On the Control of the "Ball and Beam" System: A Trajectory Planning Approach [1]

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Abstract

An approximate linearization, in combination with a suitable off-line trajectory planning, is proposed for the equilibrium to equilibrium regulation of the popular "ball and beam" system. The results are illustrated by means of digital computer simulations.

Trajectory planning, Approximate Linearization

1 Introduction

The control of underactuated systems has received sustained attention in the past. Theoretical developments as well as numerous design examples have been produced in the last few years, which point to resolve the main difficulties in controlling this class of systems. For the state of the art, the reader is referred to a recently gathered special issue of the *International Journal of Nonlinear and Robust Control*.

The " ball and the beam" system has been the object of numerous research articles. Important developments from the viewpoint of approximate feedback linearization were given by Hauser *et al* in [1], where the ill-defined nature of the relative degree of the ball position was discussed. A constructive approach based on Lyapunov stability theory was developed in the book by Sepulchre *et al* [2]. The system was also recognized to be non-differentially flat in the work of Fliess and his coworkers, [3], where an interesting procedure was proposed to approximate the system by means of a *flat* system, using a high frequency control and averaging approach (see also [4]).

In this article, from the perspective of trajectory planning, we undertake the problem of feedback controller design for the "ball and beam" system. A feedback regulation scheme is proposed for a rest-to-rest equilibrium maneuver, which does not pass through the (singular) origin of generalized coordinates. The system admits an *exact* trajectory planning for the beam angular's motions in terms of the desired nominal displacement of the ball along the beam. However, the differential equation which allows the angular displacement exact trajectory planning, exhibits a discontinuous right hand side. This fact does not allow further use of this trajectory planning option since, in this case, second order derivatives of the nominal angular motions are required for obtaining the nominal input reference trajectory. By approximating the system to a differentially flat system, an approximate nominal state and control input trajectories can then be off-line computed. We show that the obtained approximation is remarkably close to the exactly computed one, as far as the beam angular motions are concerned. The feedback scheme is completed by using an incremental time-varying linear feedback controller, obtained by recently developed algebraic techniques in an article by Fliess and Rudolph [5]. The proposed controller is shown to sustain substantial initial state perturbations. An output feedback controller, based on a time-varying state observer, is also designed, and its performance tested by means of computer simulations.

2 The ball and the beam system and some of its properties

Consider the ball and the beam system shown in Figure 1. Adopting as generalized coordinates, $q = [r, \theta]^T$, the kinetic co-energy function is then given by

$$\mathcal{T}(r, \dot{r}, \dot{\theta}) = \frac{1}{2}\left[(J + J_B + mr^2)\dot{\theta}^2 + (m + \frac{J_B}{R^2})\dot{r}^2\right]$$

where J is the beam's moment of inertia around the rotating pivot, J_B is the ball's moment of inertia with

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respect to its center, R is the ball radius and m is its mass.

On the other hand, the potential energy of the system is given by $V(q) = mgr\sin\theta$.

The Lagrangian of the system is therefore expressed as

$$\mathcal{L}(r,\dot{r},\theta,\dot{\theta}) = \frac{1}{2}\left[(J+J_B+mr^2)\dot{\theta}^2 + (m+\frac{J_B}{R^2})\dot{r}^2\right] - mgr\sin\theta \quad (1)$$

Applying the Euler-Lagrange equations it is readily found that the mathematical model for the "ball and beam" system is given by,

$$\begin{bmatrix} m+\frac{J_B}{R^2} & 0 \\ 0 & mr^2+J+J_B \end{bmatrix}\begin{bmatrix} \ddot{r} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} 0 & -mr\dot{\theta} \\ mr\dot{\theta} & mr\dot{r} \end{bmatrix}\begin{bmatrix} \dot{r} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} mg\sin\theta \\ mgr\cos\theta \end{bmatrix} = \begin{bmatrix} 0 \\ \tau \end{bmatrix} \quad (2)$$

It is easily verified that the following two crucial properties are valid.

Property 1.

The matrix $\overline{D}(q,\dot{q}) = \dot{D}(q) - 2C(q,\dot{q})$ is skew-symmetric.

Property 2.

The map

$$\Sigma : \begin{bmatrix} 0 \\ \tau \end{bmatrix} \mapsto \begin{bmatrix} \dot{r} \\ \dot{\theta} \end{bmatrix}$$

is passive.

