ACC02-IEEE1437

Control of the Furuta pendulum based on a linear differential flatness aproach

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Abstract— **In this article, we present a flatness based control approach for the stabilization and equilibrium-to-equilibrium transfers, via trajectory tracking, of the Furuta pendulum. We introduce three feedback controller design options for the stabilization and rest-to-rest trajectoring problems; a direct pole placement approach, a hierarchical high-gain approach and Generalized Proportional Integral (GPI) approach, based only on measured inputs and outputs.**

Keywords Flatness, High-gain control, Generalized Proportional Derivative control, Furuta pendulum system.

I. Introduction

In this paper, we present a differential flatness approach for the stabilization and trajectory tracking, in a Furuta pendulum system. The adopted model assumes that the inverted pendulum travels within a sufficiently small vicinity of its unstable equilibrium point, while the horizontal arm covers a large angle maneuver in a prespecified time interval. Differential flatness of the tangent approximation is then exploited in three different proposed controller design approaches: a pole placement-based controller, a hierarchical high-gain controller, related to the celebrated backstepping design procedure, and finally, a Generalized Proportional Integral (**GPI**) approach, recently developed by Fliess and his coworkers in [2] and [3] . The latter technique exploits traditional linear state feedback, but uses no state measurements and no asymptotic state observers. Rather, the state vector is replaced by a linear combination of iterated integrals of the input and of the output, which purposely neglects initial conditions. The effect of the initial conditions is later compensated, thanks to the superposition principle, via a sufficient number of iterated integrations of the output tracking error. The dynamic feedback control scheme is found to be internally exponentially asymptotically stable via a suitable and natural nesting of the open loop integrations.

Section 2 presents a flatness-based linear pole placement-based controller for the stabilization of the underactuacted Furuta pendulum system. The equilibrium-to-equilibrium transfer of the system, within a finite time interval, is also accomplished by a pole placement-based trajectory tracking controller exploiting the flatness property. In order to obtain a more robust controller design, a hierarchical high-gain feedback controller scheme is also derived in Section 3. In section 4 we present a **GPI** feedback controller for the stabilization and trajectory tracking problems for the Furuta pendulum. The proposed **GPI** controller requires only precise input and output measurements which, in this case, are represented by the applied input torque and the obtained angular position of the arm. In the three controller design cases, we present computer simulation results depicting the performance of the system to the particular feedback controller option. Section 5 is devoted to present the conclusions of the article.

II. The Furuta Pendulum

The Furuta pendulum, is a system consisting of an inverted pendulum connected to an horizontal rotating arm (see Figure 1) acted upon by a direct dc motor drive. The nonlinear model of the mechanical part of the system, which can be derived from either Newton, or the Euler-Lagrange formalism (See Lozano and Fantoni

This research was supported by the Centro de Investigación en Computación of the Instituto Politécnico Nacional (CIC-IPN), by the Centro de Investigación y Estudios Avanzados del IPN, (CINVESTAV-IPN) and by the Consejo Nacional de Ciencia y Tecnología (CONACYT-México), under Research Grant 32681-A.

[1]), is given by

$$
\begin{aligned}
f &= [I_0 + m_1(L_0^2 + l_1^2 \sin^2\theta_1)]\, \ddot{\theta}_0 + m_1 l_1 L_0 \cos\theta_1 \ddot{\theta}_1 \\
&\quad + m_1 l_1^2 \sin(2\theta_1)\dot{\theta}_0\theta_1 - m_1 l_1 L_0 \sin\theta_1 \dot{\theta}_1^2; \\
0 &= m_1 l_1 L_0 \cos\theta_1 \ddot{\theta}_0 + [J_1 + m_1 l_1^2]\, \ddot{\theta}_1 \\
&\quad - m_1 g l_1 \sin\theta_1 - m_1 l_1^2 \sin\theta_1 \cos\theta_1 \dot{\theta}_0^2
\end{aligned}
\tag{1}
$$

where θ_0 is the rotational angle of the arm, θ_1 is the rotational angle of the pendulum, and f is the input torque applied to the the joint of the horizontal arm. The torque f acts as the only control input. The parameters m_1 and J_1 stand for the mass of the pendulum and the inertia of the pendulum around its center of gravity, respectively. J_0 is the inertia of the horizontal arm. L_0 and l_1 are the total lengths of the arm and the vertical distance of the center of gravity of the pendulum to its pivot point on the extreme of the horizontal arm.

