

On the passivity of general nonlinear systems *

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Abstract

In this article, we revisit the definitions of passivity and feedback passivity in the context of general continuous-time single input, single output, systems which are jointly nonlinear in the states and the control input. Necessary conditions are given for the characterization of passive systems and extend the well known Kalman-Yakubovich-Popov (KYP) conditions. Passivity concepts are used for studying the stabilization problem of general nonlinear systems. We extend the 'Energy Shaping and Damping Injection' (ESDI) controller design methodology to the studied class of systems. A semi-canonical form for nonlinear systems which is of the Generalized Hamiltonian type, including dissipation terms, is also proposed. Passive and strictly passive systems are shown to be easily characterized in terms of such a canonical form.

1 Introduction

The study of the behavior of a system in terms of its stored, or dissipated, energy has an extraordinary value, since it is directly related to the system's stability properties. The control of a system in terms of stored energy considerations, known as *Passivity-Based Control* (PBC), exploits the system's physical properties in connection to its *energy managing* and *dissipation enhancement* possibilities. The technique

is known to bestow simplicity and robustness to the obtained feedback controller designs.

We can distinguish three stages in the evolution of passivity concepts and its application to control problems. The initial developments of dissipativity and passivity concepts were presented as a generalization of those found in early circuit theory. The development of passivity and the connections with feedback stabilization of a system, from a general operator theoretic viewpoint, were introduced in the early 70's in the work of Willems ([29, 30]). A different research line was initiated by the work of Wu and Desoer ([31]), cast in terms of the system input-output properties. The development of passivity concepts, in relation to stability, was undertaken in the works of Vidyasagar [28] and Zames [32]. The extension of Willems' results to the case of nonlinear systems, which are affine in the control input, was given by Hill and Moylan in ([15, 5, 6, 7]). The idea of making a system passive by means of a state static feedback and all the related geometry was given in the work by Byrnes *et al* ([1]). These results were later complemented, in connection with the concept of *feedback positive* systems, in the work of Kokotovic and Sussman ([10]). Definite connections between passivity, Lyapunov-based control and inverse optimal feedback can be found in the book by Sepulchre *et al* [23]. The conditions under which a nonlinear system is rendered passive, via state static state feedback were extended by Santosuosso ([22]) to the case of an affine feedthrough system. The study of feedback dissipativity, for the linear case, has been addressed in the work of Picci and Pinzoni (see [16]). The corresponding developments for the passivity of discrete-time nonlinear systems were given by Lin and Lin and Byrnes in [12, 13, 14]. For a global perspective of the area, the reader is referred to the books by van der Schaft ([27]), Sepulchre *et al* ([23]), Ortega *et al* ([20]) and Khalil ([9]).

The application of the passivity approach to systems control is mainly found in the fields of mechanical, electromechanical and electrical systems, among some others. For applications

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to robotics see the works of Ortega ([17, 19]). Electromechanical systems were addressed in Ortega ([18]). Passivity based controllers for power electronics were given in ([25]). PBC has been shown to be suitably applied to other classes of nonlinear systems (see, [24] and the works of Egeland and Godhavn [3], Johanssen and Egeland [8] and Fossen and Strand [4]).

A fundamental control problem is that of stabilization of the system trajectories around desired equilibria. To this purpose, the *Energy Shaping plus Damping Injection* (ESDI), control design methodology has been developed on the basis of modifying the stored energy of the system, to take into account the desired equilibrium, and the addition, through state or output feedback, of the required dissipation in order to enhance the dissipation structure of the underlying stabilization error system. Generalities and details of this technique can be found in the work of Ortega *et al* [19], the work of Sira-Ramírez *et al*, [25], and the recent article by Ortega *et al* [21]).

In this article, we revisit the general definitions of passivity in the context of general nonlinear SISO dynamic controlled systems. The Kalman-Yakubovich-Popov (KYP) conditions are derived in general terms. The ‘Energy Shaping and Damping Injection’ (ESDI) controller design method is also suitably extended to the general case. We propose a general semi-canonical form for nonlinear systems which is of the Generalized Hamiltonian type. Passive and strictly passive systems are shown to be easily characterized in terms of such a canonical form. Our work is motivated by the developments presented in the articles by Lin (see [11] where an attempt to generalize the KYP conditions were carried out for the continuous-time case).

