SLIDING MODE TRAJECTORY TRACKING FOR A DISCRETIZED FLEXIBLE JOINT MANIPULATOR *

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In this article, a *difference flatness* approach is used for the rest-to-rest stbilization, based on off-line trajectory planning, of an approximately (Euler) discretized model of a nonlinear, single link, flexible joint manipulator.

1 Introduction

Flexible, joint manipulators have been extensively studied in the past by many researchers. We refer the reader to the recent book chapter by De Luca [1], for historical and technical details of this area. Most of the contributions include continuous-time models. Controller design, in these instances, has been greatly facilitated by the fact that most of treated models are *exactly linearizable* and, hence, *differentially flat.* In spite of being of crucial importance in the experimental implementation of designed nonlinear controllers, no articles, within our knowledge, deal with either exact or, approximately, discretized models of flexible joint manipulators.

Difference flatness for discrete time nonlinear systems is a concept that directly stems from the concept of *differential flatness*, introduced, within the context of continuous nonlinear controlled systems by Prof. Michel Fliess and his colleages in a series of articles (Fliess *et al*, [2]). Flatness allows for a complete *difference parameterization* of all system variables, including the inputs, in terms of a special set of independent variables, called the *flat outputs*, exhibiting the same cardinality as the set of control inputs, which are *difference functions* of the state, i.e. they are functions of the state and of a finite number of advances of the state.

Section 2 presents the discretized model of a single-link flexible joint manipulator and demonstrates its flatness. We proceed to specify a desired trajectory for the flat output entitling a rest-to-rest maneuver covering a large angular displacement, devoid of oscillations. A sliding mode based feedback

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control strategy is then proposed to have the system accurately track the proposed off-line planned trajectory and exhibit a certain degree of robustness with respect to initial and unmodeled perturbations causing temporary deviations from the nominal trajectory. Section 3 presents the simulation results. Section 4 is devoted to the conclusions and suggestions for further research in this field. An appendix collects the basic background results on a new sliding surface nonlinear dynamics paradigm which guarantees finite time reachability of the sliding surface and exhibits robustness with respect to bounded perturbations.

2 The Flexible Joint Robot

Consider the following Euler-discretization model of a flexible joint robot, shown in Figure 1,

$$\begin{aligned} x_{1,k+1} &= x_{1,k} + Tx_{2,k} \\ x_{2,k+1} &= x_{2,k} + \frac{mgLT}{I}\sin x_{1,k} - \frac{K_aT}{I}(x_{1,k} - x_{3,k}) \\ x_{3,k+1} &= x_{3,k} + Tx_{4,k} \\ x_{4,k+1} &= x_{4,k} + \frac{K_aT}{J}(x_{1,k} - x_{3,k}) + \frac{T}{J}u_k \end{aligned} \tag{1}$$

where x_1 is the link angular position, x_2 is the link angular velocity, x_3 is the motor axis angular position and x_4 is the motor axis angular velocity. The control input u represents the motor applied torque. The fixed parameter T is the duration of the sampling interval.

2.1 Difference Flatness of the Flexible joint Manipulator

The system is difference flat, with flat output given by the link angular position x_1. This means, in particular, that all system variables, including the input u, are expressible as difference functions of x_1. The system equations (1) lead to the following *difference parametrization,*

$$\begin{aligned} x_{1,k} &= x_{1,k} \\ x_{2,k} &= \frac{x_{1,k+1} - x_{1,k}}{T} \\ x_{3,k} &= x_{1,k} + \frac{I}{K_a}\left(\frac{x_{1,k+2} - 2x_{1,k+1} + x_{1,k}}{T^2}\right) - \frac{mgL}{K_a}\sin x_{1,k} \end{aligned}$$

$$
\begin{aligned}
x_{4,k} = & \frac{(x_{1,k+1} - x_{1,k})}{T} + \frac{I}{K_a}\left(\frac{x_{1,k+3} - 3x_{1,k+2} + 3x_{1,k+1} - x_{1,k}}{T^3}\right) \\
& - \frac{mgL}{K_a T}(\sin x_{1,k+1} - \sin x_{1,k}) \\
u_k = & (J+I)\left(\frac{x_{1,k+2} - 2x_{1,k+1} + x_{1,k}}{T^2}\right) \\
& + \frac{JI}{K_a}\left(\frac{x_{1,k+4} - 4x_{1,k+3} + 6x_{1,k+2} - 4x_{1,k+1} + x_{1,k}}{T^4}\right) \\
& - \frac{J}{K_a}\frac{mgL}{T^2}\left(\sin x_{1,k+2} - 2\sin x_{1,k+1} + \sin x_{1,k}\right) - mgL \sin x_{1,k} \quad (2)
\end{aligned}
$$

