A flatness based Generalized PI control approach to liquid sloshing regulation in a moving container. [1]

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Abstract

In this article, we provide, on the basis of a popular fourth order dynamic model, a solution to the problem of liquid sloshing regulation, in horizontally moving containers. The approach uses the viewpoint of *differential flatness* and *Generalized Proportional Integral (GPI) control*. The performance of the derived feedback control laws are assessed on a multi-modal 10th order model of the system.

Keywords: Generalized PI control, Flat systems.

1 Introduction

Liquid sloshing in moving containers has received sustained interest by mathematicians, physicists, control and ocean engineers in the last 37 years.

The underling feedback control problem, related to the liquid oscillations in moving containers, is to suppress, at least at the beginning and the end of the container point to point transfer, the sloshing, or undesired vibrations, of the carried liquid while accomplishing the given displacement task for the container within a limited time interval. Very recently, the work of Grundelius [4]-[5] stands out as a rather complete treatment of the modeling and control problems, approached also from an optimal control viewpoint, which attempts to solving the liquid sloshing problem in an actual industrial experimental setting with very encouraging results.

In this article, we present a flatness based GPI control approach for the regulation and rest-to-rest stabilization of liquid sloshing when a container movement is required from an initial to a terminal position in a finite pre-specified amount of time. Flatness based controllers are shown to represent an alternative to traditional optimal control based regulation schemes. We show that the liquid sloshing suppression problem can be viewed as one of appropriate *flat output* trajectory planning. This viewpoint guarantees simultaneous rest-to-rest maneuvers for, both, the container horizontal position and the liquid surface position. The control problem, as in many articles of the existing literature, uses a simplified, under-actuated, fourth order linear system model. GPI control is a recently introduced linear system feedback control technique which uses iterated output and input integral error compensations on linear state feedback laws, synthesized in terms of ***integral state reconstructors***. These are, broadly speaking, iterated integral parameterizations of the state using only the inputs and the outputs of the system (See Fliess *et al* [2] for a module theoretic approach to the subject). GPI control has been initially developed by Fliess and Marquez in [1], in the context of linear predictive control. Some the-

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oretical developments, related to module localizations, can be found in Fliess in citefliesscifa. GPI control has also been used in actual experimental applications with excellent results (See Marquez and Fliess [6]).

Section 2 presents the liquid sloshing system model, taken from the work of Grundelius [4]-[5]. We establish the flatness of the adopted simplified model and define the sloshing suppression problem as a rest-to-rest transfer problem for the flat output. A GPI controller is developed in section 4. Section 5 assesses the feedback laws on a more realistic, 10th order, model of the sloshing system Throughout the article we present simulation results that depict the performance of the proposed feedback controllers. In section 3, we propose a flatness based controller which serves as the basis for a GPI controller.

2 The slosh system model

In several recent studies of the motion control of open containers carrying liquid, Grundelius, [4]-[5], proposed, and experimentally justified, the following simplified *linear model* for use in the optimal control-based actual regulation of the height of the liquid surface inside the moving container.

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -2\zeta\omega x_2 - \omega^2 x_1 + \frac{a\omega^2}{2g}u\\ \dot{x}_3 &= x_4\\ \dot{x}_4 &= u \qquad (2.1)\end{aligned}$$

where x_1 represents the liquid height, above the resting level, measured at one point located at the container's extreme wall, opposite to the direction of movement of the container (see Figure 1). The variable x_2 is the corresponding rate of the liquid height at the measurement point. The variable x_3 represents the position of the container and x_4 its associated velocity. The control input u is the applied acceleration input. The constant parameters ω and ζ represent the liquid natural oscillation frequency and damping factor corresponding to the first harmonic approximation to the liquid movement subsystem (this system is found to be poorly damped). The parameter a is the width of the moving container, g is the gravity acceleration and if we let h denote the resting liquid height, with the system in perfect equilibrium, then the oscillation frequency of the slosh is found to be, after Venugopal and Bernstein [7]