Section 2 revisits the general definitions about passive systems and feedback passive systems. In this section we derive the generalized KYP conditions and set the stage for the extension of the ESDI controller design methodology to the general nonlinear systems case. Section 3 presents the extension of the ESDI design method. Section 4 deals with the derivation of a Generalized Hamiltonian type canonical form for nonlinear systems. The conclusions and suggestions for further research are presented in the last section.

2 Passivity of General Nonlinear Systems

2.1 Generalities

We consider nonlinear single input, single output, systems of the form

$$\begin{aligned}\dot{x} &= f(x, u); \quad x \in R^n, \quad u \in R \\ y &= h(x, u); \quad y \in R\end{aligned}\quad (1)$$

where $f : R^n \times R \rightarrow R^n$, is a smooth mapping of its arguments which is zero at the origin i.e. $f(0, 0) = 0$. All considerations will be restricted to an open set of the form $\mathcal{W} = \mathcal{X} \times \mathcal{U}$ containing the origin $(0, 0)$ of $R^n \times R$. The C^1 function $h : \mathcal{X} \times \mathcal{U} \rightarrow R$, takes values on a subset \mathcal{Y} of the real line, R .

Associated with system (1), we consider a positive definite, C^1 function, addressed as the *storage function*, $V : \mathcal{X} \rightarrow R^+$ whose row gradient, with respect to x , is denoted by $\partial V / \partial x^T$.

We also consider another C^1 function, called the *supply function*, denoted by $s(y, u)$, with $s : \mathcal{Y} \times \mathcal{U} \rightarrow R$. This function satisfies the following properties

$$\begin{aligned}s(0, u) &= 0 \quad \text{for all } u \in \mathcal{U} \\ s(y, 0) &= 0; \quad \text{for all } y \in \mathcal{Y}\end{aligned}\quad (2)$$

With some abuse of notation, arising from the fact that $f(x, u)$ is not a proper *vector field* for an unspecified control input, we, nevertheless, use Lie derivatives to briefly express time derivatives of scalar functions of the state. Thus, we use

$$\dot{V} = \frac{\partial V}{\partial x^T} f(x, u) = L_{f(x, u)} V \quad (3)$$

Similarly, we write

$$\frac{\partial \dot{V}}{\partial u} = \frac{\partial}{\partial u} \left(\frac{\partial V}{\partial x^T} f(x, u) \right) = \frac{\partial V}{\partial x^T} \left(\frac{\partial f}{\partial u} \right) = L_{\frac{\partial f}{\partial u}} V \quad (4)$$

Definition 2.1 [1] *System (1) is said to be passive (resp. strictly passive) with respect to the storage function and supply function pair $\{V(x), s(y, u)\}$ if there exists a positive semi-definite (respectively, positive definite) function $\phi : \mathcal{X} \rightarrow R^+$, such that, for any $u \in \mathcal{U}$, for any t_0 and any $t_f > t_0$, the following equality is satisfied, irrespectively of the initial value of the state $x(t_0)$,*

$$\begin{aligned}V(x(t_f)) - V(x(t_0)) &= \int_{t_0}^{t_f} [s(y(\sigma), u(\sigma)) - \\ &\quad - \phi(x(\sigma))] d\sigma \quad \forall (x, u) \in \mathcal{W}\end{aligned}\quad (5)$$

The above equality (5) is addressed as the *stored energy balance equation*. Equivalently, we can say that a system is passive if

$$\begin{aligned}V(x(t_f)) - V(x(t_0)) &\leq \int_{t_0}^{t_f} s(y(\sigma), u(\sigma)) d\sigma, \\ &\quad \forall (x, u) \in \mathcal{W}\end{aligned}\quad (6)$$

If the above inequality is a strict inequality the system is *strictly passive*.