Proof

Consider the total energy function $H(q,\dot{q}) = \frac{1}{2}\dot{q}^T D(q)\dot{q} + V(q)$. The time derivative of this function, along the trajectories of the system, is given by

$$\begin{aligned} \dot{H}(q,\dot{q}) &= \dot{q}^T D(q)\ddot{q} + \frac{1}{2}\dot{q}^T\dot{D}(q)\dot{q} + \dot{q}^T\frac{\partial V}{\partial q} \\ &= \frac{1}{2}\dot{q}^T\left[\dot{D}(q) - 2C(q,\dot{q})\right]\dot{q} - \dot{q}^T\frac{\partial V}{\partial q} \\ &\quad + \dot{q}^T\frac{\partial V}{\partial q} + \dot{q}^T Q \\ &= \frac{1}{2}\dot{q}^T\,\overline{D}\,(q,\dot{q})\dot{q} + \dot{q}^T Q \\ &= \dot{q}^T Q \end{aligned}$$

Thus, the model is of the general form: $\mathcal{D}\ddot{q} + \mathcal{C}(q,\dot{q})\dot{q} + \mathcal{K}(q) = \mathcal{F}$, with

$$\begin{aligned} \mathcal{D}(q) &= \begin{bmatrix} \frac{J_B}{R^2}+m & 0 \\ 0 & mr^2+J+J_B \end{bmatrix}, \\ \mathcal{C}(q,\dot{q}) &= \begin{bmatrix} 0 & -mr\dot{\theta} \\ mr\dot{\theta} & mr\dot{r} \end{bmatrix}, \\ \mathcal{K}(q) &= \begin{bmatrix} mg\sin\theta \\ mgr\cos\theta \end{bmatrix}, \quad \mathcal{F} = \begin{bmatrix} 0 \\ \tau \end{bmatrix} \end{aligned}$$

The ball position variable on the beam, r, has been shown to have an ill defined relative degree ("in fact, between 3 and 4") (see [1]). The system is, therefore, non feedback linearizable by means of static state feedback. Hence, it is also not linearizable by means of dynamic state feedback. The equilibrium manifold of the system corresponds to a perfectly horizontal position of the beam $\theta = 0$, with no translational motions of the ball, which can be located at any distance from the origin, $r = constant$.

3 Off-line Trajectory Planning

The lack of feedback linearizability of the system precludes a *direct* trajectory planning approach for the stablization of the system motions about the equilibrium manifold, of which we must, necessarily, exclude the origin.

Suppose it is desired to realize, in a finite time interval $[t_i, t_f]$, an equilibrium to equilibrium transfer maneuver of the sliding ball by inducing a controlled displacement, from a given initial equilibrium position: $(r(t_i), \theta(t_i)) = (r_{init}, 0)$, towards a second equilibrium position, $(r(t_f), \theta(t_f)) = (r_{final}, 0)$, with, say, $r_{final} < r_{initial}$ and such that, $0 \notin [r_{init}, r_{final}]$. Suppose that a nominal trajectory for the required transfer is specified for the position coordinate of the ball, r as,

$$r^*(t) = \begin{cases} r_{init} & \text{for } t < t_i \\ r_{init} + (r_{final} - r_{init})\psi(t,t_i,t_f) & \text{for } t_i \leq t \leq t_f \\ r_{final} & \text{for } t > t_f \end{cases}$$

where the function, $\psi(t,t_i,t_f)$, is a sufficiently smooth polynomial spline satisfying: $\psi(t_i,t_i,t_f) = 0$ and $\psi(t_f,t_i,t_f) = 1$.

3.1 Exact off-line trajectory planning

From the first equation in (2) we obtain the following *implicit* differential equation for the required nominal angular displacement, $\theta^*(t)$, in terms of the nominal longitudinal displacement, $r^*(t)$

$$mr^*(t)[\dot{\theta}^*(t)]^2 = mg\sin(\theta^*(t)) + \left(\frac{J_B}{R^2} + m\right)\ddot{r}^*(t) \quad (3)$$

The following *explicit* differential equation for $\theta^*(t)$ is then readily obtained from (3) by elementary physical, and simple mathematical feasibility, considerations on the system.

$$\begin{aligned}\dot{\theta}^*(t) &= -\left[\sqrt{\left|\frac{mg\sin(\theta^*(t)) + \left(\frac{J_B}{R^2}+m\right)\ddot{r}^*(t)}{mr^*(t)}\right|}\right] \\ &\quad \times \text{sign}\left\{\frac{mg\sin(\theta^*(t)) + \left(\frac{J_B}{R^2}+m\right)\ddot{r}^*(t)}{mr^*(t)}\right\} \end{aligned} \quad (4)$$

where "sign" stands for the *signum* function.