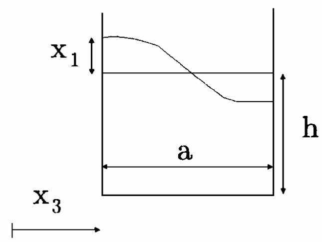

Figure 1: Container and liquid

$$\omega = \sqrt{\frac{g\pi}{a}tanh\left(\frac{h\pi}{a}\right)}$$

2.1 Slosh System Flatness

The system is found to be controllable and, hence, flat, with flat output given by the following linear combination of the state variables:

$$\begin{aligned}F &= \left[\frac{2}{\omega^4 a}\left(-1+4\zeta^2\right)g\right]x_1 + \left(\frac{4\zeta}{a\omega^5}g\right)x_2\\ &+ \left(\frac{1}{\omega^2}\right)x_3 - \left(\frac{2}{\omega^3}\zeta\right)x_4 \qquad (2.2)\end{aligned}$$

A differential parameterization of all system variables, in terms of the flat output and its time derivatives, is obtained as

$$\begin{aligned}x_1 &= \frac{a\omega^2}{2g}\ddot{F}\\ x_2 &= \frac{a\omega^2}{2g}F^{(3)}\\ x_3 &= \omega^2 F + 2\zeta\omega\dot{F} + \ddot{F}\\ x_4 &= \omega^2\dot{F} + 2\zeta\omega\ddot{F} + F^{(3)}\\ u &= \omega^2\ddot{F} + 2\zeta\omega F^{(3)} + F^{(4)} \qquad (2.3)\end{aligned}$$

Under perfect equilibrium conditions the above parameterization yields the following static relations:

$$\begin{aligned}\overline{x}_1 &= 0\\ \overline{x}_2 &= 0\\ \overline{x}_3 &= \omega^2\overline{F}\\ \overline{x}_4 &= 0\\ \overline{u} &= 0 \qquad (2.4)\end{aligned}$$

2.2 Rest-to-rest transfer problem formulation

In an equilibrium to equilibrium maneuver for the slosh system, one typically wants to solve the following problem.

Devise a feedback control strategy so that the container is taken, within a finite prescribed time interval $[t_{initial}, t_{final}]$, from the initial resting position $\overline{x}_{3\ initial} = 0$, valid for all $t \leq t_{initial}$, towards a final resting position $\overline{x}_{3\ final} = X$, valid for all $t \geq t_{final}$, so that the liquid height x_1 equally starts from a resting position $x_1(t_{initial}) = 0$ and ends in a resting position $x_1(t_{final}) = 0$, at the end of the prescribed time interval. In terms of the flat output, F, and according to the static parameterization of the system's equilibria, the problem entitles taking this special output from an initial value of zero towards a final value, given by X/ω^2, where X represents the final desired position of the container. This transfer is to take place while following a desired trajectory, $F^*(t)$, with, at least, four time derivatives being zero at the initial time, $t = t_{initial}$, and at the end of the transfer, at $t = t_{final}$.

2.3 Off-line trajectory planning

As described above the problem of a suitable rest-to-rest transfer which guarantees a desired container displacement with a reasonable liquid height excursion can be formulated as one of tracking a specified rest-to-rest trajectory for the sloshing system flat output. The flat output trajectory, $F^*(t)$ may, therefore, be specified as an interpolating function between the initial and the final flat output values, as follows:

$$F^*(t) = F_{initial} + (F_{final} - F_{initial})\varphi(t, t_{initial}, t_{final})$$

with $\varphi(t, t_{initial}, t_{final})$ chosen to be an time polynomial, of the Bézier type, smoothly rising from the initial value of zero towards the final value of 1, in the time interval $[t_{initial}, t_{final}]$.

In order to adopt a suitable reference trajectory for the flat output, F, we performed several off-line simulation trials in order to determine an acceptable transfer time interval $[t_{initial}, t_{final}]$ and the associated maximum nominal liquid height excursion, above its resting level. We assumed, as in [4], that the total required container displacement $x_3(t_{final}) - x_3(t_{initial})$ is *fixed* to a certain value, which we specified by X, and proceeded to off line obtain the liquid height profile and the horizontal container acceleration for several time interval transfers. Our primary goals were to avoid control input saturation (i.e. to avoid values of u larger that $9.8[\mathrm{m/s^2}]$) and to obtain a reasonable maximum liquid height, $x_{1,max}$, which we set to be around 0.03 [m], during the container transfer. We arbitrarily set the initial time as $t_{initial} = 2$ [s] and set the desired displacement as $x_3(t_{final}) = X = 0.2$ [m]. The simulations were performed with the following system parameters found in [4]:

$$\zeta = 0, \quad \omega = 21.0\,\mathrm{rad/s} \quad a = 0.07\,\mathrm{m} \quad h = 0.2\,\mathrm{m},$$