The last relation is known as the *fundamental passivity inequality*. It is easy to see that for C^1 energy functions, the energy balance and the passivity inequality adopt the following *infinitesimal* forms,

$$\dot{V}(x) = s(y, u) - \phi(x); \quad \dot{V}(x) \leq s(y, u) \quad (7)$$

Remark 2.2 *It is important to emphasize the need for a general output function which explicitly depends on the input u in a nonlinear fashion, i.e. $y = h(x, u)$. For, if we consider a system of the form $\dot{x} = f(x, u)$ with $y = h(x)$, and, as it is customary in the literature, we adopt $s(y, u) = yu$, as the*

supply function, then, the partial derivative, with respect to u , of the passivity equality (7), would read,

$$\frac{\partial V}{\partial x^T} \left(\frac{\partial f}{\partial u} \right) (x, u) = h(x) \quad (8)$$

which is contradictory.

We assume that the condition

$$L_{\frac{\partial f}{\partial u}} V - \frac{\partial s}{\partial y} \frac{\partial h}{\partial u} \neq 0, \quad \forall (x, u) \in \mathcal{W} \quad (9)$$

is locally satisfied everywhere in \mathcal{W} . This condition is addressed to as the transversality condition.

Example 2.3 For systems affine in the control, of the form $\dot{x} = f(x) + g(x)u$, $y = h(x)$ and $s(y, u) = yu$ the transversality condition reduces to

$$\frac{\partial V}{\partial x^T} g(x) = L_g V(x) \neq 0, \quad \forall x \in \mathcal{X}$$

which has the clear geometric interpretation of having a control vector field g which is locally nowhere tangent to the level sets $\{x \mid V(x) = \text{constant}\}$. For systems of the form $\dot{x} = f(x) + g(x)u$, $y = h(x) + k(x)u$, and $s(y, u) = yu$ the transversality condition adopts the form

$$L_g V(x) - k(x)u \neq 0 \quad \forall (x, u) \in \mathcal{W}$$

Notice that this condition implies that, as before, $L_g V(x) \neq 0$ in \mathcal{X} .

The immediate consequences of passivity are referred to the stability of the system evolving under no control actions applied to the system and the nature of the stability of the zero dynamics associated to the zero value, for an indefinite period of time, of the output function y .

Theorem 2.4 [1] For systems with positive definite storage functions, the trajectories of the passive (respectively strictly passive) uncontrolled system, $\dot{x} = f(x, 0)$, are stable (resp. asymptotically stable) around (resp. towards) the origin. Similarly, if the output y of a passive system is held to be zero, in an indefinite fashion, by means of an appropriate control input, then the zero dynamics is stable (resp. asymptotically stable).

2.2 The Kalman-Yakubovich-Popov conditions

Let a system of the form (1) be passive, then the following two consequences of passivity are usually known as the Kalman-Yakubovich-Popov (KYP) conditions

$$\begin{aligned} L_{f(x,0)} V &\leq 0 \\ L_{\frac{\partial f}{\partial u}} V &= \frac{\partial s}{\partial u} + \frac{\partial s}{\partial y} \frac{\partial h}{\partial u} \end{aligned} \quad (10)$$

The proof of these two relations is as follows. The first one directly follows from the infinitesimal version of the definition of passivity (7) and the properties of the supply function

given in (2). Indeed, using the fact that $\dot{V} = L_{f(x,u)} V \leq s(y, u)$ we let $u = 0$. The second equality of the KYP conditions follows by taking partial derivatives with respect to u in the first equation of (7).

Notice that in the second equality of the KYP conditions one clearly identifies two terms; a term due to the general form of the supply function and a term due to the feed-forward presence of the input in the output equation.

Example 2.5 Consider the case in which $f(x, u) = f(x) + g(x)u$ with $y = h(x)$ and $s(y, u) = yu$, then (10) takes the form,

$$L_f V(x) \leq 0, \quad L_g V(x) = h(x) = y \quad (11)$$

which is a well known form of the KYP conditions (see [1]). For systems affine in the control form, with $y = h(x) + k(x)u$, the KYP conditions are obtained as

$$L_f V(x) \leq 0, \quad L_g V(x) = h(x) + 2k(x)u = y + k(x)u \quad (12)$$

2.3 Feedback Passivity

Let $\gamma : \mathcal{W} \rightarrow \mathcal{U}$ be a C^1 function of its arguments. A nonlinear static state feedback control law is denoted by the expression $u = \gamma(x, v)$ with $v \in \mathcal{U} \subset \mathcal{R}$.

