Synchronizaton of Chaotic Systems : A Generalized Hamiltonian Systems Approach [1]

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Abstract

A reapproachment to chaotic systems synchronization is presented from the perspective of passivity-based state observer design in the context of Generalized Hamiltonian systems including dissipation and de-stabilizing vector fields. The synchronization and lack of synchronization of several well studied chaotic systems is re-explained in these terms.

Keywords: Synchronization, Chaotic Systems, Passivity based observers

1 Introduction

Synchronization of chaotic systems has received a lot of attention from mathematicians, physicists and control engineers in the last decade. Three special issues of major journals (see [6], [7], [8]) have been devoted to the problem of chaos, in general, and synchronization and control of chaotic systems, in particular. Aside from several edited books on the subject, a staggering collection of references has been collected by Professor G. Chen in [2]. The enormous interest in the topic of synchronization arises from the possibilities of encoding, or masking, messages using as analog "carriers" the chaotic signal generated as a state, or as an output, of a chaotic system, called the "transmitter". The effectively random nature of the carrier signal, additively, or multiplicatively modulated by the masked message signal, makes it, to say the least, "dis-encouraging" to attempt the decoding of the message from the intercepted signal (see the article by Cuomo *et al* [3]).

In this article, we are only concerned with the synchronization issue for chaotic systems, from the perspective of Generalized Hamiltonian systems including non-conservative terms. It turns out that the great majority of chaotic systems can be placed in such a Generalized Hamiltonian canonical form, from where the reconstructibility of the state vector, from a defined output signal, may be assessed from the observability or, in its absence, the detectability of a pair of *constant* matrices. The Generalized Hamiltonian structure of most known chaotic systems allows one to clearly decide on the nature of the synchronizing (output) signal on the basis of the system dissipation and conservative energy managing structure and a need for elimination, at the receiver end, of the locally, or globally, de-stabilizing vector field.

2 Nonlinear observer design for a Class of Systems in Generalized Hamiltonian Form

We consider a special class of Generalized Hamiltonian systems with de-stabilizing vector fields and linear output map , y, given by

$$\begin{aligned} \dot{x} &= \mathcal{J}(y)\frac{\partial H}{\partial x} + (\mathcal{I}+\mathcal{S})\frac{\partial H}{\partial x} + \mathcal{F}(y), \quad x \in R^n \\ y &= \mathcal{C}\frac{\partial H}{\partial x}, \quad y \in R^m \end{aligned} \tag{2.1}$$

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where $\mathcal{S}$ is a constant symmetric matrix, not necessarily of definite sign. The matrix $\mathcal{I}$ is a constant skew symmetric matrix. The vector variable y is referred to as the system *output*. The matrix $\mathcal{C}$ is a constant matrix.

We denote the *estimate* of the state vector x by ξ, and consider the Hamiltonian energy function $H(\xi)$ to be the particularization of H in terms of ξ. Similarly, we denote by η the estimated output, computed in terms of the estimated state ξ. The gradient vector $\partial H(\xi)/\partial\xi$ is, naturally, of the form $\mathcal{M}\xi$ with $\mathcal{M}$ being a, constant, symmetric positive definite matrix.

A dynamic nonlinear state observer for the system (2.1) is readily obtained as

$$\begin{aligned} \dot{\xi} &= \mathcal{J}(y)\frac{\partial H}{\partial \xi} + (\mathcal{I}+\mathcal{S})\,\frac{\partial H}{\partial \xi} + \mathcal{F}(y) + K(y-\eta) \\ \eta &= \mathcal{C}\frac{\partial H}{\partial \xi} \end{aligned} \tag{2.2}$$

where K is a constant vector, known as the *observer gain*.