We first proceed to normalize the equations of the above system. Define the scaled variables

$$
\tau = \sqrt{\frac{g}{l_1}}t, \quad u = \frac{f}{g m_1 l_1} \tag{2}
$$

and the positive constants,

$$
\begin{aligned}
\varepsilon = \frac{I_0}{m_1 l_1^2}, &\qquad \mu = \frac{L_0}{l_1} \\
\eta = \frac{J_1}{m_1 l_1^2} + 1, &\quad \square = \varepsilon + \mu^2
\end{aligned}
\tag{3}
$$

We obtain, after some algebraic manipulations, the following normalized system:

$$
\begin{aligned}
u &= \left[\square + \sin^2\theta_1\right] \ddot{\theta}_0 + \mu\cos\theta_1 \ddot{\theta}_1 + \sin(2\theta_1)\dot{\theta}_0\dot{\theta}_1 \\
&\quad - \mu \sin\theta_1 \dot{\theta}_1^2; \\
0 &= \mu\cos\theta_1 \ddot{\theta}_0 + \eta\ddot{\theta}_1 - \sin\theta_1 \cos\theta_1 \dot{\theta}_0^2 - \sin\theta_1
\end{aligned}
\tag{4}
$$

where, with an abuse of notation, the " dot " stands for differentiation with respect to the normalized (dimensionless) time τ.

Linearization of the normalized system around the unstable equilibrium point $\theta_0 = X$=constant, $\theta_1 = 0$, $\dot{\theta}_0 = 0$ and $\dot{\theta}_1 = 0$ is achieved by defining the incremental errors as follows: $\theta_{0\delta} = \theta_0 - X, \theta_{1\delta} = \theta_1$ and $u_\delta = u$, we obtain:

$$
\begin{aligned}
\square\ddot{\theta}_{0\delta} + \mu\ddot{\theta}_{1\delta} &= u_\delta \\
\mu\ddot{\theta}_{0\delta} + \eta\ddot{\theta}_{1\delta} &= \theta_{1\delta}
\end{aligned}
\tag{5}
$$

The incremental flat output, denoted by F_δ, is, in this case, given by the following relation

$$
F_\delta = \mu\theta_{0\delta} + \eta\theta_{1\delta}.
$$

Indeed all the variables can be parameterized in terms of the flat output F_δ and a finite number of its time derivatives, as follows

$$
\begin{aligned}
\theta_{1\delta} &= \ddot{F}_\delta; \\
\theta_{0\delta} &= \tfrac{1}{\mu}\left[F_\delta - \eta\ddot{F}_\delta\right]; \\
u_\delta &= \tfrac{\square}{\mu}\ddot{F}_\delta + \tfrac{\sigma}{\mu}F_\delta^{(4)}.
\end{aligned}
\tag{6}
$$

where $\sigma = \mu^2 - \eta\square$

Finally, note that the differential parameterization (6), allows for some analysis of the behavior of the various system incremental variables. For instance, the incremental angular position, $\theta_{0\delta}$ is a non-minimum phase output of the system since its zero dynamics, corresponding to $\theta_{0\delta} = 0$, is given by the unstable dynamics:

$$
\ddot{F}_\delta = F_\delta/\eta.
$$

The zero dynamics associated with the incremental angular position, $\theta_{1\delta}$ satisfies the differential equation

$$
\ddot{F}_\delta = 0,
$$

which has the origin as a critically stable equilibrium.