Definition 2.6 We say that a feedback control law $u = \gamma(x, v)$ is locally regular if for all $(x, v) \in \mathcal{W}$ it follows that $\partial \gamma / \partial v \neq 0$.

Regularity is regarded as a highly convenient property of static state feedback control laws since they lead to locally invertible input coordinate transformations. By the closed loop system we mean the system $\dot{x} = f(x, \gamma(x, v))$, which we may also denote by $\dot{x} = \bar{f}(x, v)$. We also denote by, $\bar{h}(x, v)$, the function: $h(x, \gamma(x, v))$.

Definition 2.7 Consider a system of the form (1) with an associated set of scalar functions $\{V(x), s(y, u)\}$ as defined above. The system is said to be feedback passive, or it is said to be rendered passive by means of static state feedback, if there exists a regular static state feedback control law of the form, $u = \gamma(x, v)$, such that the closed loop system is passive with respect to the pair $\{V, s(y, v)\}$. In other words, there exists a positive semi-definite function $\phi(x)$ such that

$$L_{\bar{f}(x, v)} V = s(y, v) - \phi(x) \quad (13)$$

with $y = h(x, \gamma(x, v))$.

Similarly, if $\phi(x)$ is a strictly positive definite function, then the system is said to be feedback strictly passive.

The existence of a feedback control law, of the form $u = \gamma(x, v)$, for which the system is rendered passive must be assessed from the existence of solutions, for the control input u , of the following algebraic equation,

$$L_{f(x, u)} V = s(h(x, u), v) - \phi(x) \quad (14)$$

The following theorem states the conditions under which feedback passivity is possible.

Theorem 2.8 Let $\phi(x)$ be a given positive semi-definite scalar function in X . Suppose that the following two conditions are satisfied:

1. There exists a pair of state functions, $u = u_0(x)$, and, $v = v_0(x)$, for which the equality (14) holds true, i.e.

$$L_{f(x, u_0(x))} V = s(h(x, u_0(x)), v_0(x)) - \phi(x), \quad \forall x \in X \quad (15)$$

2. the transversality condition,

$$L_{\frac{\partial f}{\partial u}} V - \frac{\partial s}{\partial y} \frac{\partial h}{\partial u} \neq 0 \quad \forall (x, u) \in W \quad (16)$$

holds locally valid in W .

Then, there exists a unique static state feedback control law of the form, $u = \gamma(x, v)$, such that the closed loop system $\dot{x} = \bar{f}(x, v)$, $y = \bar{h}(x, v)$ is feedback passive with respect to the pair $\{V(x), s(y, v)\}$.

Proof

The proof follows directly from the implicit function theorem.

□

Example 2.9 In the case where $\dot{x} = f(x) + g(x)u$ and $y = h(x)$, with $s(y, u) = yu$, the condition (16) reduces to the transversality condition,

$$L_g V(x) \neq 0$$

Under the validity of such a transversality condition, the existence of a feedback control law of the form, $u = \alpha(x) + \beta(x)v$, is guaranteed. This is obtained from

$$L_f V(x) + u L_g V(x) = h(x)v - \phi(x) \quad (17)$$

and, hence

$$u = \alpha(x) + \beta(x)v = -\frac{L_f V(x) + \phi(x)}{L_g V(x)} + \frac{h(x)}{L_g V(x)} v \quad (18)$$

The closed loop system

$$\begin{aligned} \dot{x} &= \left(I - g(x) \frac{\partial V(x)/\partial x^T}{L_g V(x)} \right) f(x) - \\ &- g(x) \frac{\phi(x)}{L_g V(x)} + g(x) \frac{h(x)}{L_g V(x)} v \end{aligned} \quad (19)$$

satisfies,

$$\begin{aligned} \dot{V} &= \frac{\partial V}{\partial x^T} \left[\left(I - g \frac{\partial V/\partial x^T}{L_g V(x)} \right) f(x) - \right. \\ &- \left. g(x) \frac{\phi(x)}{L_g V(x)} + g(x) \frac{h(x)}{L_g V(x)} v \right] \\ &= h(x)v - \phi(x) \end{aligned} \quad (20)$$

2.4 Passivity and Stability

A nonlinear regular static state feedback control law of the form: $u = \gamma(x, v)$, which achieves either passivity or strict passivity by means of state feedback, induces an implicit damping injection which makes the system stable (resp. asymptotically stable) for certain particular values of the transformed control input. The meaning of this assertion is clarified in the following theorems.