Figure 2 shows a typical simulation of the differential equation (4) with the following parameters: $m = 0.11$ [Kg], R=0.015 [m], L=0.5 [m], $J_B = 10^{-5}$ [N-m/s^2], $J = 10^{-2}$ [N-m/s^2], $r_{init} = 0.8$ [m], $r_{final} = 0.3$ [m], and a polynomial spline "smoothly" interpolating between 0 and 1,

$$\begin{aligned}\psi(t,t_i,t_f) &= \left(\frac{t-t_i}{t_f-t_i}\right)^5 \left[\gamma_1 - \gamma_2\left(\frac{t-t_i}{t_f-t_i}\right) + \cdots \right. \\ &\quad \left. -\gamma_6\left(\frac{t-t_i}{t_f-t_i}\right)^5 \right] \end{aligned} \quad (5)$$

$$\gamma_1 = 252,\ \gamma_2 = 1050,\ \gamma_3 = 1800,\ \gamma_4 = 1575,\ \gamma_5 = 700,$$
$$\gamma_6 = 126$$

and $t_i = 2$ [s], $t_f = 5.5$ [s].

Unfortunately, any traditional controller design strategy based on the previous off-line trajectory planning proves to be unfeasible, due to the lack of differentiability of the planned angular velocity, needed to off-line compute the nominal torque input. However, the exact solution obtained from (4) allows one to partially validate approximate solution schemes, such as the one proposed in the next section.

3.2 Approximate off-line trajectory planning

If we disregard the centripetal force term : $-mr\dot{\theta}^2$, in the first equation of the system (2), an approximate trajectory planning can then be performed in terms of the following resulting relation,

$$\theta_a^*(t) = -\arcsin\left[\frac{1}{g}\left(1 + \frac{J_b}{mR^2}\right)\ddot{r}^*(t)\right] \quad (6)$$

Figure 3 shows that, for a sufficiently slow transfer maneuver which guarantees small centripetal forces, the approximation (6) is rather good, when compared with the exact angular trajectory planning. In the simulation obtaining this figure we used the same parameters as before which guarantee a small centripetal force.

The approximate relation (6), makes of the ball position, r, a flat output which allows us to compute a nominal reference trajectory for the torque input variable. Using the second equation in (2), we obtain

$$\begin{aligned}\tau^*(t) &= mgr^*(t)\cos(\theta_a^*(t)) + 2mr^*(t)\dot{r}^*(t)\dot{\theta}_a^*(t) \\ &\quad +(m[r^*(t)]^2 + J + J_B)\ddot{\theta}_a^*(t) \end{aligned} \quad (7)$$

4 Feedback controller design

4.1 A state feedback controller

Define incremental state and input variables as: $r_{1,\delta} = r - r^*(t)$, $r_{2,\delta} = \dot{r} - \dot{r}^*(t)$, $\theta_{1,\delta} = \theta - \theta_a^*(t)$, $\theta_{2,\delta} = \dot{\theta} - \dot{\theta}_a^*(t)$ and $u_\delta = \tau - \tau^*(t)$.

The linearization of the system around the nominal trajectory $(r^*(t), \dot{r}^*(t), \theta_a^*(t), \dot{\theta}_a^*(t), u^*(t))$ is of the form, $\dot{x}_\delta = A(t)x_\delta + b(t)u_\delta$, with,

$$\begin{aligned}\dot{x}_\delta &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ a_{21}(t) & 0 & a_{23}(t) & a_{24}(t) \\ 0 & 0 & 0 & 1 \\ a_{41}(t) & a_{42}(t) & a_{43}(t) & a_{44}(t) \end{bmatrix} x_\delta \\ &\quad + \begin{bmatrix} 0 \\ 0 \\ 0 \\ b_4(t) \end{bmatrix} u_\delta, \\ x_\delta &= \begin{bmatrix} r_{1,\delta} \\ r_{2,\delta} \\ \theta_{1,\delta} \\ \theta_{2,\delta} \end{bmatrix} \end{aligned} \quad (8)$$