III. A Flatness-Based Pole Placement Approach for Stabilization and Tracking

A. *Stabilization around the unstable equilibrium point*

The state dependent input-coordinate transformation,

$$
u_\delta = \frac{\square}{\mu}\ddot{F}_\delta + \frac{\sigma}{\mu}v_\delta, \tag{7}
$$

shows that the linearized system (6), is equivalent to the following chain of integrations:

$$
F_\delta^{(4)} = v_\delta. \tag{8}
$$

A stabilizing feedback controller may be readily obtained by setting

$$
\begin{aligned}
v_\delta = -k_0\left[\mu\theta_{0\delta} + \eta\theta_{1\delta}\right] - k_1\left[\mu\dot{\theta}_{0\delta} + \eta\dot{\theta}_{1\delta}\right] \\
- k_2\theta_{1\delta} - k_3\dot{\theta}_{1\delta}
\end{aligned}
\tag{9}
$$

where the set of coefficients $\{k_0, k_1, k_2, k_3\}$ is chosen such that the closed loop characteristic polynomial of the linearized system, defined as:

$$
p(s) = s^4 + k_3 s^3 + k_2 s^2 + k_1 s + k_0
$$

is a Hurwitz polynomial.

B. Simulation Results

Figure 2 shows the closed loop responses, to the derived feedback controller, when applied to the original nonlinear system (1). A significant initial deviation of the angular position of the pendulum and the angular position of the arm were hypothesized to be, $\theta_{0\delta} = -0.2$ [rad] and $\theta_{0\delta} = 0.15$ [rad] In the simulation we used the following numerical values of the physical parameters:

$$I_0 = 0.175\ [m] \quad L_0 = 0.25\ [m] \quad m_1 = 0.15\ [Kg]$$
$$l_1 = 0.25\ [m] \quad J_1 = 0.19\ [N - m - s^2]$$

The closed loop characteristic polynomial was chosen to be

$$p(s) = (s^2 + 2\zeta w_n s + w_n^2)^2$$

with $\zeta = 0.707$ and $w_n = 0.9$.

C. Trajectory tracking for a rest-to-rest maneuver

Based on the linearized system model (5), we can attempt a trajectory tracking approach to the problem of changing the angular position of the horizontal arm between an initial and final value while keeping the pendulum around its unstable equilibrium position. The rest-to-rest maneuver is to be accomplished within a finite time interval, following a smooth trajectory with no restriction on the magnitude of the horizontal arm angular displacement.

We first assume that the horizontal arm is located, at time $\tau = \tau_{initial}$, at a resting point $\theta_0(\tau_{initial}) = X_{initial}$, with the pendulum standing in its unstable equilibrium position $\theta_1(\tau_{initial}) = 0$. The corresponding incremental state equilibrium point is charácterized by:

$$\theta_{0\delta}(\tau_{initial}) = 0, \quad \dot{\theta}_{0\delta}(\tau_{initial}) = 0,$$
$$\theta_{1\delta}(\tau_{initial}) = 0, \quad \dot{\theta}_{1\delta}(\tau_{initial}) = 0.$$

It is desired to move the angular position of the horizontal arm to a new position $\theta_0(\tau_{final}) = X_{final}$ within a finite time interval $T = \tau_{final} - \tau_{initial}$, in such way that when the arm reaches the desired final position, the pendulum exhibits no oscillations around the unstable equilibrium point. This mean that the horizontal arm angular position $\theta_{0\delta}$ and the pendulum angular position $\theta_{1\delta}$ adopt the following equilibrium point, at time $\tau = \tau_{final}$

$$\theta_{0\delta}(\tau_{final}) = X_{final} - X_{initial}, \quad \dot{\theta}_{0\delta}(\tau_{final}) = 0,$$
$$\theta_{1\delta}(\tau_{final}) = 0, \quad \dot{\theta}_{1\delta}(\tau_{final}) = 0$$

We next assume that we can accomplish the maneuver with a small angular deviation of the pendulum from the vertical line, so that we can still use the linearized system (5) for deriving the feedback controller. Naturally, we test the performance of the resulting linear feedback controller on the nonlinear system model (1).

