SLIDING MODE GENERALIZED PI TRACKING CONTROL OF A DC-MOTOR-PENDULUM SYSTEM. *

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In this article, a suitable combination of Sliding Modes and Generalized Proportional-Integral (GPI) control is presented for feedback trajectory tracking in a single pendulum actuated by a DC motor. The controller is developed in the context of an angular position trajectory tracking problem. The feedback controller exploits integral reconstructors of the angular position and the angular velocity variables which are obtained on the basis of electrical measurements alone. The proposed scheme has no need for measurements of the pendulum angular position or its corresponding angular velocity nor for the asymptotic dynamic estimation of such mechanical states. The validity of the results are tested via digital computer simulations.

1 Introduction.

Position control related to nonlinear mechanical systems, invariably require position measurement and accurate velocity estimation in order to realize most of the developed state feedback control schemes appearing

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in the published literature (See, for instance [2], [3] and [14]). Generally speaking, nonlinear asymptotic state estimators, with linear error dynamics, although highly attractive in theory, are somewhat difficult to synthesize for a given nonlinear mechanical system, even if the underlying system proves to be locally, or globally, observable (see [10]). Aside from this difficulty, there is always the lack of robustness with respect to the simplest type of uncertain perturbation, namely; constant load perturbations. In recent developments, a new control strategy, called Generalized PI control (GPI) has been proposed in [5], as an alternative dynamic input-output based feedback controller design for constructible (observable) linear systems. The main emphasis here is placed not in accurately (i.e. asymptotically) estimating the states of the system on the basis of measured outputs, but, simply, generating "structurally correct" state estimates, or *integral state reconstructors*, based only on input and output measurements, which can be used in a previously designed full state feedback controller. The GPI control scheme, involves linear combinations of iterated integrals of inputs and outputs to generate the unmeasured states. The integral state reconstructions are known to differ from the actual values of the states by error time-functions which are either constant, ramps, parabolas, or finite iterated integrals of such quantities. Nevertheless, the effects of such unstable state reconstruction errors is suitably compensated via the addition of a finite number of iterated integrals of output, or input, tracking errors in the controller. Such iterated integral error compensation can always be carried out in a suitably *nested* manner which avoids internal instabilities in the dynamic feedback controller. It should be remarked that the developed GPI control theory, and its reported applications thus far, mainly deal with linear systems cases. Extensions of GPI control to the realm of nonlinear systems has been, so far, limited to the control of DC-to-DC power converters and some nonlinear mechanical systems (see [11, 12]). The GPI control strategy has been applied in, both, stabilization and trajectory tracking problems in DC motor actuated *linear* mechanical systems, as demonstrated in [9] and [8]. The fundamental advantage of the GPI control scheme in inertia-loaded DC motor systems, lies in the fact that it requires no position or velocity sensors, nor traditional asymptotic observers. These facts bear considerable hardware savings in mechanical sensors (coders and tachometers) and the use of low cost analog electronics replacing expensive Digital Signal Processing cards or digital computers in the control loop. Experimental GPI control of rotational spring inertia loaded DC-motors, have also been reported in [6] where robust performance features were obtained. In this article, we explore an extension of GPI control for the regulation and trajectory tracking of a nonlinear mechanical system constituted by the combination of a DC-motor acting on a single link pendulum. We show that a GPI feedback controller scheme can be suitably, and advantageously, combined with the robustifying features of a sliding mode controller. The results are, however, local in two respects: Local reachability of the sliding surface can be achieved and, also, restricted angular position trajectories can be ideally tracked.