The last equation immediately suggests the following state-dependent input coordinate transformation:

$$
\begin{aligned}
u_k = & (J+I)\left(\frac{x_{1,k+2} - 2x_{1,k+1} + x_{1,k}}{T^2}\right) \\
& + \frac{JI}{K_a}\left(\frac{v_k - 4x_{1,k+3} + 6x_{1,k+2} - 4x_{1,k+1} + x_{1,k}}{T^4}\right) \\
& - \frac{J}{K_a}\frac{mgL}{T^2}\left(\sin x_{1,k+2} - 2\sin x_{1,k+1} + \sin x_{1,k}\right) - mgL \sin x_{1,k} \quad (3)
\end{aligned}
$$

where v_k represents the new, or transformed, input coordinate. Thus, the system is seen to be equivalent, after state feedback and an input coordinate transformation, to the following *linear* system,

$$x_{1,k+4} = v_k \quad (4)$$

2.2 Off-line Trajectory Planning

Suppose it is desired to bring the link angular position variable x_1 from an initial equilibrium value, $\overline{x}_1^{initial}$, at time $t = K_1$, towards a final equilibrium position, $\overline{x}_1^{final}$, at time $t = K_2$, along a prescribed path $x^*_{1,k}$ satisfying the initial and final conditions. We prescribe such a desired trajectory as

$$x^*_{1,k} = \overline{x}_1^{initial} + \left(\overline{x}_1^{final} - \overline{x}_1^{initial}\right)\varphi(k, K_1, K_2) \quad (5)$$

with $\varphi(K_1, K_1, K_2) = 0$ and $\varphi(K_2, K_1, K_2) = 1$.

Specifically, we choose, as an interpolating polynomial, a *Bezier polynomial* in discrete time. The expression

$$\varphi(k, K_1, K_2) = \left(\frac{k - K_1}{K_2 - K_1}\right)^5 \left[r_1 - r_2\left(\frac{k - K_1}{K_2 - K_1}\right) + \cdots - r_6\left(\frac{k - K_1}{K_2 - K_1}\right)^5\right]$$

$$r_1 = 252,\ r_2 = 1050,\ r_3 = 1800,\ r_4 = 1575,\ r_5 = 700,\ r_6 = 126 \tag{6}$$

defines a rather "smooth" interpolation between the initial value of zero, at time $k = K_1$, and the final value of 1, at time $k = K_2$.

Figure 2 shows the shape of the prescribed angular trajectory, $x^*_{1,k}$, for the flat output x_1. In this particular instance, the initial value of the angular position was taken to be $x_{1,K_1} = \pi/2$ and the final value of the angular position was taken to be $x_{1,K_2} = -\pi/2$, with $K_1 = 4$ s and $K_2 = 6$ s.

Simulations were performed to obtain the open loop behavior of the system state variables and control inputs in accordance with the off-line planned trajectory $x^*_{1,k}$, as given by (5), (6). A flexible joint manipulator model with the following parameter values was used for the simulations.

$$m = 0.4 \text{ Kg},\quad g = 9.81 \text{ m/s}^2,\quad L = 0.185 \text{ m},\quad J = 0.002 \text{ N} - \text{ms}^2/\text{rad}$$

$$I = 0.0059\text{N} - \text{ms}^2/\text{rad},\quad K_a = 1.61 \text{ N} - \text{m} - \text{s/rad}$$

The proposed maneuver entitled starting the motions at time $K_1 = 4$, from an equilibrium position located at $x_{1,K_1} = \pi/2$, with no initial velocity, and to perform a rotation of the link, during a time interval of only 2 s, towards a final position given by $x_{1,K_2} = -\pi/2$, arriving at the new position also with zero angular velocity. In order to properly initialize the states of the manipulator, the initial angular position for the motor axis, corresponding to the link equilibrium, was computed from the equilibrium condition,

$$x_{3,K_1} = x^*_{1,K_1} - \frac{mgL}{K_a} \sin x^*_{1,K_1} \tag{7}$$

This value turned out to be $x_{3,K_1} = 1.12$ rad. The final resting equilibrium position for the motor axis, at time $k = K_2$, can be similarly computed. This yields, $x_{3,K_2} = -1.12$ rad.

Figure 3 depicts the nominal (open loop) trajectories of the state variables and the control input variables behavior for the given planned angular position maneuver on the described flexible joint manipulator.

