Symposium on Nonlinear Control Systems* NOLCOS'98 Enschede, The Netherlands, July 1998.

[16] K. Y. Pettersen and H. Nijmeijer, "Global Practical Stabilization and Tracking for an underactuated Ship- A Combined Averaging and Backstepping Approach" *Proceedings of the 1998 IFAC Conference on Systems Structure and Control*, Nantes, France. July 1998, pp. 59-64.

[17] M. Reyhanoglu, "Control and stabilization of an underactuated surface vessel" In *Proceedings of the 35th IEEE Conference on Decision and Control*, Kobe, Japan. pp. 2371-2376.

[18] H. Sira-Ramírez, " On the control of the underactuated ship : A trajectory planning approach", *37th IEEE Conference on Decision and Control*, Phoenix, AZ, December 1999. (to appear).

[19] H. Sira-Ramírez and C. Aguilar-Ibáñez, "On the control of the hovercraft system" *Dynamics and Control*, (accepted for publication, to appear).

[20] H. Sira-RamíRez, A. Levant and L. Fridman, "Regulation of the RTAC system via flatness and second order sliding" in *Proc. of the Variable Structure Systems Workshop*, Rockhampton, Australia, December 2000. (to appear).

[21] V. I. Utkin, *Sliding Modes and their Applications in the Theory of Variable Structure Systems*, MIR Editors, Moscow, 1977.

Acknowledgments The author has greatly benefitted from discussions with Dr. G. Silva-Navarro of CINVESTAV-IPN and also from e-mail discussions with Dr. Arie Levant and Professor L. Fridman. Their criticism and encouragement is gratefully acknowledged.