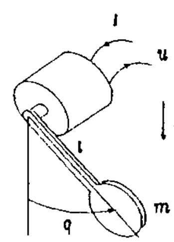

Figure 1: DC-motor Pendulum system

2 Dynamic Model of a DC Motor-Pendulum System

Consider the DC motor actuated pendulum, shown in Figure 1. It is well known (see [13]) that the dynamic model of the composite DC-Motor-pendulum system is given by:

$$L\dot{I} + RI + k_e\dot{q} = u, \quad J\ddot{q} + B\dot{q} + G\sin q = k_m I \tag{1}$$

$$J = J_m + ml^2, \quad B = B_m + B_L, \quad G = mgl \tag{2}$$

where q stands for pendulum angular position and m, l, B_L and g are, respectively, pendulum mass, length, viscous friction coefficient and gravity constant. The variables I, u, are, respectively, the armature circuit electric current and the applied voltage. L, R, k_e stand, respectively, for the armature circuit inductance, the armature circuit resistance and the back electro-motive force constant. The parameters, J_m, B_m, k_m denote rotor mechanical inertia, viscous friction coefficient and DC-motor torque constant, respectively. We define the state of the DC-motor-pendulum system as constituted by the pendulum angular position q, the corresponding angular velocity $\dot{q}$, and the armature circuit electric current I. It is easy to see that these state variables, and the system input u as well, can all be expressed in terms of the pendulum angular position q and a finite number of its time derivatives. This shows that the composite system exhibits the flatness property (See [4]), with the position variable q being a flat output.

3 A Sliding Mode Generalized PI Controller

We are interested in feedback controlling the pendulum angular position variable q towards a pre-specified desired trajectory $q_d(t)$. The feedback control actions are to be based only on the availability of the DC motor armature circuit current, I, and the corresponding armature circuit input voltage, u. Such a control scheme evidently avoids the need for mechanical sensors. We regard the electric current, I, as the measured system output and denote it by $y = I$. We denote by $y_d = I_d = I_d(t)$ the corresponding desired output current variable, computed on the basis of the known signals, $q_d(t)$, $\dot{q}_d(t)$ and $\ddot{q}_d(t)$, in accordance to the system flatness property, expressed by equation (1).

Define the following sliding surface coordinate function:

$$\hat{\sigma} = (y - y_d) + k_1(\hat{q} - q_d) + k_0 \int_0^t (y - y_d) d\tau \tag{3}$$

$$\hat{q} = \frac{1}{k_e}\left(\int_0^t (u - Ry) d\tau - Ly\right), \quad q = \hat{q} + q(0) \tag{4}$$

where k_1 and k_0 are constant design parameters and $\hat{q}$ represents the *integral reconstruction* of the position variable q, given according to the system dynamics, and $q(0)$ is the unknown initial pendulum position.

The integral term in (3) is intended to compensate, under ideal sliding conditions, for the unknown but constant difference between the actual value of q and its reconstructed value, $\hat{q}$, depicted in (4). Differentiating (3) once, and using the following discontinuous feedback control law:

$$u = L\left[-(k_0 - \frac{R}{L})y + \frac{dy_d}{dt} + k_1\dot{q}_d + k_0 y_d\right] - LW \, sign(\hat{\sigma}) \tag{5}$$

where $W = \delta + \eta$, with $\eta > 0$, $\delta > \mid (-\frac{k_e}{L} + k_1)\dot{q} \mid \geq 0$, leads to the following closed loop sliding surface coordinate function dynamics:

$$\dot{\hat{\sigma}} = -(\frac{k_e}{L} - k_1)\dot{q} - W\text{sign}\ (\hat{\sigma}) \tag{6}$$

Hence, we locally obtain $\hat{\sigma} = 0$. Thus, the sliding surface, is locally reachable in finite time. Thus, a *sliding regime* has been shown to indefinitely exist on the sliding surface $\hat{\sigma} = 0$. Under ideal sliding surface invariance conditions, $\hat{\sigma} = 0$, $\dot{\hat{\sigma}} = 0$, and thanks to the flatness property, (1), we obtain, after some algebraic manipulations, the following *ideal sliding dynamics*:

$$\begin{aligned} e_q^{(3)} &+ a_2\ddot{e}_q + a_1\dot{e}_q = b_1\dot{u}_e + b_0 u_e \qquad (7) \\ e_q &= q - q_d, \quad u_e = \nu - \nu_d, \quad \nu = -\sin q, \quad \nu_d = -\sin q_d \\ a_2 &= \frac{B + Jk_0}{J}, \quad a_1 = \frac{k_1 k_m + Bk_0}{J}, \quad b_1 = \frac{G}{J}, \quad b_0 = \frac{Gk_0}{J} \end{aligned}$$