Let us recall, from (6), the control input differential parameterization of the linearized system

$$u_\delta = \frac{\Box}{\mu}\ddot{F}_\delta + \frac{\sigma}{\mu}F_\delta^{(4)}$$

A controller which achieves the equilibrium-to equilibrium transfer, via trajectory tracking, is given by the following expression

$$u_\delta = \frac{\Box}{\mu}\ddot{F}_\delta -$$
$$\frac{\sigma}{\mu}\Big[-[F_\delta^*(t)]^{(4)} + k_0(F_\delta - F_\delta^*(t)) + k_1(\dot{F}_\delta - \dot{F}_\delta^*(t))$$
$$+k_2\left(\ddot{F}_\delta - \ddot{F}_\delta^*(t)\right) + k_3\left(F_\delta^{(3)} - [F_\delta^*(t)]^{(3)}\right)\Big]$$

where the set of coefficients $\{k_0, k_1, k_2, k_3\}$ is chosen so that the closed loop characteristic polynomial of the linearized system, given by, $p(s) = s^4 + k_3 s^3 + k_2 s^2 + k_1 s + k_0$, is Hurwitz.

In terms of the normalized incremental state variables, the tracking feedback controller is given by

$$u_\delta = \frac{\Box}{\mu}\theta_{1\delta} - \frac{\sigma}{\mu}\Big[k_0(F_\delta - F_\delta^*(t)) + k_1\left(\dot{F}_\delta - \dot{F}_\delta^*(t)\right) + k_2\left(\theta_{1\delta} - \ddot{F}_\delta^*(t)\right) + k_3\left(\dot{\theta}_{1\delta} - [F_\delta^*(t)]^{(3)}\right)\Big] \tag{10}$$

D. Simulation Results

In the presented simulations, the equilibrium-to-equilibrium trajectory $F_\delta^*(t)$,for the flat output F_δ,was implemented by the following function

$$F_\delta^*(\tau) = F_\delta(\tau_{initial}) + [F_\delta(\tau_{final}) - F_\delta(\tau_{initial})] * \psi(\tau, \tau_{initial}, \tau_{final}) \tag{11}$$

where $\psi(\tau, \tau_{initial}, \tau_{final})$ is a polynomial in the normalized time variable τsatisfying,

$$\psi(\tau_{initial}, \tau_{initial}, \tau_{final}) = 0$$
$$\psi(\tau_{final}, \tau_{initial}, \tau_{final}) = 1$$

We have prescribed such a polynomial, smoothly interpolating between 0and 1, as follows

$$\psi(\tau, \tau_{initial}, \tau_{final}) = \left(\frac{\tau - \tau_{initial}}{T}\right)^5 * \sum_{i=1}^{6}(-1)^{i-1} r_i \left(\frac{\tau - \tau_{initial}}{T}\right)^{i-1}$$

with

$$r_1 = 252 \quad r_2 = 1050 \quad r_3 = 1800$$
$$r_4 = 1575 \quad r_5 = 700 \quad r_6 = 126$$

Figure 3 shows the closed loop responses for a one time intervals $T = 3.4$seconds.

IV. A HIERARCHICAL HIGH-GAIN APPROACH FOR STABILIZATION AND TRACKING

The previously derived feedback controller (10), is not quite robust with respect to un-modeled perturbations affecting the arm motions. Due to this fact, the control design should be split into a two level strategy as proposed by Fliess *et al* in [3] for the crane control problem, which, incidentally, belongs to the same family of under-actuacted systems as our treated example.