For trajectories not including the origin of coordinates, the linear time-varying system (8) is found to be uniformly controllable according to the rank criterion established by Silverman and Meadows [6]. The rank criterion is tested on the controllability matrix:

$$\mathcal{C} = \left[\ b(t),\ \ (A - \tfrac{d}{dt})b(t),\ \ \cdots,\ \ (A - \tfrac{d}{dt})^3 b(t)\ \right]$$

We propose, following [5], an incremental linear time-varying state feedback controller, for the stabilization of the linearized system, which is of the form,

$$\begin{aligned}u_\delta &= -k^T(t)x_\delta = -k_1(t)r_{1,\delta} - k_2(t)r_{2,\delta} - k_3(t)\theta_{1,\delta} \\ &\quad -k_4(t)\theta_{2,\delta}\end{aligned}$$

with time-varying gains, $\{k_1(t), k_2(t), k_3(t), k_4(t)\}$, chosen so that the instantaneous characteristic polynomial of the closed loop system matrix, computed as: $p(\lambda) = \det[\lambda I - A(t) + b(t)k^T]$, exhibits ***constant*** coefficients which coincide with those of a ***Hurwitz polynomial*** of the form, $p(\lambda) = \lambda^4 + \gamma_4\lambda^3 + \gamma_3\lambda^2 + \gamma_2\lambda + \gamma_1$, whose roots are located sufficiently deep into the left portion of the complex plane. This procedure, used in [5], within the context of linear time-varying Luenberger observer design, is valid -and it is known to guarantee exponential stability of the origin of the closed loop linearized system- thanks to the fact that the coefficients of the open loop linearized system matrices, $A(t), b(t)$, belong to a *Hardy* field. A Hardy field being one in which the largest "comparability class" is constituted by exponential functions of time (see [5] for further details).

The time-varying gains are computed as

$$\begin{aligned}
k_1(t) &= \frac{1}{(a_{23}^2 - a_{21}a_{24}^2)b_4}\left[a_{21}^2(a_{23} - a_{24}\gamma_4)\right. \\
&\quad + a_{23}(a_{21}\gamma_3 + \gamma_1 + a_{41}a_{23}) \\
&\quad \left. - a_{21}a_{24}(a_{41}a_{24} + \gamma_2)\right] \\
k_2(t) &= \frac{1}{(a_{23}^2 - a_{21}a_{24}^2)b_4}\left[a_{23}(a_{21}\gamma_4 + a_{23}a_{42} + \gamma_2)\right. \\
&\quad \left. - a_{21}a_{24}(a_{21} + a_{42}a_{24} + \gamma_3) - \gamma_1 a_{24}\right] \\
k_3(t) &= \frac{1}{(a_{23}^2 - a_{21}a_{24}^2a_{23})b_4}\left[\left(a_{21}(a_{23} - a_{24}\gamma_4)\right.\right. \\
&\quad \left.\left. - a_{24}\gamma_2 + (a_{43} + \gamma_3)a_{23}\right) + a_{24}^2(\gamma_1 - a_{43}a_{21})\right] \\
k_4(t) &= \frac{a_{44} + \gamma_4}{b_4}
\end{aligned}$$

The feedback controller for the nonlinear system is then synthesized as,

$$\tau = \tau^*(t) + u_\delta = \tau^*(t) - k^T(t)(x - x^*(t))$$

Figure 4 shows the performance of the proposed feeback controller based on approximate linearization around the off-line planned trajectory. The constant closed loop poles were all set to be located at the real value -2. As it can be seen, the proposed controller has a good performance even if significant initial deviations are allowed for the ball and the beam positions from the nominal initial unstable equilibrium value of the desired trajectory.