The state estimation error, defined as $e = x - \xi$ and the output estimation error, defined as $e_y = y - \eta$, are governed by

$$\begin{aligned} \dot{e} &= \mathcal{J}(y)\frac{\partial H}{\partial e} + [\mathcal{I}+\mathcal{S}-\mathcal{K}\mathcal{C}]\,\frac{\partial H}{\partial e}, \quad e \in R^n \\ e_y &= \mathcal{C}\frac{\partial H}{\partial e}, \quad e_y \in R^m \end{aligned} \tag{2.3}$$

where the vector, $\partial H/\partial e$ actually stands, with some abuse of notation, for the gradient vector of the *modified* energy function, $\partial H(e)/\partial e = \partial H/\partial x - \partial H/\partial \xi = \mathcal{M}(x-\xi) = \mathcal{M}e$. Below, we set, when needed, $\mathcal{I}+\mathcal{S} = \mathcal{W}$.

Definition 2.1 *Given a pair of constant matrices $(\mathcal{C},\mathcal{A})$, respectively of dimensions $m \times n$ and $n \times n$. The pair is said to be detectable if the matrix*

$$\begin{bmatrix} \mathcal{C} \\ sI - \mathcal{A} \end{bmatrix} \tag{2.4}$$

has full rank n for all values of s in the open right half of the complex plane. The system is said to be observable if the above matrix is full rank for all values of s in the complex plane.

Theorem 2.2 *The state x of the nonlinear system (2.1) can be globally exponentially asymptotically estimated by the state ξ of an observer of the form (2.2), if the pair of matrices $(\mathcal{C},\mathcal{W})$, or the pair $(\mathcal{C},\mathcal{S})$, is either observable or, at least, detectable.*

An observability condition on the either the pair $(\mathcal{C},\mathcal{W})$, or the pair $(\mathcal{C},\mathcal{S})$, is clearly a ***sufficient*** but not necessary condition for asymptotic state reconstruction. The following simple example readily demonstrates this issue.

Example 2.3 *The pair of matrices*

$$\mathcal{S} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \mathcal{C} = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

constitutes a non-observable, although it is a detectable pair. Nevertheless, setting $\mathcal{K} = 0$ already renders the sum, $2\left[\mathcal{S} - \frac{1}{2}(\mathcal{K}\mathcal{C} + \mathcal{C}^T\mathcal{K}^T)\right] = 2\mathcal{S}$, as a negative definite *matrix.*

Theorem 2.4 *The state x of the nonlinear system (2.1) can be globally exponentially asymptotically estimated, by the state ξ of the observer (2.2) if and only if there exists a constant matrix $\mathcal{K}$ such that the symmetric matrix*

$$\begin{aligned} [\mathcal{W}-\mathcal{K}\mathcal{C}] + [\mathcal{W}-\mathcal{K}\mathcal{C}]^T &= [\mathcal{S}-\mathcal{K}\mathcal{C}] + [\mathcal{S}-\mathcal{K}\mathcal{C}]^T \\ &= 2\left[\mathcal{S} - \frac{1}{2}(\mathcal{K}\mathcal{C} + \mathcal{C}^T\mathcal{K}^T)\right] \end{aligned} \tag{2.5}$$

is negative definite.

3 Applications to Synchronization of Chaotic Circuits

3.1 Chen's Chaotic Attractor

Consider now Chen's chaotic attractor. This system is described by the following set of differential equations

$$\begin{aligned} \dot{x}_1 &= a(x_2 - x_1) \\ \dot{x}_2 &= (c-a)\,x_1 - x_1x_3 + cx_2 \\ \dot{x}_3 &= x_1x_2 - bx_3 \end{aligned} \tag{3.1}$$

Taking as a Hamiltonian energy function the scalar function

$$H(x) = \frac{1}{2}\left[x_1^2 + x_2^2 + x_3^2\right] \tag{3.2}$$

we write the system in Generalized Hamiltonian Canonical form as

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} &= \begin{bmatrix} 0 & a-c/2 & 0 \\ -a+c/2 & 0 & -x_1 \\ 0 & x_1 & 0 \end{bmatrix} \frac{\partial H}{\partial x} \\ &+ \begin{bmatrix} -a & c/2 & 0 \\ c/2 & c & 0 \\ 0 & 0 & -b \end{bmatrix} \frac{\partial H}{\partial x} \end{aligned} \tag{3.3}$$