$$g = 9.8\,\mathrm{m/s^2}$$

The corresponding initial and final equilibrium values for F were readily obtained, from the static parameterization (2.4) as: $F_{initial} = 0$, $F_{final} = X/\omega^2 = 4.5351 \times 10^{-4}$. The time polynomial function $\varphi(t, t_{initial}, t_{final})$ was prescribed as:

$$\varphi(t, t_{initial}, t_{final}) = \theta^5(t)\Big[r_1 - r_2\theta(t) + r_3\theta^2(t) - r_4\theta^3(t) + r_5\theta^4(t) - r_6\theta^5(t)\Big]$$

with the function $\theta(t)$ defined as

$$\theta(t) = \frac{t - t_{initial}}{t_{final} - t_{initial}}$$

and

$$r_1 = 252, \quad r_2 = 1050, \quad r_3 = 1800, \quad r_4 = 1575,$$

$$r_5 = 700, \quad r_6 = 126$$

Figure 2 shows computer simulation runs for three different transfer time intervals, $T = t_{final} - t_{initial} = 2$, 1 and 0.5 [s] respectively. The figures depict the nominal surface height evolution x_1 and the corresponding nominal input acceleration u.

A nominal transfer time of $T = 0.44$ [s] results in the use of a maximum available input acceleration of $9.8[\mathrm{m/s^2}]$ at some point of the transfer maneuver with a maximum liquid height of 0.04 [m]. From the simulations shown in Figure 2, a transfer time of 0.05 [s] results in a reasonable nominal surface height excursion of about 0.03 [m], without any risk of nominal controller saturation. In order to depict the complete open loop system behavior, for such a choice of the transfer time, we recall that the differential parameterization (2.3) allows one to specify a complete set of nominal trajectories for the state components of the state vector, $x^*(t)$, and of the control input, $u^*(t)$, on the basis of a specified nominal desired trajectory for the flat output, $F^*(t)$. Figure 3 shows such a complete set of nominal state and control input trajectories.

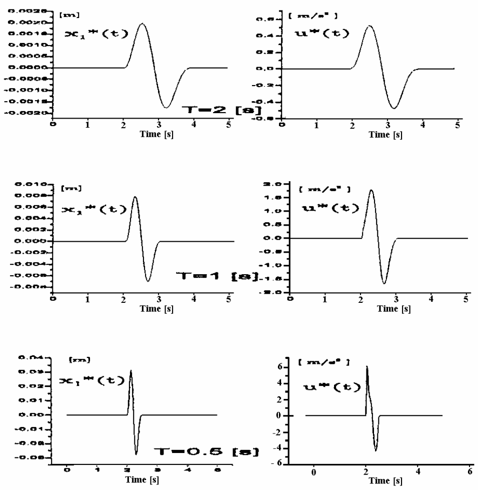

Figure 2: Influence of transfer time interval on liquid height and input acceleration

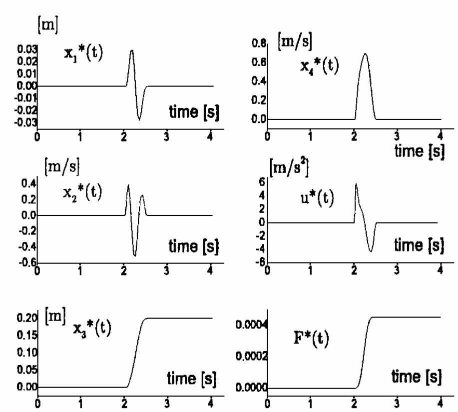

Figure 3: Nominal state and control input behavior corresponding to a planned flat output desired rest-to-rest transfer with $T = 0.5$ [s]

3 An integral state reconstructor-based controller

Consider then the slosh system with output variables $y_1 = x_1$, $y_2 = x_3$,

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -2\zeta\omega x_2 - \omega^2 x_1 + \frac{a\omega^2}{2g} u \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= u \\ y_1 &= x_1, \quad y_2 = x_3 \end{aligned} \tag{3.1}$$