The high-gain hierarchical approach is as follows: We first use a lower level high-gain feedback controller to regulate the position of the arm towards the trajectory of an auxiliary variable reference signal, $q_\delta(\tau)$. We now consider the influence of the variable set point $q_\delta(\tau)$,on the "slow dynamics" of the system and proceed to determine the law of variation of this reference signal so that the flat output converges, towards a desired stabilizing reference trajectory.

From the differential parameterization (6), in terms of the flat output F_δ, we may obtain the following two relations:

$$\begin{aligned} \ddot{F}_\delta &= \frac{1}{\eta}\left(F_\delta - \mu\theta_{0\delta}\right); \\ \ddot{\theta}_{0\delta} &= \frac{\mu}{\sigma}\theta_{1\delta} - \frac{\eta}{\sigma}u_\delta. \end{aligned} \tag{12}$$

Consider then the following high gain controller

$$u_\delta = \frac{\sigma}{\eta}\left[k_1\dot{\theta}_{0\delta} + k_2(\theta_{0\delta} - q_\delta(t))\right] \tag{13}$$

where $q_\delta(t)$ is and auxiliary variable reference trajectory for the normalized, incremental arm angular position $\theta_{0\delta}$.Then the closed loop horizontal arm dynamics is governed by

$$\ddot{\theta}_{0\delta} + k_1\dot{\theta}_{0\delta} + k_2[\theta_{0\delta} - q_\delta(t)] = \frac{\mu}{\sigma}\theta_{1\delta}. \tag{14}$$

If the gain $k_1$and k_2,are sufficiently large, the motion of the closed loop system (14) asymptotically converges towards the following reduced order dynamics, valid on the controlled system' slow manifold,

$$\dot{\theta}_{0\delta} = -\frac{k_2}{k_1}[\theta_{0\delta} - q_\delta(t)] \tag{15}$$

and, thus, $\theta_{0\delta}$ exponentially converges to $q_\delta(t)$.Now the pendulum dynamics, represented by the first equation of (12), is governed by the reference signal $q_\delta(t)$. We then have,

$$\ddot{F}_\delta = \frac{1}{\eta}F_\delta - \frac{\mu}{\eta}q_\delta(t). \tag{16}$$

We may now regard the auxiliary reference signal $q_\delta(t)$as a virtual control input, which is to be chosen to guarantee that the flat output F_δ follows the desired reference trajectory $F_\delta^*(t)$.We then have,

$$\begin{aligned} &q_\delta(t) = \frac{1}{\mu}(\mu\theta_{0\delta} + \eta\theta_{1\delta}) \\ &+\tfrac{u}{\mu}\Big[-\ddot{F}_\delta^*(t) + 2\zeta w_n(\mu\dot{\theta}_{0\delta} + \eta\dot{\theta}_{1\delta} - \dot{F}_\delta^*(t)) \\ &\;+(\mu\theta_{0\delta} + \eta\theta_{1\delta} - F_\delta^*(t))w_n^2\Big]; \end{aligned} \tag{17}$$

A. Simulations results

Figure 4 shows the performance of the designed hierarchical feedback control system acting on the nonlinear system (1). The values of, $k_1$and k_2, were chosen to be: $k_1 = 25$and $k_2 = 10$.The reference trajectory, $F_\delta^*(\tau)$ for the flat output, $F_\delta(\tau)$, was set to be identically zero, as required by the stabilization task. The rest of the controller parameters were set to be, $\zeta = 0.70707$, $w_n = 0.9$.In this case, we obtained a control force which was nearly 5 times larger than in the previous design.

Figure 5 shows the performance of the nonlinear system (1) to the proposed hierarchical feedback controller (17) designed for tracking purposes. The reference trajectory, $F_\delta^*(\tau)$,for the flat output, $F_\delta(\tau)$,was set to be the same interpolating polynomial (11) used in the previous section. This trajectory causes the angular position of the arm to go from the initial value 0 [rad], to that of 1 [rad], for one equilibrium-to equilibrium transfer time intervals, $T = 3.2$[s].