Theorem 2.10 Let $\phi(x)$ be a locally positive definite scalar function in X . Suppose there exists a feedback control law, $u = \gamma(x, v)$, which achieves strict passivity of the closed loop system, with respect to the pair $\{V(x), s(y, v)\}$, (where, $V(x) > 0$), then, the control law $u = \gamma(x, 0)$ locally asymptotically stabilizes the system trajectories to zero.

Proof

Indeed, consider the time derivative of $V(x)$,

$$\begin{aligned} \dot{V} &= \frac{\partial V}{\partial x^T} f(x, \gamma(x, v)) = s(h(x, \gamma(x, v)), v) - \\ &- \phi(x) < s(h(x, \gamma(x, v)), v) \end{aligned} \quad (21)$$

If $V(x)$ is positive definite, and use is made of the above feedback control law with $v = 0$, then the resulting closed loop system is asymptotically stable. Indeed, for $v = 0$ we have

$$\begin{aligned} \dot{V} &= \frac{\partial V}{\partial x^T} f(x, \gamma(x, 0)) \\ &= s(h(x, \gamma(x, 0)), 0) - \phi(x) = -\phi(x) < 0 \end{aligned} \quad (22)$$

□

Corollary 2.11 Let the system $\dot{x} = f(x, u)$, $y = h(x, u)$, with the associated functions $\{V(x), s(y, u)\}$ (where $V(x) > 0$), be such that it is rendered passive by means of the regular nonlinear static state feedback control law, $u = \gamma(x, v)$, with v being an arbitrary external input. Then, the control law $u = \gamma(x, 0)$ renders the zero solution $x(t) = 0$ of the system locally stable.

Proof Since the system is feedback passive, then there exists a positive semi-definite function, $\phi(x)$, such that

$$\begin{aligned} \dot{V} &= \frac{\partial V}{\partial x^T} f(x, \gamma(x, v)) = s(h(x, \gamma(x, v)), v) - \phi(x), \\ &\forall (x, v) \in W \end{aligned} \quad (23)$$

In particular, for $v = 0$ we have

$$\dot{V}(x) = \frac{\partial V}{\partial x^T} f(x, \gamma(x, 0)) = -\phi(x) \leq 0 \quad (24)$$

The result of the theorem follows.

□

3 The Energy Shaping plus Damping Injection Method

Consider the nonlinear system,

$$\begin{aligned}\dot{x} &= f(x, u), \quad x \in R^n, \quad u \in R \\ y &= h(x, u), \quad y \in R\end{aligned}\quad (25)$$

Associated with the system (25), we hypothesize a positive definite C^1 storage function $V : \mathcal{X} \rightarrow R^+$. We then have the following theorem.

Theorem 3.1 *Let the system (25) be a nonlinear system which may be rendered passive, with respect to the pair $\{V(x), s(y, u)\}$ with $V(x) > 0$, by means of a regular nonlinear static state feedback control law of the form $u = \gamma(x, v)$. i.e. let $\gamma(x, v) = \bar{u}$, be the regular static state feedback control representing a state dependent solution, parameterized by v , of the equality*

$$\frac{\partial V}{\partial x^T} f(x, \bar{u}) = s(h(x, \bar{u}), v) - \phi(x) \quad (26)$$

Then, the tracking error vector $e = x - \xi$, with ξ defined as

$$\begin{aligned}\dot{\xi} &= f(x, u) - f(x - \xi, \gamma(x - \xi, 0)) + \\ &+ R(x - \xi) \left[\frac{\partial V(e)}{\partial e} \Big|_{e=x-\xi} \right]\end{aligned}\quad (27)$$

with $R(e)$ being an $n \times n$ positive semidefinite matrix, such that

$$\frac{\partial V(e)}{\partial e^T} R(e) \frac{\partial V(e)}{\partial e} + \phi(e) > 0 \quad (28)$$

is asymptotically stable to zero. Moreover, the asymptotic stability result holds even if $R(e)$ is such that,

$$\frac{\partial V(e)}{\partial e^T} R(e) \frac{\partial V(e)}{\partial e} + \phi(e) \geq 0 \quad (29)$$

and

$$\begin{aligned}\{e \mid \frac{\partial V(e)}{\partial e^T} f(e, \gamma(e, 0)) = 0\} \cap \\ \{e \mid \frac{\partial V(e)}{\partial e^T} R(e) \frac{\partial V(e)}{\partial e} = 0\} = \{0\}\end{aligned}\quad (30)$$

