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Exact Delayed Reconstructors in Nonlinear Discrete-time Systems Control

Hebertt Sira-Ramírez[1] and Pierre Rouchon[2]

Cinvestav-I.P.N[1]
Dept. Ingeniería Eléctrica, Sección Mecatrónica
Avenida IPN, # 2508, Col. San Pedro Zacatenco, A.P. 14740
07300 México, D.F., México.
Centre Automatique et Systèmes[2]
Ècole des Mines de Paris
35 Rue Saint-Honoré
77305 Fontainebleau, Cedex, France

Abstract. In this article we show that the n-dimensional state of an observable discrete-time nonlinear SISO system can always be exactly synthesized by means of a "structural reconstructor" which only requires knowledge of a finite number of delayed inputs and delayed outputs. This fact, when combined with the *difference flatness* of the system, results in an effective systematic feedback control scheme which avoids the need for traditional asymptotic state observers.

1 Introduction

Availability of the state vector in the synthesis of model-based designed feedback control laws is a crucial assumption, or requirement, needed in achieving the desired closed loop behavior of a given dynamic system. The lack of knowledge of the state vector, due to the necessarily limited character of measurements on the system variables, may sometimes be replaced by the use of a complementary dynamic system, called an *asymptotic state observer*, whose state trajectories are guaranteed to converge towards those of the original plant, irrespectively of its arbitrary initial state values. The literature on asymptotic state observers for linear and nonlinear systems, of continuous or discrete-time nature, is vast and certainly out of the scope of this article for a fair review. Nevertheless, for the interested reader, we briefly mention some important contributions made in the past, all in the realm of discrete-time nonlinear systems (DTNLS), which are relevant to the problem of observer design, observability and feedback control of this important class of systems. The work of Grizzle [3], Jakubczyk and Sontag [4] and Monaco and Normad-Cyrot [5] and Fliess [1], all deal with fundamental aspects of the description of DTNLS and the relevance of particular analysis tools. The work of Aranda-Bricaire *et al* [6] devotes special attention to the problem of feedback linearizability of DTNLS, which we also use in the control aspects related to this work. The reader is also referred to the book recently edited by

Nijmeijer and Fossen for a glimpse of the current state of the art in nonlinear systems observer design (see [7]).

In this article, we present an approach, based on exact state reconstructors, to the problem of controlling constructible DTNLS. State reconstructors are based on accurate knowledge of only past inputs and outputs. This fact, which is the outcome of the *difference algebra* approach to the study of observability in nonlinear discrete time systems, was advocated in the work of M. Fliess (see Fliess [2]) thirteen years ago. In that work, exact state reconstruction is recognized as an *algebraic elimination* problem. Somehow, this idea never led, to our knowledge, to the development of particular application examples dealing with control systems design.

We emphasize that the exact state reconstruction approach is fundamentally different from the traditional asymptotic observer approach in the sense that an *exact*, either immediate or "dead-beat", recovery of the true state trajectories is always guaranteed under the assumption of ***constructibility*** of the system, a weaker condition than that of observability. The state reconstruction, in an n dimensional discrete-time nonlinear system, is *exact* from the initial time on provided a finite string, of length $n-1$, of past values of the applied inputs and the corresponding outputs are remembered, or stored. If such a delayed input and output information is not yet available at the initial time, the exact state reconstruction still takes place at the end of the next $n-1$ steps, provided the applied inputs and corresponding outputs are stored from this initial time onwards. The exact reconstruction is then independent of the initialization values arbitrarily assigned to the unavailable past inputs and past outputs in the reconstructor expression. In the latter case, the convergence of the reconstructor is, of course, also independent of any design gains or of a particularly desired asymptotic estimation error dynamics.

Section 2 establishes the main result of the article, which basically proves, through elementary considerations, that an observable nonlinear discrete-time system is also constructible. Section 3 is devoted to present a controller design example for the non-holonomic car model. The system presented is exactly discretized, from the original continuous-time version, and an approximate feedback linearizable version of the system model is adopted for dynamic feedback controller design purposes. The feedback performance results, tested on the full model, are illustrated by means of digital computer simulations. Section 4 presents the conclusions and suggests some topics for further research.