4.2 An output feedback controller

For trajectories not including the origin of coordinates, the linearized system (8) with incremental output variable, $y_\delta = r_{1,\delta} = cx_\delta = [1,0,0,0]x_\delta$, is found to be uniformly observable, according to the rank criterion of Silverman and Meadows [6], conducted on the observability matrix:

$$\mathcal{O} = \left[c^T, (A(t) - \frac{d}{dt})^T c^T, \cdots, ((A(t) - \frac{d}{dt})^T)^3 c^T\right]^T$$

A state observer of the Luenberger type, is given by,

$$\frac{d}{dt}\hat{x}_\delta = A(t)\hat{x}_\delta + b(t)u_\delta + H(t)(y_\delta - \hat{y}_\delta)$$

with $\hat{y}_\delta = \hat{r}_{1,\delta}$ and $H(t)$ being a column vector, with components $h_1(t), ..., h_4(t)$, constituted by time-varying gains. A set of observer gains, can then be computed so that the polynomial: $q(\lambda) = \det[\lambda I - A(t) + H(t)c]$, has constant coefficients coinciding with those of a Hurwitz polynomial of the form: $q(\lambda) = \lambda^4 + \beta_4\lambda^3 + \beta_3\lambda^2 + \beta_2\lambda + \beta_1$. Since, again, the coefficients of the matrices $A(t)$ and, evidently, those of c, belong to a Hardy field, the proposed observer is guaranteed to exponentially asymptotically estimate the state of the incremental system (8).

The observer gains are computed as

$$\begin{aligned}
h_1(t) &= \beta_4 + a_{44} \\
h_2(t) &= (\beta_4 + a_{44})a_{44} + \beta_3 + a_{43} + a_{42}a_{24} + a_{21} \\
h_3(t) &= \frac{m_3 a_{23} - a_{24}m_4}{a_{23}^2 + a_{24}(a_{23}a_{44} - a_{24}a_{43})} \\
h_4(t) &= \frac{m_4 a_{23} + m_3(a_{23}a_{44} - a_{24}a_{43})}{a_{23}^2 + a_{24}(a_{23}a_{44} - a_{24}a_{43})}
\end{aligned}$$

with

$$\begin{aligned}
m_3(t) &= \beta_2 + a_{42}a_{23} - a_{21}a_{44} + a_{41}a_{24} \\
&\quad + (\beta_4 + a_{44})(a_{43} + a_{42}a_{24}) \\
&\quad + a_{44}\left[(\beta_4 + a_{44})a_{44} + \beta_3 + a_{43} + a_{42}a_{24} + a_{21}\right] \\
m_4(t) &= \beta_1 - a_{21}a_{43} + a_{41}a_{23} + (\beta_4 + a_{44})a_{42}a_{23} \\
&\quad + a_{43}\left[(\beta_4 + a_{44})a_{44} + \beta_3 + a_{43} + a_{42}a_{24} + a_{21}\right]
\end{aligned}$$

The estimated state vector, $\hat{x}_\delta$, is then used in the synthesis of the state feedback law derived in the previous section

$$\tau = \tau^*(t) - k^T(t)\hat{x}_\delta$$

Figure 5 shows the simulations results of the closed loop output feedback controlled nonlinear system subject to significant initial deviations from the proposed nominal trajectory and reasonable initial estimation errors.

5 Conclusions

In this article we have provided a state as well as an output feedback regulation schemes for the ball and beam system, which are based on trajectory planning and approximate linearization around a prescribed nominal trajectory computed from a differentially flat approximation of the nonlinear system dynamics. The resulting time-varying linearized system controller and observer designs are carried out by placing the poles of the closed loop system and of the estimation error dynamics in constant locations of the complex plane sufficiently bounded away from the imaginary axis. The regulator and observer design techniques can be used thanks to the fact that the coefficients of the open loop system matrices belong to a Hardy field.

The proposed method can also be extended to deal with more difficult maneuvers including trajectory tracking problems. It can also be partially used in hybrid control schemes, geared to avoid the singularity implied in a transfer maneuver that includes passing through, or resting at, the origin of the system position coordinates.

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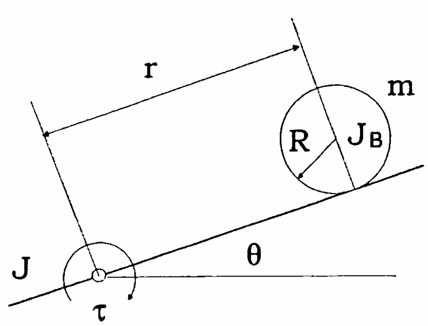

Figure 1: Ball and Beam system.