An inspection of the state and output equations, of the slosh system, reveals that one can, indeed, carry out such input-output integral parameterization of the state variables in the following manner:

$$\begin{aligned} \widehat{x}_1 &= x_1 = y_1 \\ \widehat{x}_2 &= -2\zeta\omega y_1 - \int_0^t \left(\omega^2 y_1(\sigma) - \frac{a\omega^2}{2g} u(\sigma)\right) d\sigma \\ \widehat{x}_3 &= y_2 \\ \widehat{x_4} &= \int_0^t u(\sigma) d\sigma \end{aligned} \tag{3.2}$$

where, we have explicitly given to the reconstructed variables the character of "estimates" of the corresponding state variables. Such " hat " notation is justified in view of the relations which tie these state reconstructions with the actual values of the state variables:

$$\begin{aligned} x_1 &= \widehat{x}_1 = y_1 \\ x_2 &= \widehat{x}_2 + x_2(0) \\ x_3 &= \widehat{x}_3 = y_3 \\ x_4 &= \widehat{x}_4 + x_4(0) \end{aligned} \tag{3.3}$$

where $x_2(0)$, and $x_4(0)$ are the unknown initial velocities of the liquid height and the container, respectively.

The main idea in devising a feedback controller which only needs the input, u, and the outputs, y_1, and, y_2, is to replace the state variables, in the derived flatness-based state feedback control law, with their corresponding integral reconstructors and, then, to appropriately compensate the unknown (in this case, constant) estimation errors with output tracking error integrals, so as to obtain an asymptotically exponentially stable closed loop system tracking error.

The proposed feedback control law, based on integral state reconstructors and appropriate integral compensation (represented by the term $\xi(t)$ in the following expressions), is then given by

$$u = \left(\frac{2g}{a}\right) y_1 + 4\left(\frac{g\zeta}{a\omega}\right)\widehat{x}_2 + v + \xi(t) \tag{3.4}$$

$$\begin{aligned} v = {} & [F^*(t)]^{(4)} - k_4(\frac{2g}{a\omega^2}\widehat{x}_2 - [F^*(t)]^{(3)}) \\ & -k_3(\frac{2g}{a\omega^2}y_1 - \ddot{F}^*(t)) \\ & -k_2(-\frac{4g\zeta}{a\omega^3}y_1 - \frac{2g}{a\omega^4}\widehat{x}_2 + \frac{1}{\omega^2}\widehat{x}_4 - \dot{F}^*(t)) \\ & -k_1\left(\left[\frac{2}{\omega^4 a}\left(-1+4\zeta^2\right)g\right]y_1 + \left(\frac{4\zeta}{a\omega^5}g\right)\widehat{x}_2 \right. \\ & \left. + \left(\frac{1}{\omega^2}\right)y_2 - \left(\frac{2}{\omega^3}\zeta\right)\widehat{x}_4 - F^*(t)\right) \end{aligned} \tag{3.5}$$

$$\xi(t) = -k_0\int_0^t [y_2(\sigma) - y_2^*(\sigma)]\, d\sigma \tag{3.6}$$

$$y_2^*(t) = \omega^2 F^*(t) + 2\zeta\omega\dot{F}^*(t) + \ddot{F}^*(t) \tag{3.7}$$

The flat output tracking error satisfies now the closed loop dynamics

$$e_F^{(4)} + k_4 e_F^{(3)} + k_3\ddot{e}_F + k_2\dot{e} + k_1 e_F + k_0\int_0^t \left[\omega^2 e_F(\sigma) + 2\zeta\omega\dot{e}_F(\sigma) + \ddot{e}_F(\sigma)\right] d\sigma = 0$$

which is equivalent to

$$e_F^{(5)} + k_4 e_F^{(4)} + k_3 e_F^{(3)} + (k_2 + k_0)\ddot{e}_F + (k_1 + 2\zeta\omega k_0)\dot{e}_F + k_0\omega^2 e_F = 0$$

A convenient set of coefficients: $k_0, \ldots, k_4$, may be now chosen to guarantee an asymptotically exponentially stable behavior of the tracking error e_F.

Figure 4 depicts the performance of the integral reconstructor based feedback controller for the initially perturbed rest-to-rest maneuver described in the previous section. The initial deviations entitled position and velocity errors for the liquid surface height and initial position error for the container. The controller parameters were chosen so that the closed loop characteristic polynomial of the tracking error coincided with the polynomial

$$p_{cl} = (s^2 + 2\xi\omega_n s + \omega_n^2)^2(s+\beta)$$

with $\xi = 0.707$ and $\omega_n = 18$, $\beta = 9$.

In this case, we initially obtained a temporary, non-destabilizing, saturation of the control input, which we limited to values within the interval $|u| \leq 9.8[\mathrm{m/s^2}]$.