2 State Reconstruction in Nonlinear Observable Systems

Consider the following n-dimensional MIMO nonlinear system with $k \in \{0,1,2,...\}$,

$$x_{k+1} = f(x_k, u_k), \quad x_k \in R^n, \; u_k \in R^m$$

$$y_k = h(x_k), \quad y_k \in R^p \tag{1}$$

We concentrate, for simplicity sake, on the local analysis. A global analysis can be achieved with additional technical assumptions. Note that the computations based on this analysis provide, for the nonlinear example in next section, almost global reconstruction formulae.

2.1 Basic assumptions

1. We assume that the strings of applied inputs and obtained outputs, prior to $k = 0$, are known from the time $1 - n$ on. In other words, y_k and u_k, for $1 - n < k < 0$ are known.
2. The system is assumed to lie around an equilibrium point (x_e, u_e, y_e), i.e., $x_e = f(x_e, u_e), y_e = h(x_e)$.
3. The system (1) is assumed to be locally *observable* around the constant operating point (x_e, u_e, y_e). This means that the Jacobian matrix

$$\frac{\partial\{y_k, y_{k+1}, \ldots, y_{k+(n-1)}\}}{\partial x_k} \tag{2}$$

evaluated at the constant operating point (x_e, u_e) is full column rank n.

2.2 Notation

We use the delay operator δ to express the fact that $\delta\phi_k = \phi_{k-1}$, and, correspondingly, the *advance* operator is denoted by δ^{-1}. The expression, $\delta^\mu \phi_k$, for any positive μ, stands for the identity $\delta^\mu \phi_k = \phi_{k-\mu}$ and, similarly, $\delta^{-\mu}\phi_k = \phi_{k+\mu}$. The underlined symbol δ, as in, $\underline{\delta}^\mu \phi_k$, stands for the collection: $\{\phi_{k-1}, \phi_{k-2}, \ldots, \phi_{k-\mu}\}$, i.e. $\underline{\delta}^\mu \phi_k = \{\delta\phi_k, \ldots, \delta^\mu \phi_k\}$. Evidently, $\delta^0 = \underline{\delta}^0 = \mathrm{Id}$ and $\underline{\delta}^1 = \delta$. On the other hand, $\underline{\delta}^{-\mu}\phi_k$ stands for the collection, $\{\phi_k, \phi_{k+1}, \ldots, \phi_{k+\mu}\} = \{\phi_k, \delta^{-1}\phi_k, \ldots, \delta^{-\mu}\phi_k\}$.

Note that the system equation (1) is equivalent to:

$$x_k = \delta f(x_k, u_k) = f(\delta x_k, \delta u_k) = f(x_{k-1}, u_{k-1})$$

Since, in turn, one may write:

$$x_{k-1} = f(x_{k-2}, u_{k-2}) = f(\delta x_{k-1}, \delta u_{k-1}) = f(\delta^2 x_k, \delta^2 u_k)$$

it is clear that $x_k = f(f(\delta^2 x_k, \delta^2 u_k), \delta u_k)$. We denote this last quantity by $f^{(2)}(\delta^2 x_k, \underline{\delta}^2 u_k)$. The expression $f^{(\mu)}(\delta^\mu x_k, \underline{\delta}^\mu u_k)$, for $\mu > 0$, should be clear from the recursion:

$$\begin{aligned} f^{(i)}(\delta^i x_k, \underline{\delta}^i u_k) &= f(f^{(i-1)}(\delta^{i-1} x_k, \underline{\delta}^{i-1} u_k), \delta u_k) \\ f^{(1)}(\delta x_k, \underline{\delta} u_k) &= f(\delta x_k, \delta u_k) \end{aligned} \tag{3}$$

The operators δ and $\underline{\delta}$ satisfy the following relation

$$\delta^i \underline{\delta}^{-i} \phi_k = \{\phi_k, \underline{\delta}^i \phi_k\} \tag{4}$$

Similar expressions may be defined for the advances of states.