Note that we have obtained an ideal closed loop dynamic system constituted by a linear stable system of the form

$$G(s) = \frac{b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s} \tag{8}$$

in negative feedback connection with the following static nonlinearity:

$$\psi(t, e_q) = \psi(t, q - q_d) = \psi(t, e_q) = \sin q - \sin q_d \tag{9}$$

The static nonlinearity (9) is, generally speaking, time-varying because such is the nature of the reference signal q_d. In order to use well-known classical results about absolute stability (see [7]), we proceed to analyze the *sector* properties of the nonlinear function $\psi(t, e_q)$.

Consider the set $S = \{q \in R : \mid q \mid < \pi/2\}$. In Figure 2, it can be seen that $\psi(t, e_q)$ belongs to an closed sector of the form, $[\alpha, 1]$, for some $\alpha > 0$ as long as $q_d(t)$ uniformly belongs to S and $e_q(t) \in [a, b]$, for some scalar constants a and b. Note that $\mid \partial \sin(q)/\partial q \mid \leq 1$ for all $q \in R$. However, a and b depend on the desired reference signal, $q_d(t)$, and the distance to the boundaries of the set S. In order to ensure that a sector property is always satisfied, we proceed as follows: 1) Let ϵ be the greatest lower bound of the distance between the signal values, $q_d(t)$, and any one of the two the boundaries of S, 2) Let $a > -2\epsilon$ and $b < 2\epsilon$. It is then easy to see that $\psi(t, e_q)$ satisfies the sector conditions: $\alpha e_q^2 \le e_q \psi(t, e_q) \le \beta e_q^2$, $\forall t \ge 0$, $| q_d | < \frac{\pi}{2}$, $| e_q | < 2\epsilon$, $\alpha > 0$, $\beta = 1$. To show asymptotic stability and convergence towards the desired position trajectory, $q_d(t)$ under the ideal sliding mode invariance conditions, it suffices to show absolute stability of the ideal sliding dynamics (7). According to [7], the sufficient conditions for absolute stability are $G_T(s)$ is Hurwitz and $Z_T(s)$ is Strictly Positive Real (SPR):

$$G_T(s) = \frac{b_1 s + b_0}{s^3 + a_2 s^2 + (a_1 + \alpha b_1)s + \alpha b_0} \tag{10}$$

$$Z_T(s) = \frac{N(s)}{D(s)} = \frac{s^3 + a_2 s^2 + (a_1 + \beta b_1)s + \beta b_0}{s^3 + a_2 s^2 + (a_1 + \alpha b_1)s + \alpha b_0} \tag{11}$$

The Routh criterion states that the first condition is guaranteed if and only if:

$$a_2 > 0, \quad a_2 a_1 + \alpha a_2 b_1 - \alpha b_0 > 0, \quad \alpha b_0 > 0 \tag{12}$$