Choosing the output as $y = x_1$ one obtains,

$$\mathcal{C} = [1 \ 0 \ 0], \quad \mathcal{S} = \begin{bmatrix} -a & c/2 & 0 \\ c/2 & c & 0 \\ 0 & 0 & -b \end{bmatrix},$$
$$\mathcal{I} = \begin{bmatrix} 0 & a - c/2 & 0 \\ -a + c/2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{3.4}$$

The pair of matrices $(\mathcal{C}, \mathcal{S})$ already constitute a detectable, but not observable, pair. The addition to $\mathcal{S}$ of the matrix $\mathcal{I}$ does not improve the lack of observability. In this case, clearly, the unstable nature of the observable eigenvalues of $\mathcal{S}$ requires the introduction of damping through the output error injection map and proceed to place the eigenvalues of the observable part of the dissipative structure of the reconstruction error in suitable (asymptotically) stable locations in the complex plane. This results in the receiver,

$$\begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \dot{\xi}_3 \end{bmatrix} = \begin{bmatrix} 0 & a - c/2 & 0 \\ -a + c/2 & 0 & -x_1 \\ 0 & x_1 & 0 \end{bmatrix} \frac{\partial H}{\partial \xi} + \begin{bmatrix} -a & c/2 & 0 \\ c/2 & c & 0 \\ 0 & 0 & -b \end{bmatrix} \frac{\partial H}{\partial \xi} + \begin{bmatrix} K_1 \\ K_2 \\ K_3 \end{bmatrix} (x_1 - \xi_1)$$

The synchronization error, corresponding to this receiver, is found to be

$$\begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \end{bmatrix} = \begin{bmatrix} 0 & a - (c - K_2)/2 & K_3/2 \\ -a + (c - K_2)/2 & 0 & -x_1 \\ -K_3/2 & x_1 & 0 \end{bmatrix} \frac{\partial H}{\partial e} + \begin{bmatrix} -K_1 - a & (c - K_2)/2 & -K_3/2 \\ (c - K_2)/2 & c & 0 \\ -K_3/2 & 0 & -b \end{bmatrix} \frac{\partial H}{\partial e} \tag{3.5}$$

We may now prescribe K_1, K_2 and K_3 in order to ensure asymptotic stability to zero of the synchronization error. This is achieved by setting $K_1 > c - a$, $K_2 > c/2 + 2(a + K_1)$. We may set $K_3 = 0$ since it has no influence on the observable eigenvalues of the non-conservative structure of the system.

Figure 1 shows the performance of the designed receiver with the following parameter values for the system and for the constant gains.

$$a = 35, \ b = 3, \ c = 28, \ K_1 = 2, \ K_2 = 100, \ K_3 = 0$$

3.2 The hysteretic circuit

Consider the following nonlinear circuit equations treated by Carroll and Pecora in [1]

$$\begin{aligned} \dot{x}_1 &= x_2 + \gamma x_1 + c x_3 \\ \dot{x}_2 &= -\omega x_1 - \delta x_2 \\ \epsilon \dot{x}_3 &= (1 - x_3^2)(s x_1 + x_3) - \beta x_3 \end{aligned} \tag{3.6}$$

The system can be written in Generalized Hamiltonian canonical form with the energy function given by

$$H(x) = \frac{1}{2}\left[x_1^2 + x_2^2 + \epsilon x_3^2\right] \tag{3.7}$$

Indeed,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2}(1+\omega) & \frac{1}{2\epsilon}(c - s) \\ -\frac{1}{2}(1+\omega) & 0 & 0 \\ -\frac{1}{2\epsilon}(c - s) & 0 & 0 \end{bmatrix} \frac{\partial H}{\partial x} + \begin{bmatrix} \gamma & \frac{1}{2}(1-\omega) & \frac{1}{2\epsilon}(c + s) \\ \frac{1}{2}(1-\omega) & -\delta & 0 \\ \frac{1}{2\epsilon}(c + s) & 0 & -\frac{1}{\epsilon^2}(\beta - 1) \end{bmatrix} \frac{\partial H}{\partial x} + \begin{bmatrix} 0 \\ 0 \\ -x_3^2(x_3 + s x_1) \end{bmatrix}$$