V. A GPI APPROACH FOR STABILIZATION AND TRACKING

Let us consider the linearized system and define the incremental output $y_\delta = \theta_{0\delta}$.The linearized system is observable with respect this output and hence, it is constructible. Therefore, we can obtain an integral input-output parameterization of the incremental flat output F_δ and its time derivatives, modulo the unknown initial conditions. Indeed, from (5) we obtain the following structural estimates of the incremental state variables:

$$\begin{aligned} \widehat{\theta_{1\delta}} &= \frac{1}{\mu}\left[\int_0^\tau\int_0^\lambda u_\delta(\varrho)d\varrho d\lambda - \Box y_\delta\right] \\ \widehat{\dot{\theta}_{1\delta}} &= \frac{\sigma}{\mu}\int_0^\tau y_\delta(\lambda)d\lambda + \frac{\eta}{\mu}\int_0^\tau\int_0^\lambda\int_0^\varrho u_\delta(\upsilon)d\upsilon d\varrho d\lambda \\ \theta_{0\delta} &= y_\delta \\ \widehat{\dot{y}_\delta} &= \frac{1}{\Box}\left(\int_0^\tau u_\delta(\lambda)d\lambda - \mu\widehat{\dot{\theta}_{1\delta}}\right) \end{aligned}$$

We denote by $\left(\int\Phi\right)$the quantity $\int_0^\tau \Phi(\lambda)d\lambda$.

An integral input-output parameterization of the incremental flat output and its time derivatives, are given by:

$$
\begin{aligned}
\widehat{F}_\delta &= \frac{1}{\mu}\left(\sigma y_\delta + \eta\left(\int\int\int u_\delta\right)\right)\\
\widehat{\dot{F}}_\delta &= \frac{1}{\mu}\left(\int\int\int u_\delta - \Box\int y_\delta\right),\\
\widehat{\ddot{F}}_\delta &= \frac{1}{\mu}\left(\int\int u_\delta - \Box y_\delta\right)\\
\widehat{F^{(3)}}_\delta &= \frac{\mu}{\sigma}\left(\int u_\delta\right) - \frac{\Box}{\sigma}\widehat{\dot{F}}_\delta
\end{aligned}
\tag{18}
$$

Based on the previously derived controller (7)-(9), the above structural estimates of the incremental flat output, and its time derivatives, can be used in the synthesis of a Generalized Proportional Integral (**GPI**). One simply replaces the flat output F_δ, and its time derivatives, by the obtained integral input-output parameterization presented in (18). Taking into account the neglected initial conditions we proceed to compensate their destabilizing effect via a sufficient number of integrals of the output stabilization (or tracking) error. The required stabilizing **GPI** controller is obtained as:

$$
\begin{aligned}
&\widehat{u} = \frac{\Box}{\mu}\widehat{\ddot{F}}_\delta -\\
&\frac{\sigma}{\mu}\left(k_3\widehat{F}_\delta + k_4\widehat{\dot{F}}_\delta + k_5\widehat{\ddot{F}}_\delta + k_6\widehat{F^{(3)}_\delta} - \rho_1\right);\\
&\dot{\rho}_1 = -\beta_2 y_\delta + \rho_2;\quad \rho_1(0) = 0;\\
&\dot{\rho}_2 = -\beta_1 y_\delta + \rho_3;\quad \rho_2(0) = 0;\\
&\dot{\rho}_3 = -\beta_0 y_\delta;\quad \rho_3(0) = 0.
\end{aligned}
\tag{19}
$$