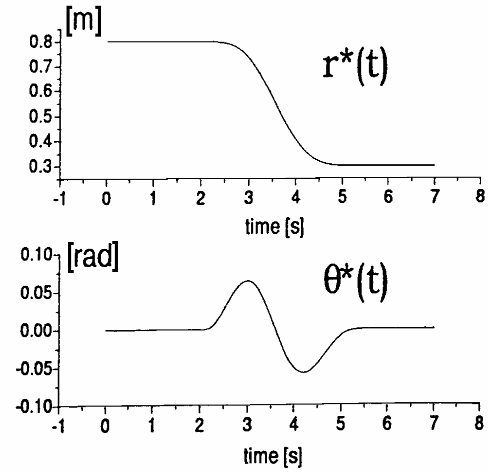

Figure 2: Off line trajectory planning.

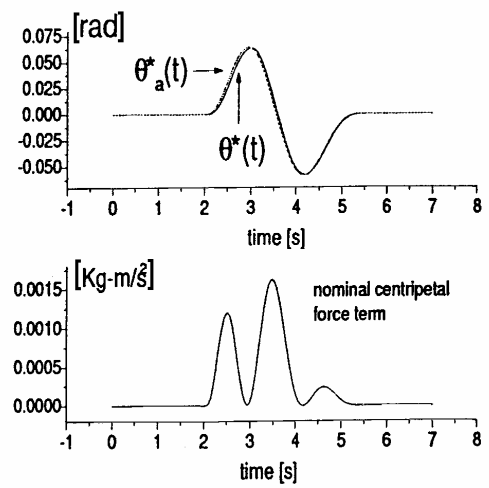

Figure 3: Approximate off-line trajectory planning and centripetal force magnitude.

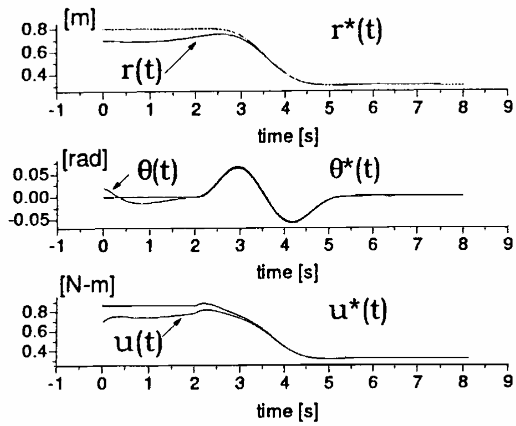

Figure 4: Closed loop performance of feedback controller based on approximate linearzation.

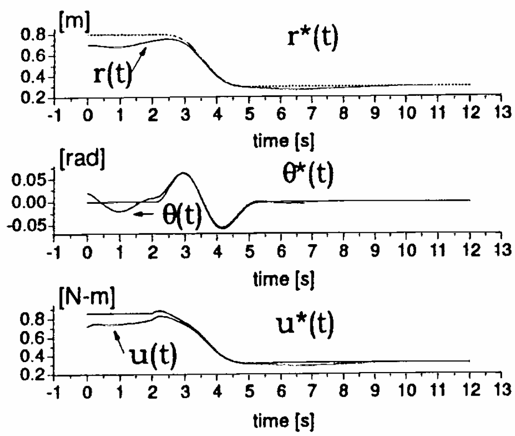

Figure 5: Closed loop performance of output feedback controller based on approximate linearzation.

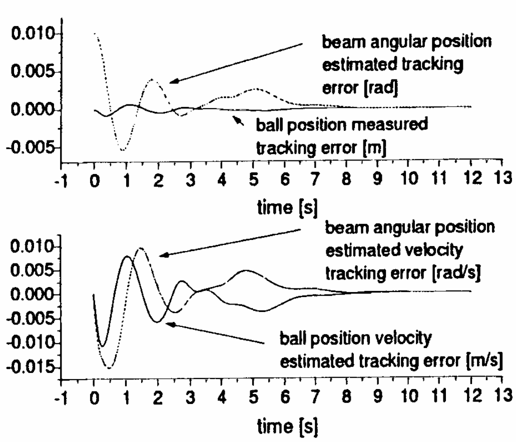

Figure 6: Estimator generated state tracking errors.