$$
\begin{aligned}
x_{k+1} &= \delta^{-1} x_k = f(x_k, u_k) = f^{[1]}(x_k, \underline{\delta}^0 u_k) \\
x_{k+2} &= \delta^{-2} x_k = f(f(x_k, u_k), u_{k+1}) = f^{[2]}(x_k \underline{\delta}^{-1} u_k) \\
x_{k+3} &= f(f^{[2]}(x_k \delta^{-1} u_k), u_{k+2}) = f^{[3]}(x_k, \underline{\delta}^{-2} u_k) \\
&\vdots \\
x_{k+i} &= f^{[i]}(x_k, \underline{\delta}^{-(i-1)} u_k)
\end{aligned}
\tag{5}
$$

We set

$$f^{[0]}(x_k \underline{\delta}^{-1} u_k) = x_k$$

2.3 An exact delayed input output state reconstructor

Using the system state equation in (1) in an iterative fashion, one finds:

$$
\begin{aligned}
x_k &= \delta f(x_k, u_k) = f(\delta x_k, \delta u_k) \\
x_k &= f(\delta(f(\delta x_k, \delta u_k)), \delta u_k) = f(f(\delta^2 x_k, \delta^2 u_k), \delta u_k) \\
&= f^{(2)}(\delta^2 x_k, \underline{\delta}^2 u_k) \\
&\vdots \\
x_k &= f^{(n-1)}(\delta^{n-1} x_k, \underline{\delta}^{n-1} u_k)
\end{aligned}
\tag{6}
$$

The elements in a finite sequence of advances of the output signal, y_k, are found to be given by

$$
\begin{aligned}
y_k &= h(x_k) = h(f^{[0]}(x_k \underline{\delta}^{-1} u_k)) \\
y_{k+1} &= \delta^{-1} h(x_k) = h(\delta^{-1} x_k) = h(f(x_k, u_k)) \\
&= (h \circ f^{[1]})(x_k, \underline{\delta}^0 u_k) \\
y_{k+2} &= \delta^{-1}(h \circ f(x_k, u_k)) = h(\delta^{-1} f(x_k, u_k)) \\
&= h(f(f(x_k, u_k), \delta^{-1} u_k)) \\
&\quad (h \circ f^{[2]})(x_k, \underline{\delta}^{-1} u_k) \\
&\vdots \\
y_{k+(n-1)} &= (h \circ f^{[n-1]})(x_k, \underline{\delta}^{-(n-2)} u_k)
\end{aligned}
$$

In the following proposition we show that a locally observable system is always locally constructible. The converse is not necessarily true.

Proposition 1. *Under assumptions 1, 2 and 3, the system is locally constructible, i.e. there exists, locally around* (x_e, u_e, y_e)*, a map* φ *(not necessarily unique) such that the state* x_k *of the system can be* exactly *expressed* *in terms of the output and a finite string of previously applied inputs and obtained outputs, in the form:*

$$x_k = \varphi(y_k, y_{k-1}, \ldots, y_{k-(n-1)}, u_{k-1}, \ldots, u_{k-(n-1)}), \quad k > 0 \tag{7}$$

provided the string of inputs and outputs $\{y_k, u_k\}$ *for* $-n+1 < k < 0$ *is completely known.*

Proof

According to the constant rank theorem and the stated hypothesis, it follows that there exists a mapping Φ such that the solution x_k of the np equations

$$\begin{bmatrix} y_k \\ y_{k+1} \\ \vdots \\ y_{k+(n-1)} \end{bmatrix} = \begin{bmatrix} h(x_k) \\ (h \circ f^{[1]})(x_k, \underline{\delta}^0 u_k) \\ \vdots \\ (h \circ f^{[n-1]})(x_k, \underline{\delta}^{n-2} u_k) \end{bmatrix} \tag{8}$$

can be expressed via a function Φ (not necessarily unique) as follows:

$$x_k = \Phi(\underline{\delta}^{-(n-1)} y_k, \underline{\delta}^{-(n-2)} u_k). \tag{9}$$

This just consists of an extraction of n equations having a full rank Jacobian versus x_k. Such extraction is not unique and thus Φ is also non unique.