Proof

Consider the modified stored energy function $V(x - \xi)$. The time derivative of the modified energy function is given by

$$\dot{V}(x - \xi) = \frac{\partial V(e)}{\partial e^T} \Big|_{e=x-\xi} (f(x, u) - \dot{\xi}) \quad (31)$$

According to the dynamics assigned to the auxiliary variable ξ , it follows that the time derivative of the modified energy

function $V(e)$ is given by

$$\begin{aligned}\dot{V}(e) &= \partial V(e) / \partial e^T \left[f(e, \gamma(e, 0)) - R(e) \frac{\partial V(e)}{\partial e} \right] \\ &= \frac{\partial V(e)}{\partial e^T} f(e, \gamma(e, 0)) - \frac{\partial V(e)}{\partial e^T} R(e) \frac{\partial V(e)}{\partial e} \\ &\leq - \left[\phi(e) + \frac{\partial V(e)}{\partial e^T} R(e) \frac{\partial V(e)}{\partial e} \right] < 0\end{aligned}\quad (32)$$

The main result of the theorem follows from fundamental results of Lyapunov stability theory.

To prove the second part of the theorem, consider the set,

$$\begin{aligned}\{e \in R^n \mid \dot{V}(e) = \frac{\partial V(e)}{\partial e^T} [f(e, \gamma(e, 0)) - \\ - R(e) \frac{\partial V(e)}{\partial e^T}] = 0\}\end{aligned}\quad (33)$$

This set clearly coincides with the set,

$$\{e \in R^n \mid \frac{\partial V(e)}{\partial e^T} f(e, \gamma(e, 0)) = \frac{\partial V(e)}{\partial e^T} R(e) \frac{\partial V(e)}{\partial e}\}$$

The left hand side of the last equality, according to the Corollary 2.11, is a negative semi-definite function, while, from the definition of $R(e)$, the right hand side is a positive semi-definite function of e . The equality can only be valid over the set of values of e where both expressions are zero. This implies that the set is given by

$$\begin{aligned}\{e \in R^n \mid \frac{\partial V(e)}{\partial e^T} f(e, \gamma(e, 0)) = 0\} \cap \\ \{e \in R^n \mid \frac{\partial V(e)}{\partial e^T} R(e) \frac{\partial V(e)}{\partial e} = 0\}\end{aligned}\quad (34)$$

It follows, according to the assumption in the theorem, that the invariance set, $\{e \mid \dot{V}(e) = 0\}$, is constituted just by the singleton represented by $\{e \in R^n \mid e = 0\}$. The asymptotic stability to zero of the trajectories of the tracking error, $e(t) = x(t) - \xi(t)$, follows as a consequence of LaSalle's invariance theorem. \square

4 A Canonical Form for Passive Systems

In this section we show that, under very mild conditions, nonlinear systems enjoy a canonical form which further generalizes that of systems in Generalized Hamiltonian form treated extensively in [2].

Theorem 4.1 *Let the following condition be satisfied over all (x, u) in the open set \mathcal{W} ,*

$$L_{\frac{\partial V}{\partial x}} V \neq 0 \quad (35)$$

Then, the system $\dot{x} = f(x, u)$, $y = h(x, u)$ can be written in the form,

$$\dot{x} = [J(x, u) + S(x, u)] \frac{\partial V}{\partial x}, \quad y = h(x, u) \quad (36)$$

with

$$J(x, u) + J^T(x, u) = 0; \quad S(x, u) = S^T(x, u), \\ \forall (x, v) \in \mathcal{W}$$