The destabilizing vector field requires two signals for complete cancellation at the receiver. Namely, the variables, x_1 and x_3. The output is then chosen as the vector $y = [y_1, y_2]^T = [x_1, \epsilon x_3]^T$. The $\mathcal{C}$ and $\mathcal{S}$ matrices are given by

$$\mathcal{C} = \begin{bmatrix} \mathcal{C}_1 \\ \mathcal{C}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$
$$\mathcal{S} = \begin{bmatrix} \gamma & \frac{1}{2}(1-\omega) & \frac{1}{2\epsilon}(c + s) \\ \frac{1}{2}(1-\omega)\epsilon & -\delta & 0 \\ \frac{1}{2\epsilon}(c + s) & 0 & -\frac{1}{\epsilon^2}(\beta - 1) \end{bmatrix}$$

The pair $(\mathcal{C}, \mathcal{S})$ is observable, and hence detectable. In order to achieve chaotic behavior, β is, in general, a small number, and the $\mathcal{S}$ matrix is therefore of indefinite sign. This means that the required receiver needs to add "multivariable" damping, through an output reconstruction error vector injection. However, one can easily avoid the multivariable pole placement problem by observing that the pair of matrices $(\mathcal{C}_1, \mathcal{S})$ is also an observable pair. An injection of the synchronization error $e_1 = x_1 - \xi_1$ suffices to have an asymptotically stable trajectory convergence. The receiver would then be designed, exploiting this last observation, as follows.

$$\begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \dot{\xi}_3 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2}(1+\omega) & \frac{1}{2\epsilon}(c - s) \\ -\frac{1}{2}(1+\omega)\epsilon & 0 & 0 \\ -\frac{1}{2\epsilon}(c - s) & 0 & 0 \end{bmatrix} \frac{\partial H}{\partial \xi}$$

$$+\begin{bmatrix} \gamma & \frac{1}{2}(1-\omega) & \frac{1}{2\epsilon}(c+s) \\ \frac{1}{2}(1-\omega)\epsilon & -\delta & 0 \\ \frac{1}{2\epsilon}(c+s) & 0 & -\frac{1}{\epsilon^2}(\beta-1) \end{bmatrix} \frac{\partial H}{\partial \xi}$$

$$+\begin{bmatrix} 0 \\ 0 \\ -\frac{1}{\epsilon^2}y_2^2(\frac{1}{\epsilon}y_2+sy_1) \end{bmatrix} + \begin{bmatrix} K_1 \\ K_2 \\ K_3 \end{bmatrix} [x_1-\xi_1]$$

Figure 2 shows the performance of the proposed synchronization scheme. The chosen parameters were set, following [5], as

$$\begin{aligned} \gamma &= 0.2,\ c=2,\ \omega=10,\ \delta=0.001,\ s=1.667, \\ & \beta=0.001,\ \epsilon=0.3 \end{aligned} \quad (3.8)$$

with receiver parameter gains: $K_1 = 7.198$, $K_2 = -17.988$ and $K_3 = 13.927$.

3.3 The Mitschke-Flüggen hybrid optical bi-stable chaotic system

In [4] an analog circuit is proposed as a model of an hybrid optical bi-stable system (see Figure 3). The circuit equations are given by

$$\begin{aligned} C\frac{dx_1}{dt} &= \frac{1}{R}\left[-x_1+\nu^2(x_3-\mu)^2\right] \\ L_m\frac{dx_2}{dt} &= -R_m x_2 - x_3 + x_1 \\ C_m\frac{dx_3}{dt} &= x_2 \end{aligned} \quad (3.9)$$

where x_1 is the voltage accross the capacitor C, x_2 is the current through the inductor and x_3 is the voltage in the second capacitor C_m The total stored energy in the system can be taken as the positive definite Hamiltonian function