LIST OF FIUGRES

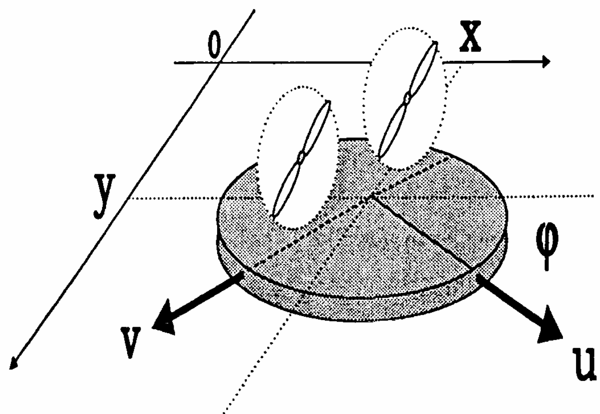

Figure 1: The simplified hovercraft system

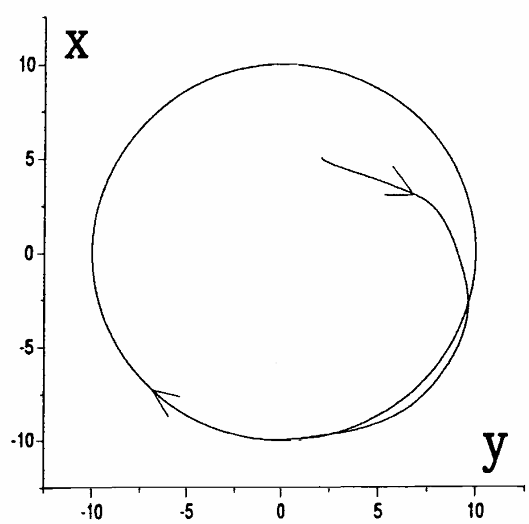

Figure 2: Feedback controlled position coordinates for circular path tracking

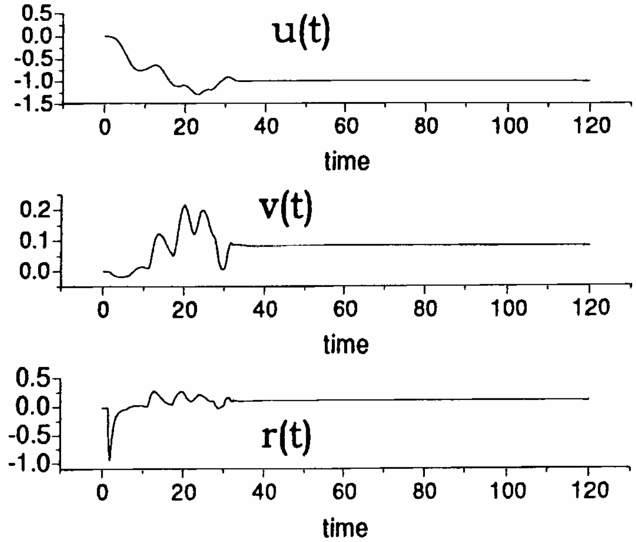

Figure 3: Feedback controlled velocity variables for circular path tracking

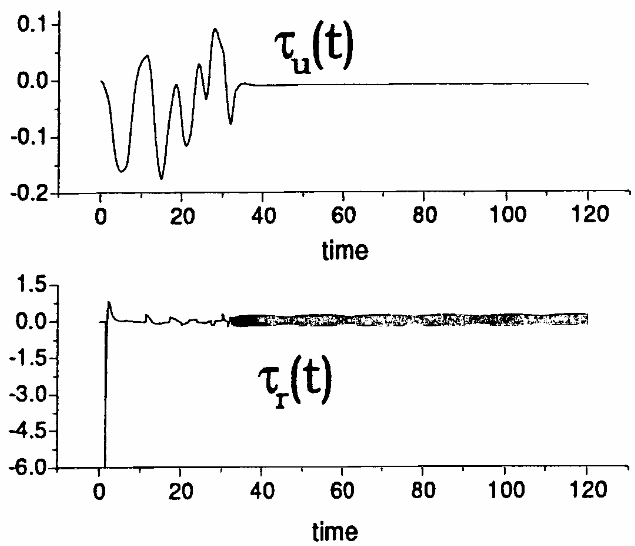

Figure 4: Applied control inputs for circular path tracking

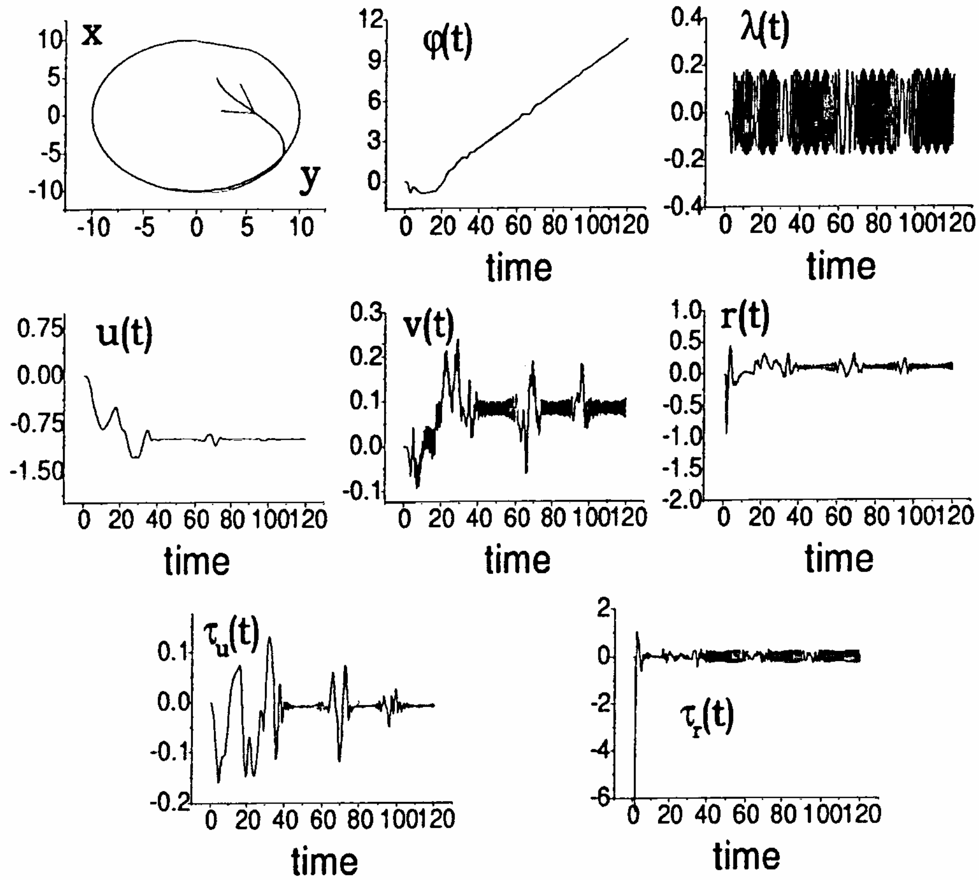

Figure 5: Circular path tracking performance under unmodeled sustained perturbations