The second condition can be tested using the theorems an definitions presented in [1]. Thus (11) is SPR if : 1) all coefficients of $Z_T(s)$ are real, 2) all the poles of $Z_T(s)$ have negative real parts and 3) the function $f(\omega)$, given by [1]

$$\begin{aligned} f(\omega) &= \omega^6 + f_4\omega^4 + f_2\omega^2 + f_0 \\ f_4 &= a_2^2 - (a_1 + \alpha b_1) - (a_1 + \beta b_1) \\ f_2 &= -\beta a_2 b_0 - \alpha a_2 b_0 + (a_1 + \beta b_1)(a_1 + \alpha b_0) \\ f_0 &= \alpha\beta b_0^2 \end{aligned} \tag{13}$$

is strictly positive for all $\omega \in R$. Condition 1) is readily satisfied and condition 2) is the same as the Routh test conditions: (12) previously stated. Regarding condition 3), we observe that $f(\omega)$ only contains even powers of ω. Hence, $f(\omega) > 0$, $\forall \omega \in R$, provided f_4, f_2 and f_0 are strictly positive. Thus, $Z_T(s)$ is SPR if the conditions (12) are satisfied and in addition:

$$f_4 > 0, \quad f_2 > 0, \quad f_0 > 0 \tag{14}$$

Based on these facts, we conclude that the ideal sliding dynamics (7) is absolutely stable within a finite domain: $| q_d(t) | < \frac{\pi}{2}$ and $| e_q | < 2\epsilon$.

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[1] $Re\{Z_T(s)\} = Re\{\frac{N(j\omega)D(-j\omega)}{D(j\omega)D(-j\omega)}\}$, thus, $Re\{Z_T(s)\} > 0$ if $f(w) = Re\{N(j\omega)D(-j\omega)\} > 0$, see [1].

Therefore, the closed loop system (1), (3), (4), (5) is locally asymptotically stable and $q \to q_d(t)$ as $t \to \infty$. Finally, the system flatness property (1) ensures that $(y - y_d) \to 0$ also. Moreover, (3) and (4) clearly demonstrate that: $\int_0^t (y - y_d) d\tau \to \frac{k_1}{k_0} q(0)$. We have proved, via the preceding developments, the following result

Theorem. 1 *Consider the composite pendulum-DC motor system (1) and a desired angular reference trajectory, $q_d(t)$, for the pendulum angular position, q, satisfying the restriction $\mid q_d(t) \mid < \frac{\pi}{2}$. Let a feedback control action be represented by the sliding mode controller (5), (3), (4), (12), (14). Where y_d is an off-line time-varying computed signal satisfying: $y_d = \frac{1}{k_m}[J\ddot{q}_d + B\dot{q}_d + G \sin q_d]$. Then, the proposed feedback sliding mode controller locally drives the controlled system state trajectories, $q(t), \dot{q}(t), I(t)$, to satisfy, in finite time, the sliding mode condition: $\widehat{\sigma} = 0$, from any initial condition located within the state space set defined by the inequality, $\delta > \mid \left(-\frac{k_e}{L} + k_1\right) \dot{q} \mid$. Moreover, the ideally restricted feedback controlled system variables remain all bounded and the position trajectories, $q(t)$, evolving on the sliding surface, $\widehat{\sigma} = 0$, locally asymptotically converge towards the desired reference angular trajectory, $q_d(t)$, from any sliding surface hitting point satisfying the boundary layer condition: $\mid q - q_d(t) \mid < 2\epsilon$, with, $\epsilon > 0$, being such that, $-a/2 < \epsilon$, and, $b/2 < \epsilon$, where $a < b$ are real constant parameters such that, for all t, $q - q_d(t) \in [a, b]$*

4 Simulation Results.

The numerical values of the DC motor-pendulum system parameters that will be used in the simulations are the following: $J = 44.8 \times 10^{-5}\text{kg} - \text{m}^2$, $G = 39.28 \times 10^{-3}\text{kgm}^2\text{s}^{-2}$, $B = 0.62419 \times 10^{-3}\text{N} - \text{m} - \text{s/rad}$, $L = 43.31\text{mH}$, $R = 30\ \Omega$, $k_m = 56.37 \times 10^{-3}\text{N} - \text{m/A}$, $k_e = 0.076\text{V} - \text{sec/rad}$. We prescribed a desired angular-position trajectory $q_d(t)$ to be tracked by the pendulum position q as $q_d(t) = 0.6(\cos(t) - 1)rad$ for $t \in [0, \pi]$ and $q_d(t) = -1.2$ for $t \geq \pi$. Figure 3 shows the simulation results when the proposed GPI plus Sliding Mode controller proposed in section 2 is used for tracking of the desired pendulum position. Initial conditions for the simulation were set to be, $q(0) = -0.5$ [rad], $\dot{q}(0) = 1$ [rad/s], $y(0) = I(0) = 0$ [A]. The sliding surface parameters parameters were set to be, $k_0 = 60$, $k_1 = 10$. We also used the following design values: $\alpha = 0.2$, $\beta =$