$$H(x) = \frac{1}{2}\left[Cx_1^2 + L_m x_2^2 + C_m x_3^2\right] \quad (3.10)$$

This leads to the following system in Generalized Hamiltonian canonical form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{2CL_m} & 0 \\ \frac{1}{2CL_m} & 0 & -\frac{1}{L_mC_m} \\ 0 & \frac{1}{L_mC_m} & 0 \end{bmatrix} \frac{\partial H}{\partial x}$$

$$+\begin{bmatrix} -\frac{1}{RC^2} & \frac{1}{2CL_m} & 0 \\ \frac{1}{2CL_m} & -\frac{R_m}{Lm^2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \frac{\partial H}{\partial x} + \begin{bmatrix} \frac{1}{R}\nu^2(x_3-\mu)^2 \\ 0 \\ 0 \end{bmatrix}$$

The de-stabilizing presence of x_3 suggests that the output of the transmitter should be the voltage variable $y = x_3$. This implies that the matrices $\mathcal{C}$, $\mathcal{S}$ and $\mathcal{I}$ are given by

$$\mathcal{C} = [0,\ 0,\ \frac{1}{C_m}],\quad \mathcal{S} = \begin{bmatrix} -\frac{1}{RC^2} & \frac{1}{2CL_m} & 0 \\ \frac{1}{2CL_m} & -\frac{R_m}{Lm^2} & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\mathcal{I} = \begin{bmatrix} 0 & -\frac{1}{2CL_m} & 0 \\ \frac{1}{2CL_m} & 0 & -\frac{1}{L_mC_m} \\ 0 & \frac{1}{L_mC_m} & 0 \end{bmatrix}$$

The pair of matrices $(\mathcal{C}, \mathcal{S})$ is not observable but it is detectable. However, The pair of matrices, $(\mathcal{C}, \mathcal{W})$, with $\mathcal{W}$ given by

$$\mathcal{W} = \begin{bmatrix} -\frac{1}{RC^2} & 0 & 0 \\ \frac{1}{CL_m} & -\frac{R_m}{Lm^2} & -\frac{1}{L_mC_m} \\ 0 & \frac{1}{L_mC_m} & 0 \end{bmatrix} \quad (3.11)$$

is found to be observable. In order to add suitable damping to the synchronization error dynamics an ouput reconstruction error injection is needed. A receiver can then be designed as

$$\begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \dot{\xi}_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{RC^2} & 0 & 0 \\ \frac{1}{CL_m} & -\frac{R_m}{Lm^2} & -\frac{1}{L_mC_m} \\ 0 & \frac{1}{L_mC_m} & 0 \end{bmatrix} \frac{\partial H}{\partial \xi}$$

$$+\begin{bmatrix} \frac{1}{R}\nu^2(y-\mu)^2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} K_1 \\ K_2 \\ K_3 \end{bmatrix} (y-\xi_3)$$

To guarantee asymptotic stability of the error dynamics, it suffices to choose K_1, K_2, K_3 as arbitrary strictly positive constants.

The synchronization error evolves according to

$$\begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{RC^2} & 0 & -K_1 \\ \frac{1}{CL_m} & -\frac{R_m}{Lm^2} & -\left(\frac{1}{L_mC_m}+K_2\right) \\ 0 & \frac{1}{L_mC_m} & -K_3 \end{bmatrix} \frac{\partial H}{\partial e}$$

4 Conclusions

In this article, we have approached the problem of synchronization of chaotic systems from the perspective of Generalized Hamiltonian systems including dissipation and destabilising terms. The approach allows to give a simple design procedure for the receiver system and clarifies the issue of deciding on the nature of the output signal to be transmitted. This may be accomplished on the basis of a simple linear detectability or observability test. Several chaotic systems were analyzed from this new perspective and their possibilities for synchronization were either confirmed, in the case of already obtained positive results, or it was explained in those cases where there is a known lack of synchronization.

References

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