Then the closed loop dynamics system is governed by the following expression:

$$
\begin{aligned}
\frac{\Box}{\mu}\ddot{F}_\delta + \frac{\sigma}{\mu}F_\delta^{(4)} = -\frac{\sigma}{\mu}\Big[&k_3 F_\delta + k_4\dot{F}_\delta\\
&+\left(k_5 - \frac{\Box}{\mu}\right)\ddot{F}_\delta + k_6 F_\delta^{(3)}\\
&+\left(\overline{k}_0 + \overline{k}_1 t + \overline{k}_2 t^2\right) - \rho_1\Big]
\end{aligned}
$$

where $\overline{k}_0\overline{k}_1$and $\overline{k}_3$,are unknown constants which depend on the initial conditions. The closed loop system is thus given by,

$$
\begin{aligned}
F_\delta^{(7)} + k_6 F_\delta^{(6)} + k_5 F_\delta^{(5)} + k_4 F_\delta^{(4)} + k_3 F_\delta^{(3)}&\\
+\beta_2\ddot{y}_\delta + \beta_1\dot{y}_\delta + \beta_0 y_\delta &= 0
\end{aligned}
\tag{20}
$$

from the second equation of (6) and recalling that $y_\delta = \theta_{0\delta}$, we have that $y_\delta = (F_\delta - \eta\ddot{F}_\delta)/\mu$. Substituting this last expression into (20), we obtain the following closed loop dynamics for the flat output:

$$
\begin{aligned}
&F_\delta^{(7)} + k_6 F_\delta^{(6)} + k_5 F_\delta^{(5)} + \left[k_4 - \frac{\eta\beta_2}{\mu}\right]F_\delta^{(4)} +\\
&\left[k_3 - \frac{\eta\beta_1}{\mu}\right]F_\delta^{(3)} + \left[\frac{\beta_2}{\mu} - \frac{\beta_0\eta}{\mu}\right]\ddot{F}_\delta + \frac{\beta_1}{\mu}\dot{F}_\delta + \frac{\beta_0}{\mu}F_\delta = 0
\end{aligned}
\tag{21}
$$

It is clear that by appropriate choice of the design parameters, $\{k_6, k_5, k_4, k_3, \beta_2, \beta_1, \beta_0\}$, we guarantee that the roots of the closed loop characteristic polynomial,CUR

$$
\begin{aligned}
s^7 + k_6 s^6 + k_5 s^5 + \left[k_4 - \tfrac{\eta\beta_2}{\mu}\right]s^4 +&\\
\left[k_3 - \tfrac{\eta\beta 1}{\mu}\right]s^3 + \left[\tfrac{\beta_2}{\mu} - \tfrac{\beta_0\eta}{\mu}\right]s^2 +&\\
\tfrac{\beta 1}{\mu}s + \tfrac{\beta_0}{\mu} &= 0
\end{aligned}
$$

may be conveniently located in the left half of the complex plane.

Finally, the rest-to-rest maneuver, by means of suitable trajectory tracking, can also be accomplished using only the angular position measurements of the arm and the applied input force, by considering the following **GPI** feedback tracking controller:

$$
\begin{aligned}
&\widehat{u}_\delta = \frac{\Box}{\mu}\widehat{\ddot{F}}_\delta - \frac{\sigma}{\mu}\Big[-[F_\delta^*]^{(4)} + k_3(\widehat{F_\delta} - F_\delta^*) +\\
&k_4(\widehat{\dot{F}}_\delta - \left[\dot{F}_\delta^*\right]) + k_5(\widehat{\ddot{F}}_\delta - \left[\ddot{F}_\delta^*\right]) +\\
&k_6(\widehat{F_\delta^{(3)}} - [F_\delta^*]^{(3)}) - \rho_1\Big];\\
&\dot{\rho}_1 = -\beta_2(y_\delta - y_\delta^*) + \rho_2;\quad \rho_1(0) = 0;\\
&\dot{\rho}_2 = -\beta_1(y_\delta - y_\delta^*) + \rho_3;\quad \rho_2(0) = 0;\\
&\dot{\rho}_3 = -\beta_0(y_\delta - y_\delta^*);\quad \rho_3(0) = 0.
\end{aligned}
$$

where $y_\delta^*$is the desired horizontal arm, rest-to-rest, trajectory.