2.3 A sliding mode based feedback controller design

Using the results of the Appendix, a sliding mode controller can be proposed which asymptotically forces the system to track the given desired trajectory, $x^*_{1,k}$, for the flat output. We consider the following sliding surface coordinate function,

$$\sigma_k = \left(x_{1,k+3} - x^*_{1,k+3}\right) + a_3\left(x_{1,k+2} - x^*_{1,k+2}\right) + a_2\left(x_{1,k+1} - x^*_{1,k+1}\right)$$
$$+ a_1\left(x_{1,k} - x^*_{1,k}\right) \tag{8}$$

Let e_k denote the flat output reference tracking error $e_k = x_k - x^*_k$. Then, if the evolution of σ_k is indefinitely constrained to zero, the corresponding zero dynamics is characterized by the following asymptotically stable linear dynamics

$$e_{k+3} + a_3 e_{k+2} + a_2 e_{k+1} + a_1 e_k = 0 \tag{9}$$

Imposing on the evolution of σ_k the nonlinear paradigm dynamics $\sigma_{k+1} = \Gamma(\sigma_k)$, described in the Appendix, one obtains from (4) the prescription of the transformed control input, v_k, as

$$\begin{aligned} v_k = {} & x^*_{1,k+4} - a_3\left(x_{1,k+3} - x^*_{1,k+3}\right) - a_2\left(x_{1,k+2} - x^*_{1,k+2}\right) \\ & - a_1\left(x_{1,k+1} - x^*_{1,k+1}\right) \\ & - \Gamma\Bigg(\left(x_{1,k+3} - x^*_{1,k+3}\right) + a_3\left(x_{1,k+2} - x^*_{1,k+2}\right) \\ & + a_2\left(x_{1,k+1} - x^*_{1,k+1}\right) + a_1\left(x_{1,k} - x^*_{1,k}\right)\Bigg) \end{aligned} \tag{10}$$

where

$$\begin{aligned} x_{1,k+1} &= x_{1,k} + Tx_{2,k} \\ x_{1,k+2} &= \left(1 - \frac{K_a T^2}{I}\right) x_{1,k} + 2Tx_{2,k} + \left(\frac{K_a T^2}{I}\right) x_{3,k} + \left(\frac{mgLT^2}{I}\right) \sin x_{1,k} \\ x_{1,k+3} &= \left(1 - 3\frac{K_a T^2}{I}\right) x_{1,k} + \left(3T - \frac{K_a T^3}{I}\right) x_{2,k} + 3\left(\frac{K_a T^2}{I}\right) x_{3,k} \\ &\quad + \left(\frac{K_a T^3}{I}\right) x_{4,k} + \frac{mgLT^2}{I}\Big(2\sin x_{1,k} + \sin\left(x_{1,k} + Tx_{2,k}\right)\Big) \end{aligned} \tag{11}$$

The complete sliding mode feedback controller is constituted by the expressions in equations (3), (5),(6), (8)-(11) and (A.12)

3 Simulation Results

Numerical simulations were carried out for assessing the closed loop responses of the sliding mode controlled flexible joint manipulator represented by the previously given parameter values. A rest-to-rest trajectory tracking task was considered which takes the link from the initial angular position $x_{1,K_1} = \pi/2$ at time $k = K_1$ towards the final desired position $x_{1,K_2} = -\pi/2$ at time $k = K_2$. The inital conditions for the numerical simulation were taken to be

$$x_{1,K_1} = 1 \text{ rad}, \quad x_{2,K_1} = 0 \text{ rad/s}, \quad x_3 = 0.62059 \text{ rad}, \quad x_4 = 0 \text{ rad/s}$$

which represent a significant initial deviation from the prescribed trajectory.

The sliding mode controller parameters, as defined in the appendix, were set to be

$$A = 0.1, \qquad B = 0.06, \quad K = 0.05$$

The auxiliary function σ was chosen in accordance with the stable characteristic polynomial coefficients given by

$$a_3 = 0.6, \quad a_2 = 0.12, \quad a_1 = 0.008$$

i.e., after the sliding surface coordinate reaches zero, the tracking error signal, $e_k = x^*_{1,k} - x_{1,k}$, evolves according to the asymptotically stable linear dynamics

$$e_{k+3} + 0.6 e_{k+2} + 0.12 e_{k+1} + 0.008 e_k = 0$$

whose characteristic polynomial has all its roots located at the point, $0.2 \pm 0\, j$, located inside the unit circle centered at the origin of the complex plane. The discretization interval was set to be $T = 0.2$ s.