If we take $n-1$ delays in this expression we clearly obtain:

$$\begin{aligned} \delta^{n-1} x_k &= \Phi(\delta^{n-1} \underline{\delta}^{1-n} y_k, \delta^{n-1} \underline{\delta}^{2-n} u_k) \\ &= \Phi(y_k, \underline{\delta}^{n-1} y_k, \underline{\delta}^{n-1} u_k) \end{aligned} \tag{10}$$

Using (10) in the last expression of equation (6) we have:

$$\begin{aligned} x_k &= f^{(n-1)}(\delta^{n-1} x_k, \underline{\delta}^{n-1} u_k) \\ &= f^{(n-1)}(\Phi(\underline{\delta}^{n-1} y_k, \underline{\delta}^{n-1} u_k), \underline{\delta}^{n-1} u_k) \\ &= \varphi(y_k, \underline{\delta}^{n-1} y_k, \underline{\delta}^{n-1} u_k) \end{aligned} \tag{11}$$

The result follows.

□

The previous proposition allows for an exact local "delayed input-output parameterization" of the state vector at time k for locally observable systems. Examples below show that, following the same line of development, such "delayed input-output parameterization" can be almost global in practice.

3 The non-holonomic car system

Consider the following (kinematic) model of the non-holonomic car system

$$\begin{aligned}
\dot{x} &= v\cos\theta \\
\dot{y} &= v\sin\theta \\
\dot{\theta} &= \frac{v}{L}\tan\varphi
\end{aligned} \tag{12}$$

where v is the forward velocity, acting as a control input and φ is the control input representing the angular direction of the front wheels with respect to the main axis of the car. The angle, θ, is the orientation angle with respect to the x-axis. The quantities x and y are the position coordinates of the rear axis of the car, which are the only measurable outputs. The parameter L is the length between the front and rear axes of the car. We define the auxiliary input ω as $\omega = (v/L)\tan\varphi$.

Defining the complex variable, $z = x + jy$, we obtain

$$\begin{aligned}
\dot{z} &= v\exp(j\theta) \\
\dot{\theta} &= \omega \\
\eta &= z
\end{aligned} \tag{13}$$

where η denotes the measurable position outputs.

An exact discretization of the complex system (13) follows by considering constant control inputs $\overline{v}$ and $\overline{\omega}$ in an arbitrary time interval $[t_0, t]$, and then proceeding to integrate the resulting differential equations. We obtain:

$$\begin{aligned}
z_{k+1} &= z_k + v_k T\left(\frac{\exp jw_kT - 1}{jw_kT}\right)\exp j\theta_k \\
\theta_{k+1} &= \theta_k + \omega_k T \\
\eta_k &= z_k
\end{aligned} \tag{14}$$

where $z_k = z(t_k)$, $\theta_k = \theta(t_k)$, and $v_k = v(t_k) = \overline{v}$, $\omega(t_k) = \omega_k = \overline{\omega}$ and $T = t_{k+1} - t_k$.

System (14) is observable for $v_k \neq 0$. Following the procedure outlined in the proof of Proposition 1, we first rewrite the system dynamics as

$$\begin{aligned}
z_k &= z_{k-1} + v_{k-1}\left(\frac{\exp j\omega_{k-1}T - 1}{j\omega_{k-1}T}\right)\exp j\theta_{k-1} \\
\theta_k &= \theta_{k-1} + \omega_{k-1}T
\end{aligned} \tag{15}$$

The state of the system in terms of advances of the inputs and the outputs is given by

$$\begin{aligned}
z_k &= \eta_k \\
\theta_k &= \arg(\eta_{k+1} - \eta_k) - \arg\left(\exp(j\omega_k T) - 1\right) + \frac{\pi}{2}
\end{aligned} \tag{16}$$

Combining equations (15) and (16), we obtain an exact delayed input-output parameterization of the state of the discretized system (14) in the following terms

$$\begin{aligned} z_k &= \eta_k \\ \theta_k &= \arg(\eta_k - \eta_{k-1}) - \arg(\exp(j\omega_{k-1}T) - 1) + \frac{\pi}{2} + \omega_{k-1}T \end{aligned} \tag{17}$$

The exact delayed reconstructor (17) will be used for feedback purposes.