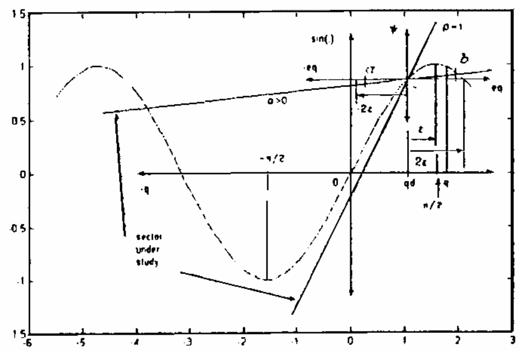

Figure 2: Sector properties of $\psi(t, q - q_d)$.

1. Note that, as expected, the actual pendulum position q asymptotically converges towards the desired sinusoid trajectory $q_d(t)$. Simulation results show good tracking performance features. As expected, the control input signal, u, remains bounded throughout the trajectory tracking task.

5 Conclusions.

A sliding mode GPI feedback control scheme has been presented for the local stabilization and trajectory tracking of a nonlinear system constituted by a DC motor and a rigid pendulum. A locally reachable sliding surface, which only requires input and output variables of the electric type, was shown to lead to an ideal sliding dynamics characterized by a linear stable system in negative feedback connection with a static nonlinearity of the sector type. Local asymptotic stability of the closed loop ideal sliding dynamics was demonstrated by resorting to classical absolute stability results. Integral state reconstructors have been shown to be useful for the controller synthesis in a particular nonlinear mechanical system tracking controller design example. Hence, for local position stabilization and for limited excursion trajectory tracking tasks, in the described nonlinear pendulum-DC motor system, additional hardware for position and velocity measurements has been shown to become largely unnecessary. Sliding mode control, which is known to be quite robust, while enjoying sufficiently simple implementation demands, has been shown to be eas-

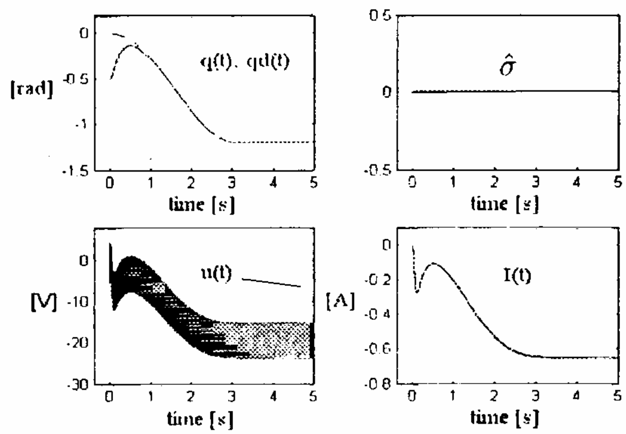

Figure 3: Pendulum DC motor system responses, and applied control input, to SMGPI feedback tracking controller.

ily combined with integral reconstructors and iterated integral tracking error compensation schemes, thus providing the overall control approach with enhanced robustness and simplicity. Global trajectory tracking results have been recently obtained for the composite system studied in this article when angular position measurements alone are allowed still within a GPI scheme which needs no velocity, nor acceleration, measurements. These and other results, which lie outside the realm of sliding mode control, will be reported in a forthcoming publication.

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