4 Conclusions

In this article, we have examined the relevance of difference flatness in the regulation, via planned trajectory tracking and sliding mode control, of an approximately discretized flexible joint manipulator model. Difference flatness facilitates a systematic procedure for feedback controller synthesis directly from the associated difference parameterization provided by the flatness property. The difference parameterization represents an off line computational asset for trajectory planning linked to the possibilities of complying with state variables and control input trajectory restrictions.

Appendix

In this appendix, we present some generalities about sliding mode control of nonlinear systems. Our developments are based on establishing a nonlinear autonomous dynamic system "paradigm" which exemplifies the sliding surface coordinate behavior. The idea is then to force a particular system output, like the flat output, to mimic the proposed preferred dynamics, with the aid of a suitable feedback control action.

Let K, A, B be three strictly positive numbers with $A > B$. Consider a scalar nonlinear discrete-time dynamical system given by the following set of relations:

$$\sigma_{k+1} = \Gamma(\sigma_k) = \begin{cases} K \text{ sign } \sigma_k & \text{for } |\sigma_k| > A \\ \frac{K}{A-B}\left(|\sigma_k| - B\right) \text{ sign } \sigma_k & \text{for } B < |\sigma_k| < A \\ 0 & \text{for } |\sigma_k| < B \end{cases} \quad \text{(A.12)}$$

where "sign" stands for the *signum* function. We then have the following result

Theorem 4.1 *The trajectories of system (A.12) are globally asymptotically stable to zero in finite time if and only if,*

$$K < A$$

Moreover, σ_k globally converges to zero in just one step (i.e., after $k = 1$), if and only if $K < B$.

Proof

Consider a Lyapunov function candidate given by

$$V(\sigma) = \sigma^2 \quad ; \quad \text{with } V_k = V(\sigma_k) \quad \text{(A.13)}$$

Notice that V_k is strictly positive and it is bounded below by zero. Then, according to the scalar system dynamics (A.12), we have

$$V_{k+1} - V_k = \begin{cases} K^2 - \sigma_k^2 & \text{for } |\sigma_k| > A \\ \frac{K^2}{(A-B)^2}\left(|\sigma_k| - B\right)^2 - \sigma_k^2 & \text{for } B < |\sigma_k| < A \\ 0 - \sigma_k^2 & \text{for } |\sigma_k| < B \end{cases} \quad \text{(A.14)}$$

Suppose that, at some instant k, $|\sigma_k| > A$, then for the Lyapunov function candidate V_k to be strictly decreasing while this condition is valid, it is sufficient that $|K| < |A|$, since, then, $|K| < |A| < |\sigma_k|$ and therefore $V_{k+1} - V_k = K^2 - \sigma_k^2 < K^2 - A^2 < 0$. Under the above conditions, the evolution of σ_k reaches the region $B < |\sigma_k| < A$ in a single step. In this region, the condition $|K| < |A|$ implies that

$$\begin{aligned} V_{k+1} - V_k &= \frac{K^2}{(A-B)^2}\left(|\sigma_k| - B\right)^2 - \sigma_k^2 \\ &= \frac{K^2}{(A-B)^2}\left[\left(|\sigma_k| - B\right)^2 - \frac{(A-B)^2}{K^2}\sigma_k^2\right] \end{aligned}$$

$$< \frac{K^2}{(A-B)^2}\left[(|\sigma_k|-B)^2 - \frac{(A-B)^2}{A^2}\sigma_k^2\right] < 0 \qquad \text{(A.15)}$$

Therefore, in the region $B < |\sigma_k| < A$, the magnitude of $|\sigma|$ monotonically decreases with finite negative steps given by

$$\Delta_k = \sigma_{k+1} - \sigma_k = \left(\frac{K}{A-B} - 1\right)\sigma_k - \frac{K}{A-B}B \qquad \text{(A.16)}$$

As it can be seen from (A.16), in the region $B < |\sigma_k| < A$, each element of the sequenc of negative steps $\{\Delta_k\}$ is found within the interval,

$$\min\{K-A, -B\} < \Delta_k < \max\{K-A, -B\}$$

The magnitude of $|\sigma|$ thus decreases until it eventually satisfies the condition $|\sigma(\mathcal{K})| < B$ at some finite instant $\mathcal{K}$. From the definition of the dynamics it follows that $\sigma_k = 0$ for $k = \mathcal{K}+1, \mathcal{K}+2, \ldots$ and the system is globally asymptotically stable in finite time.