3.1 Feedback controller design based on approximate flatness

As the continuous time system (13) is differentially flat, its exact discretization (14) is difference flat, but with a very different and not easy to use flat output. In order to obtain a suitable, and simpler, controller for the system, we proceed to approximate the exactly discretized system (14) by a more useful difference flat system. This is achieved by assuming $\omega_k T$ to be sufficiently small, thus yielding the approximation, $\exp j\omega_k T \approx 1 + j\omega_k T$. We obtain the following system, which entirely coincides with the Euler discretization of system (13),

$$\begin{aligned} z_{k+1} &= z_k + Tv_k \exp j\theta_k \\ \theta_{k+1} &= \theta_k + \omega_k T \\ \eta_k &= z_k \end{aligned} \tag{18}$$

The system (18) is also observable for $v_k \neq 0$ and evidently difference flat, with flat outputs given by by the measurable outputs z_k. We can thus express all system variables in terms of the complex output z and some of its advances.

$$\begin{aligned} v_k &= \frac{1}{T} \mid z_{k+1} - z_k \mid \\ \theta_k &= \arg(z_{k+1} - z_k) \\ \omega_k &= \frac{1}{T}[\arg(z_{k+2} - z_{k+1}) - \arg(z_{k+1} - z_k)] \end{aligned} \tag{19}$$

The expressions in (19) are useful in obtaining a feedback controller. Note that in order to have an invertible relation between the largest advance of the complex flat output z and the control inputs, we must introduce an extension to the system input v_k, by defining it as an auxiliary state, ξ_k, and proceed to consider the following (complex) dynamic input coordinate transformation:

$$\begin{aligned} v_k &= \xi_k \\ \xi_{k+1} &= \frac{1}{T} \mid u_k - z_{k+1} \mid \\ \omega_k &= \frac{1}{T}[\arg(u_k - z_{k+1}) - \arg(z_{k+1} - z_k)] \end{aligned}$$

with u_k being a new complex control input.

The transformed system is seen to be equivalent to the following linear system

$$z_{k+2} = u_k \tag{20}$$

The specification of a prescribed trajectory for the position variables as z_k^* yields the following auxiliary controller:

$$\begin{aligned} u_k &= z_{k+2}^* - k_1(z_{k+1} - z_{k+1}^*) - k_2(z_k - z_k^*) \\ &= z_{k+2}^* - k_1(z_k + T\xi_k \exp j\theta_k - z_{k+1}^*) - k_2(z_k - z_k^*) \end{aligned} \tag{21}$$

The dynamic feedback controller is then obtained as

$$\begin{aligned} v_k &= \xi_k \\ \xi_{k+1} &= \frac{1}{T} \mid z_{k+2}^* - k_2(z_k - z_k^*) - z_{k+1}^* \\ &\quad -(1+k_1)(z_k + T\xi_k \exp j\theta_k - z_{k+1}^*) \mid \\ \omega_k &= \frac{1}{T}\{ \arg \left[z_{k+2}^* - k_2(z_k - z_k^*) - z_{k+1}^* \right. \\ &\quad \left. -(1+k_1)(z_k + T\xi_k \exp j\theta_k - z_{k+1}^*)\right] - \theta_k \} \end{aligned} \tag{22}$$

For the implementation of the designed dynamic feedback controller (22) on the exactly discretized system (14), we use the previously obtained state reconstructor (17). Knowledge of the car position at time $k = 0$, and the applied control inputs at time $k = -1$ results in a dynamic feedback controller capable of satisfactorily tracking the prescribed trajectory.

3.2 Simulation Results

We prescribe a 3-*leaved rose* as a desired trajectory in the (x, y) plane. This function is described in polar coordinates as:

$$\rho = a \cos(m\vartheta) \tag{23}$$

where a is the radius of the circle in which the rose is inscribed and ϑ, the angle of a representative point of the rose in the plane (x, y). The integer m represents the number of "leaves" of the rose. We set the time parameterization of the angle ϑ as a linear growing function of time of the form: $\vartheta(t) = p + q(t - t_0)$, with p and q being suitable constant parameters.