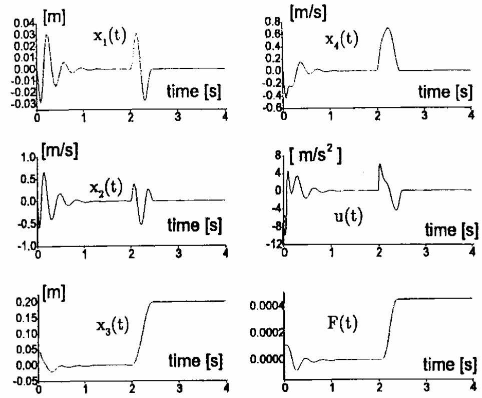

Figure 4: Integral reconstructor based control performance for significant initial conditions deviations

4 Assessment of control laws performance on a higher dimensional slosh system model

In order to assess the effectiveness of the GPI feedback control laws, proposed in the previous section, we now evaluate the performance of such feedback actions on a higher dimensional model which approximates, within several oscillation modes, the liquid slosh. The model derivations make direct use of the Bernoulli equations as explained in Grundelius [4]. We make a slight modification of the model presented in that work to adapt it to our notation and take only the first four oscillation modes. We assume that surface height measurements are made at the leftmost wall of the container.

$$\begin{aligned} \dot{x}_{1m} &= x_{2m} \\ \dot{x}_{2m} &= -\omega_1^2 x_{1m} + \frac{4\omega_1^2\sqrt{2a^3}}{\pi^2}u \\ \dot{x}_{3m} &= x_{4m} \\ \dot{x}_{4m} &= -\omega_3^2 x_{3m} + \frac{4\omega_3^2\sqrt{2a^3}}{9\pi^2}u \\ \dot{x}_{5m} &= x_{6m} \\ \dot{x}_{6m} &= -\omega_5^2 x_{5m} + \frac{4\omega_5^2\sqrt{2a^3}}{25\pi^2}u \\ x_{7m} &= x_{8m} \\ \dot{x}_{8m} &= -\omega_7^2 x_{7m} + \frac{4\omega_7^2\sqrt{2a^3}}{49\pi^2}u \\ \ddot{x}_3 &= u \\ x_1 &= \frac{1}{g}\sqrt{\frac{2}{a}}\,(x_{1m} + x_{3m} + x_{5m} + x_{7m}) \end{aligned}$$

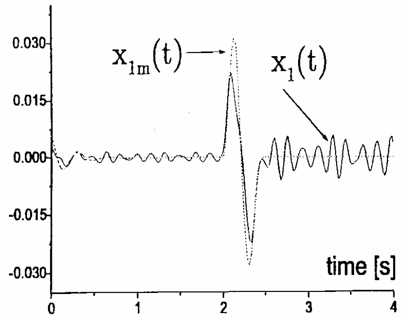

Figure 5: GPI based feedback controlled performance of multi-modal slosh system model for significant initial conditions deviations

$$+\left[\frac{a}{2g}-\frac{4a}{\pi^2 g}\left(1+\frac{1}{9}+\frac{1}{25}+\frac{1}{49}\right)\right]u \tag{4.1}$$

with

$$\omega_i=\sqrt{\frac{ig\pi}{a}\tanh(\frac{ih\pi}{a})},\quad i=1,3,5,.$$

Figure 5 depicts the closed loop responses of the slosh model (4.1) to the GPI based feedback control law (3.4)-(3.7), with significant initial deviations from the rest equilibrium for the surfaçe height, but no initial error on the container position.

5 Conclusions

In this article we have presented a flatness based controller for liquid sloshing regulation in an horizontally moving container. A typical fourth order simplified linear model of the "slosh system" was adopted, as fully described in [4]. It was found that such a model is controllable, and, hence, differentially flat. The flatness property of the model allows for a rather direct approach to the trajectory planning task involved in a rest-to-rest container position maneuver with a corresponding rest-to-rest induced liquid surface height evolution. Flatness allowed also for an off-line assessment of the nominal dynamic effects of the required container position transfer maneuver on, both, the required control input magnitude and the resulting maximum liquid height excursion exhibited during the proposed displacement maneuver.

In order to avoid velocity measurements or its asymptotic estimations via dynamic observers, a Generalized PI controller, based on integral input-output state reconstructors, was also derived. The GPI approach evades calculated estimates based on output sampling and it only uses the measured values of inputs and outputs (i.e. realistically speaking, it could be synthesized measuring only the liquid height and the armature input voltage of the d-c motor acting on the container's transportation belt).

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