To prove necessity, suppose the system is globally asymptotically stable to zero in finite time. It follows that, for each k, there exists a subsequence of integers $j_k \geq 1$, such that $V(k+j_k) - V_k < 0$. Then there exists a finite $\mathcal{K}$ such that for all $k > \mathcal{K}$ thesliding surface coordinate σ_k becomes zero after reaching the region $|\sigma| < B$. Suppose, contrary to what we want to establish that $|K| > |A|$, then motions starting on the region $|\sigma| > (A^2 + B(K-A))/K$ will never leave this region since $V_{k+1} - V_k \geq 0$ and the magnitude of $|\sigma|$ becomes constant (and equal to K) after the first step. We have a contradiction since the system is not globally asymptotically stable to zero.

It is clear that for any given initial condition, $\sigma(0)$, at $k = 0$, the next value of the surface satisfies $\sigma(1) \leq K$. Therefore, if $K < B$ then $\sigma_k = 0$ for $k \geq 2$.

References

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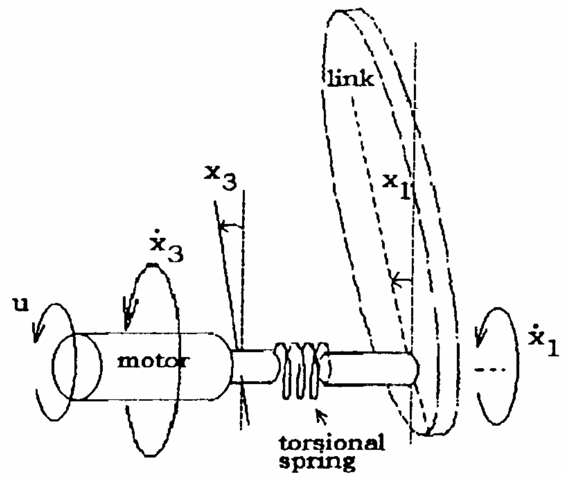

Figure 1. Single link flexible joint manipulator.

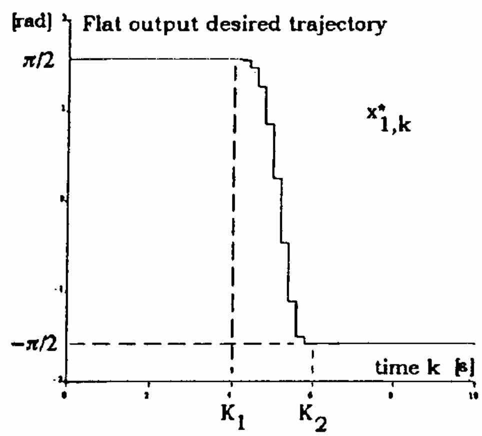

Figure 2. Nominal rest-to-rest maneuver for link angular position (flat output).

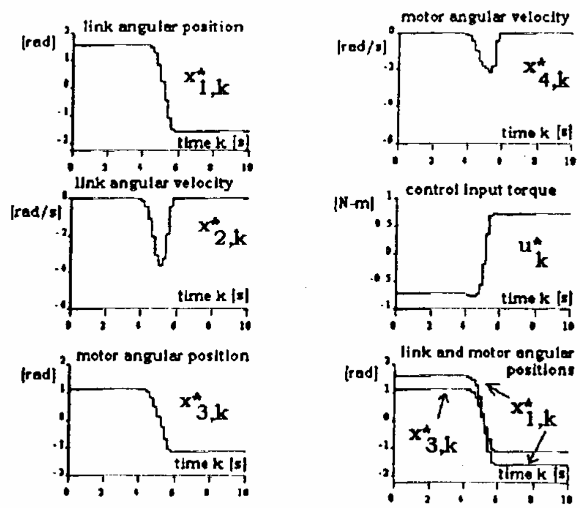

Figure 3. Open loop state and input responses for rest-to-rest angular maneuver.

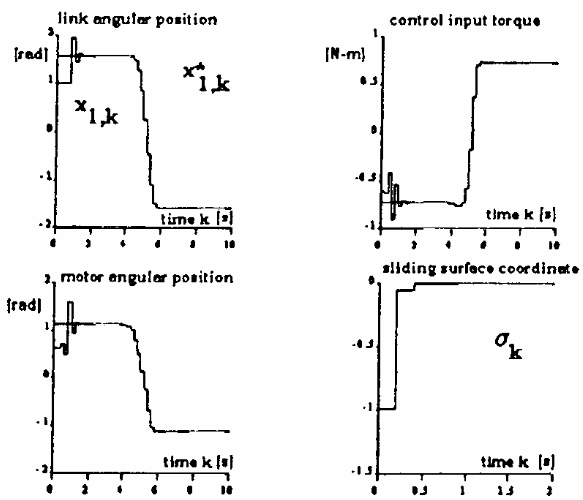

Figure 4. Closed loop sliding mode controlled responses for rest-to-rest angular maneuver.