Figure 6 shows the closed loop system responses to **GPI** feedback stabilizing corntroller.

VI. Conclusions

Differential flatness allows one to systematically solve a number of interesting nonlinear control problem. In this instance, we have exploited flatness for the efficient regulation and tracking of the Furuta pendulum, which happens to be locally controllable (i.e. flat) around its unstable equilibrium. Since the system is also observable, and hence, constructible, an integral input-output parameterization of the incremental state variables and of the incremental flat output and its various time derivatives is possible. This allows us to obtain a trajectory tracking controller for the incremental flat output behavior guaranteeing the desired rest-to-rest maneuver. The input-output integral parametrization allows one to locally control the system, by simply resorting to a standard pole placement procedure on a higher order system devoid of internal instabilities. This feedback regulation scheme is carried out, without use of state measurements, or without devising dynamic systems devoted to solve the state estimation problem, as traditionally provided by asymptotic state observers.

References

[1] I. Fantoni and R. Lozano. "Control of the Furuta Pendulum Based on an Energy Approach." European Control conference ECC, 1999, 1999.

[2] M. Fliess, R. Márquez, E. Delaleau and H. Sira-Ramírez, "Correcteurs Proportionnels-Intégraux Généralisés, *ESAIM: Control, Optimization and Calculus of Variations*, URL: http://www.emath.fr/cocv/.(accepted for publication, to appear).

[3] M. Fliess, H. Sira-Ramírez and R. Márquez, "Regulation of non-minimun phase autputs: a flatness based approach" in *Perspectives in Control : Theory and Applications*, D. Normand-Cyrot Ed. pp. 143-163, Springer-Verlag, London, 1998.

FIGURES

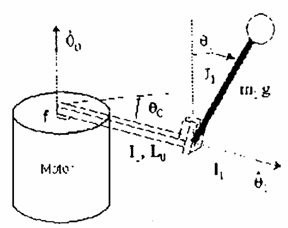

Figure 1: The Furuta pendulum system (**FP**).

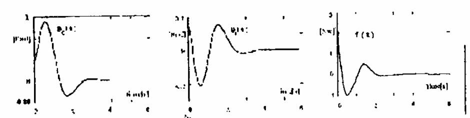

Figure 2: Closed loop responses of flatness-based controlled **FP** to stabilizing controller

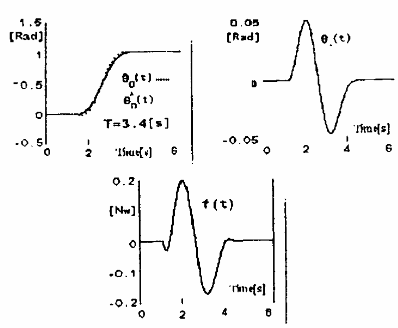

Figure 3: Closed loop system responses for a large horizontal arm angular position rest-to-rest excursion

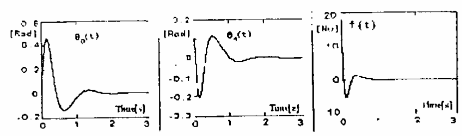

Figure 4: High-gain hierarchical feedback controller performance in a stabilization task.

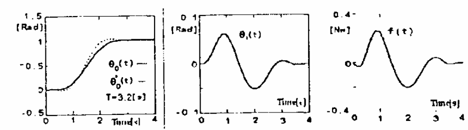

Figure 5: High-gain hierarchical feedack trajectory tracking controller performance for a t=3.2.

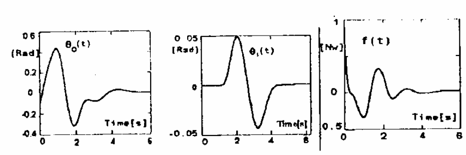

Figure 6: **GPI** Controlled stabilization responses of the **FP**.