Figures 1 and 2 show the performance of the approximate flatness-based dynamic feedback controller implemented on the exactly discretized system (14). The exact state reconstructor (17) was used, in the controller implementation, providing it with precise knowledge of the prior values of the inputs and the outputs. The controller gains were set to be $k_1 = -1.3$ and $k_2 = 0.5825$. This choice of gains placed the roots of the closed loop characteristic polynomial of the linearized tracking error system at the values $z = 0.65 \pm 0.4j$. The sampling time was set to $T = 0.4$. For generating the 3-leaved rose figure, the values $a = 10$, $m = 3$, $p = \pi/2$, $q = 0.05$ were used. In order to test the controller performance, we set the initial position values x_0, y_0 far away from the origin, at the values $x_0 = 14$, $y_0 = 3$, with an initial orientation of the car given by the angle 2π [rad].

4 Conclusions

In this article we have presented an approach to the problem of controlling a nonlinear discrete time system without measurements of all the components of the state vector. The approach is based on using an exact state reconstructor which requires only knowledge of inputs, outputs and a finite string of delayed applied inputs and obtained outputs. We have tested the "observer-less" control scheme in connection with flatness based controllers for a typical nonlinear system: a discretized non-holonomic multivariable car. The performance of the proposed feedback controller scheme, based on the exact delayed resconstructor, was shown to be good and with quite natural recovery features, specially in those cases where knowledge of applied inputs and corresponding obtained outputs, prior to the initial instant of time, was not allowed and the reconstructor had to be arbitrarily initialized.

An important aspect is that of providing robustness to the proposed exact reconstructor-based feedback, for the case of external perturbation inputs and other classes of uncertain influences on the given plant. This will be the topic of future work.

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FIGURES

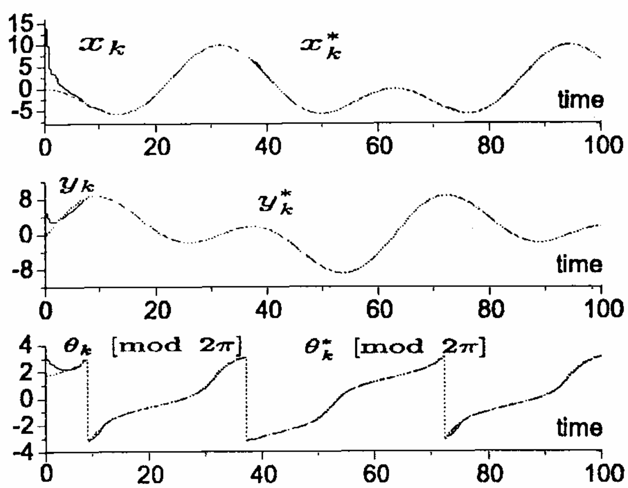

Fig. 1. Trajectories of non-holonomic car controlled with a state reconstructor having perfect knowledge of inputs and outputs prior to $k = 0$.

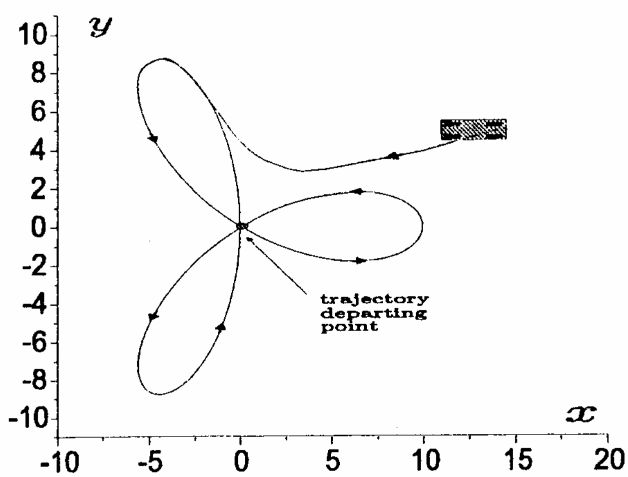

Fig. 2. Performance of controlled car following a "3-leaved rose" trajectory