Proof

Consider the following string of equalities

$$\begin{aligned} \dot{x} &= f(x, u) \\ &= f(x, u) + \frac{\partial f(x, u)}{\partial u} \left[\frac{\partial V / \partial x^T}{L \frac{\partial f}{\partial u} V} f(x, u) \right] - \\ &\quad - \frac{\partial f(x, u)}{\partial u} \left[\frac{\partial V / \partial x^T}{L \frac{\partial f}{\partial u} V} f(x, u) \right] = \\ &= \left(I - \frac{\partial f(x, u)}{\partial u} \left[\frac{\partial V / \partial x^T}{L \frac{\partial f}{\partial u} V} \right] \right) f(x, u) + \\ &\quad + \frac{\partial f(x, u)}{\partial u} \left[\frac{\partial V / \partial x^T}{L \frac{\partial f}{\partial u} V} f(x, u) \right] = \\ &= \frac{1}{L \frac{\partial f}{\partial u} V} [f(x, u) \frac{\partial f^T(x, u)}{\partial u} - \\ &\quad - \frac{\partial f(x, u)}{\partial u} f^T(x, u)] \frac{\partial V}{\partial x} + \\ &\quad + \frac{\partial f(x, u)}{\partial u} \left[\frac{\partial V / \partial x^T}{L \frac{\partial f}{\partial u} V} f(x, u) \right] \end{aligned} \quad (37)$$

On the other hand, the term

$$\begin{aligned} &\frac{\partial f(x, u)}{\partial u} \left[\frac{\partial V / \partial x^T}{L \frac{\partial f}{\partial u} V} f(x, u) \right] = \\ &= \frac{1}{L \frac{\partial f}{\partial u} V} \left[\frac{\partial f(x, u)}{\partial u} f^T(x, u) \right] \frac{\partial V}{\partial x} \end{aligned} \quad (38)$$

can be decomposed as the sum of two further terms; one containing a symmetric matrix and a second one having a skew-symmetric matrix.

$$\begin{aligned} &\frac{1}{L \frac{\partial f}{\partial u} V} \left[\frac{\partial f(x, u)}{\partial u} f^T(x, u) \right] \frac{\partial V}{\partial x} = \\ &= \frac{1}{2L \frac{\partial f}{\partial u} V} \left\{ \left[\frac{\partial f(x, u)}{\partial u} f^T(x, u) + \right. \right. \\ &\quad + f(x, u) \frac{\partial f^T(x, u)}{\partial u} \left. \right] + \\ &\quad + \left[\frac{\partial f(x, u)}{\partial u} f^T(x, u) - \right. \\ &\quad \left. \left. - f(x, u) \frac{\partial f^T(x, u)}{\partial u} \right] \right\} \frac{\partial V}{\partial x} \end{aligned} \quad (39)$$

Combining this last expression with those in (37), one obtains,

$$\dot{x} = \frac{1}{2L \frac{\partial f}{\partial u} V} [f(x, u) \frac{\partial f^T(x, u)}{\partial u} -$$

$$\begin{aligned} &- \frac{\partial f(x, u)}{\partial u} f^T(x, u)] \frac{\partial V}{\partial x} + \\ &+ \frac{1}{2L \frac{\partial f}{\partial u} V} [f(x, u) \frac{\partial f^T(x, u)}{\partial u} + \\ &+ \frac{\partial f(x, u)}{\partial u} f^T(x, u)] \frac{\partial V}{\partial x} \\ &= [J(x, u) + S(x, u)] \frac{\partial V}{\partial x} \end{aligned} \quad (40)$$

therefore, one has

$$\begin{aligned} J(x, u) &= \frac{1}{2L \frac{\partial f}{\partial u} V} [f(x, u) \frac{\partial f^T(x, u)}{\partial u} - \\ &- \frac{\partial f(x, u)}{\partial u} f^T(x, u)] \\ S(x, u) &= \frac{1}{2L \frac{\partial f}{\partial u} V} [f(x, u) \frac{\partial f^T(x, u)}{\partial u} + \\ &+ \frac{\partial f(x, u)}{\partial u} f^T(x, u)] \end{aligned} \quad (41)$$

□

According to the last theorem, any nonlinear system for which the condition (35) is satisfied, can be written in the above canonical form. If the system is feedback passive (respectively, feedback strictly passive), one can more specifically characterize the resulting closed loop matrix, $S(x, \gamma(x, v)) = \bar{S}(x, v)$, as a symmetric negative semidefinite (respectively negative definite) matrix. This is the topic of the next theorem where we also denote $J(x, \gamma(x, v))$ as $\bar{J}(x, v)$.

Theorem 4.2 Let $\dot{x} = \bar{J}(x, v)$, $y = \bar{h}(x, v)$ be a passive (resp. strictly passive) system with respect to the pair $\{V(x), s(y, v)\}$. Suppose that the condition

$$L \frac{\partial f}{\partial u} V \neq 0 \quad (42)$$

Then, there exists a neighborhood $\bar{\mathcal{W}} \subset \mathcal{W}$ where the closed loop system can be written as

$$\dot{x} = [\bar{J}(x, v) + \bar{S}(x, v)] \frac{\partial V}{\partial x} \quad (43)$$

with

$$\begin{aligned} &\bar{J}(x, v) + \bar{J}^T(x, v) = 0, \\ &\bar{S}(x, v) = \bar{S}^T(x, v) \leq 0, \quad (\text{resp. } < 0) \end{aligned} \quad (44)$$

Proof

The fact that the closed loop system can be written in the Generalized Hamiltonian form follows from the assumption (42) and from the result of the previous theorem 4.1. To prove the fact that $\bar{S}(x, v)$ is negative semidefinite or negative definite, consider the time derivative of the storage function $V(x)$, and assume, for simplicity, that the closed loop

system is passive (the argument is the same when the closed loop system is strictly passive),

$$\begin{aligned}\dot{V} &= \frac{\partial V}{\partial x^T} \bar{S}(x, v) \frac{\partial V}{\partial x} = \frac{\partial V}{\partial x^T} \bar{f}(x, v) = \\ &= s(h(x, \gamma(x, v)), v) - \phi(x)\end{aligned}\quad (45)$$

Letting $v = 0$, one has

$$\frac{\partial V}{\partial x^T} \bar{S}(x, 0) \frac{\partial V}{\partial x} = -\phi(x) \leq 0 \quad \forall x \in \mathcal{X} \quad (46)$$

Therefore, there exists a neighborhood \bar{U} of the origin in \mathcal{U} , and an open subset, $\bar{\mathcal{X}}$ of \mathcal{X} , i.e. an open neighborhood \bar{W} in the product set \mathcal{W} , with $\bar{U} \times \bar{\mathcal{X}} = \bar{W} \subset \mathcal{W}$, where the following assertion is valid,

$$\frac{\partial V}{\partial x^T} \bar{S}(x, v) \frac{\partial V}{\partial x} \leq 0 \quad \forall (x, v) \in \bar{W} \quad (47)$$

□

Example 4.3 For affine systems of the form $\dot{x} = f(x) + g(x)u$, $y = h(x)$, where $L_g V \neq 0$, the canonical form simply reads as $\dot{x} = (J(x) + S(x))\partial H/\partial x + g(x)u$, with $J + J^T = 0$ and $S = S^T$. The local input coordinate transformation, $u = (h(x)/L_g V)v$, renders the output equation $y = \bar{g}^T(x)\partial H/\partial x$, with $\bar{g}(x) = g(x)(h(x)/L_g V)$, which is clearly a Generalized Hamiltonian system (including dissipation and, possibly, de-stabilizing terms).

5 Conclusions

In this article we have extended the notions of passivity to general single input single output nonlinear systems. The results completely generalize well established results for the cases of systems which are affine in the control and for those systems which have control input feed-forward terms in the output expression. The Kalman-Yakubovich-Popov conditions are also suitably generalized. The energy shaping plus damping injection controller design methodology has been extended to general nonlinear systems. Also, under mild conditions, a general canonical form, of the Generalized Hamiltonian form, has been clearly found indicating the conservative, the dissipation, the de-stabilizing and the external energy acquisition term. This canonical form has been shown to be valid for any nonlinear system (see also Sira-Ramírez [26]) and, in particular, it points to an interesting "energy managing structure", which drives the state velocity in terms of contributions of stored energy gradient projections. The canonical form conforms to the characterization of passive and strictly passive systems and it should be quite helpful in nonlinear feedback controller design tasks.

The obtained results have an important bearing on the general case of discrete-time systems where an affine in the control form of the system dynamics is rather questionable. The results here obtained can be extended, with some technical difficulties, to the case of general multivariable nonlinear systems.

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